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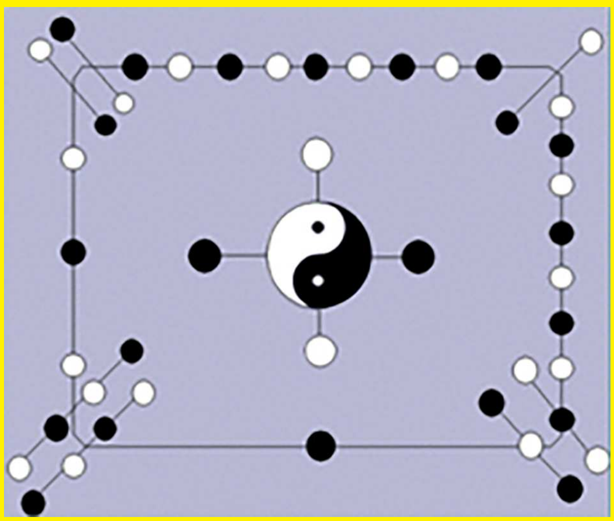

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Errors can not stand failure, but truth is not afraid of failure.

By R.Tagore, an Indian polymath, poet, musician and artist.

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On the Involute D̃-scroll in Euclidean 3-Space

Şeyda Kılıçoğlu

(Baskent University, Ankara, Turkey)

Süleyman ŞENYURT

(Ordu University, Ordu, Turkey)

E-mail: seyda@baskent.edu.tr, senyurtsuleyman@hotmail.com

Abstract: In this paper we consider two special ruled surfaces associated to a space curve α with curvature $k_1 \neq 0$ and its involute curve β. We will define and work on $\tilde{D}-scroll$, which is known as the rectifying developable surface, of any curve α and the *involute* $\tilde{D}-scroll$ of the curve α. Also we have examined the normal vectors of these special ruled surfaces $\tilde{D}-scroll$ and *involute* $\tilde{D}-scroll$, associated to each other. Further, as an example, we examined the positions of the $\tilde{D}-scroll$ and the *involute* $\tilde{D}-scroll$ relative to each other of a cylindrical helix.

Key Words: Darboux vector, involute curve, ruled surface, helix.

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§1. Introduction and Preliminaries

Deriving curves based on the other curves is a subject in geometry. Involute-evolute curves, Bertrand curves are this kind of curves. By using the similar method we produce a new ruled surface based on the other ruled surface. It is well-known that, if a curve is differentiable in an open interval, at each point, a set of mutually orthogonal unit vectors can be constructed. And these vectors are called Frenet frame or moving frame vectors. The rates of these frame vectors along the curve define curvatures of the curves. The set, whose elements are frame vectors and curvatures of a curve α, is called Frenet-Serret apparatus of the curves. Involvents play a part in the construction of gears. The evolute is the locus of the centers of tangent circles of the given planar curve, [12]. Let Frenet vector fields be V_1, V_2, V_3 of the curve α and let the first and second curvatures of the curve α be k_1 and k_2, respectively. The quantities $\{V_1, V_2, V_3, k_1, k_2\}$ are collectively Frenet-Serret apparatus of the curves. Also the Darboux vector provides a concise way of interpreting curvature k_1 and torsion k_2 geometrically; curvature is the measure of the rotation of the Frenet frame about the binormal unit vector, and torsion is the measure of the rotation of the Frenet frame about the tangent unit vector. For any unit speed curve α,

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in terms of the Frenet-Serret apparatus, the Darboux vector D can be expressed as, [10]

$$D(s) = k_2(s)V_1(s) + k_1(s)V_3(s). \tag{1.1}$$

Let a vector field be

$$\tilde{D}(s) = \frac{k_2}{k_1}(s)V_1(s) + V_3(s) \tag{1.2}$$

along $\alpha(s)$ under the condition that $k_1(s) \neq 0$ and it is called the modified Darboux vector field of α [6]. We will work on the special ruled surface $\tilde{D} - scroll$ which is also the rectifying developable surface, of the curves *evolute* α, and *involute* β. Further we will define and introduce *involute* $\tilde{D} - scroll$ of α. Also *involute* $\tilde{D} - scroll$ of α will be examined in terms of the Frenet-Serret apparatus of the curve α.

A ruled surface can always be described (at least locally) as the set of points swept by a moving straight line. A ruled surface is one which can be generated by the motion of a straight line ([2],[3]). Choosing a directrix on the surface, i.e. a smooth unit speed curve $\alpha(s)$ orthogonal to the straight lines, and then choosing $v(s)$ to be unit vectors along the curve in the direction of the lines, the velocity vector α_s and v satisfy

$$\left\langle \alpha', v \right\rangle = 0.$$

To illustrate the current situation, we bring here the famous example of L. K. Graves, [4], so called the $B - scroll$. The special ruled surfaces $B - scroll$ over null curves with null rulings in 3-dimensional Lorentzian space form has been introduced by L. K. Graves. The Gauss map of B-scrolls has been examined in [1]. The properties of the $B - scroll$ are also examined and $n - space$ and in Lorentzian $3 - space$ and $n - space$ with time-like directrix curve and null rulings ([7], [8], [9]). Also *involutive* $B - scroll$ (binormal scroll) of the curve α is defined as in the following definition and examined in [13]. In [14] the Differential geometric elements of the Involute $\tilde{\mathrm{D}}$-scroll is examined too.

Definition 1.1 *Let α and β be the curves . The tangent lines to a curve α generate a surface called the tores of α. If the curve β which lies on the tores intersect the tangent lines orthogonally is called an involute of α. If a curve β is an involute of α, then by definition α is an evolute of β. Hence given α, its evolutes are the curves whose tangent lines intersect α orthogonally.*

If the curve $\beta(s)$ is the involute of $\alpha(s)$, then we have that

$$\beta(s) = \alpha(s) + (c - s)V_1(s) \tag{1.3}$$

and $d(\alpha(s), \beta(s)) = |c - s|$, where $\forall s \in I$, $c = constant$, [5].

Theorem 1.1([5]) *$\alpha, \beta \subset \boldsymbol{E}^3$, α and β are the arclengthed curves with the arcparametres. Let β be the involute of the curve α. The quantities $\{V_1, V_2, V_3, k_1, k_2\}$ and $\{V_1^*, V_2^*, V_3^*, k_1^*, k_2^*\}$ are collectively Frenet-Serret apparatus of the curve α and the involute β, respectively. The Frenet-Serret apparatus of the involute β, in terms of the Frenet-Serret apparatus of the its* *evolute curve α are*

$$\begin{cases} V_1^* = V_2, \\ V_2^* = -\frac{k_1}{\sqrt{k_1^2+k_2^2}} V_1 + \frac{k_2}{\sqrt{k_1^2+k_2^2}} V_3, \\ V_3^* = \frac{k_2}{\sqrt{k_1^2+k_2^2}} V_1 + \frac{k_1}{\sqrt{k_1^2+k_2^2}} V_3 \end{cases} \tag{1.4}$$

$$k_1^* = \frac{\sqrt{k_1^2+k_2^2}}{(c-s)k_1}, \quad k_2^* = -\frac{k_2^2\left(\frac{k_1}{k_2}\right)'}{(c-s)k_1\left(k_1^2+k_2^2\right)}. \tag{1.5}$$

Corollary 1.1 *If the second curvature k_2 of the curve $\alpha(s)$ is a nonzero constant, i.e. $k_2' = 0$, then second curvature of involute β is*

$$k_2^* = \frac{-k_1' k_2}{(c-s)\, k_1\left(k_1^2+k_2^2\right)}. \tag{1.6}$$

Theorem 1.2 *Let β be the involute of the curve α. Let the first and second curvatures of the curve α be k_1 and k_2, respectively. The modified Darboux vector field of the involute β is*

$$\tilde{D}^* = \frac{k_2}{\sqrt{k_1^2+k_2^2}} V_1 - \frac{k_2^2\left(\frac{k_1}{k_2}\right)'}{\left(k_1^2+k_2^2\right)^{\frac{3}{2}}} V_2 + \frac{k_1}{\sqrt{k_1^2+k_2^2}} V_3. \tag{1.7}$$

Proof Since the definition of the modified Darboux vector field $\tilde{D}^* = \frac{k_2^*}{k_1^*} V_1^* + V_3^*$ and Theorem 1.2 it is trivial. □

Corollary 1.2 *If the second curvature k_2 of the curve α is constant but not equal to zero, then $k_2' = 0$. Hence, we have that the modified Darboux vector field of the involute β is*

$$\tilde{D}^* = \frac{k_2}{\sqrt{k_1^2+k_2^2}} V_1 - \frac{k_1' k_2}{\left(k_1^2+k_2^2\right)^{\frac{3}{2}}} V_2 + \frac{k_1 V_3}{\sqrt{k_1^2+k_2^2}}. \tag{1.8}$$

Definition 1.2([13]) *Let α and β be the arclengthed curves. Let $\beta(s)$ be the involute of the curve $\alpha(s)$. The equation*

$$\varphi^*(s,v) = \beta(s) + v V_3^*(s) \tag{1.9}$$

is the parametrization of the ruled surface which is called involutive V_3^* − scroll (binormal scroll) of the curve β.

Definition 1.3 *The ruled surface*

$$\varphi(s,u) = \alpha(s) + u\tilde{D}(s)$$
$$\varphi(s,u) = \alpha(s) + u\frac{k_2}{k_1}(s) V_1(s) + u V_3(s)$$

is the parametrization of the ruled surface which is called rectifying developable surface of the *curve α in [6]. Here, it is referred to as $\tilde{D}-scroll$ cause of generator vector is modified Darboux vector field $\tilde{D}$.*

Definition 1.4 *Let the curve β be involute of α, hence*

$$\varphi^*(s,v)=\beta(s)+v\left(\frac{k_2^*}{k_1^*}(s)V_1^*\left(s\right)+V_3^*\left(s\right)\right) \tag{1.4}$$

is the parametrization of the $\tilde{D}-scroll$ of involute β. Further this rectifying developable surface is called involute $\tilde{D}-scroll$ of α.

We can write the parametrization of the $\tilde{D}-scroll$ of *involute* β, in terms of the Frenet-Serret apparatus of the curve α, as in the following theorem. Hence it can be called *involute* $\tilde{D}-scroll$ of the curve α.

§2. On the Involute $\tilde{\mathbf{D}}$-scroll in Euclidean 3-Space

In this section to determine the positions of the $\tilde{D}-scroll$ and *involute* $\tilde{D}-scroll$, we questioned their normal vector vectors.

Theorem 2.1 *If β is the involute curve of the curve α, then the parametrization of the involute $\tilde{D}-scroll$ of the curve α in terms of the Frenet-Serret apparatus of the curve α is*

$$\varphi^*\left(s,v\right)=\alpha+\left(\lambda+\frac{vk_2}{\sqrt{k_1^2+k_2^2}}\right)V_1-\frac{k_2^2\left(\frac{k_1}{k_2}\right)'}{(k_1^2+k_2^2)^{\frac{3}{2}}}V_2+\frac{vk_1}{\sqrt{k_1^2+k_2^2}}V_3. \tag{2.1}$$

Proof Substituting equation (2.1) into equations (1.3) and (1.8), the proof is complete. □

Corollary 2.1 *If the second curvature k_2 of the curve α is constant but not equal to zero, then $k_2'=0$. Hence, the parametrization of involute $\tilde{D}-scroll$ is*

$$\varphi^*\left(s,v\right)=\alpha+\left(c-s+\frac{vk_2}{\sqrt{k_1^2+k_2^2}}\right)V_1-\frac{vk_1'k_2}{(k_1^2+k_2^2)^{\frac{3}{2}}}V_2+\frac{vk_1}{\sqrt{k_1^2+k_2^2}}V_3. \tag{2.2}$$

Theorem 2.2 *The equation $\varphi\left(s,u\right)=\alpha\left(s\right)+u\tilde{D}\left(s\right)$ is the parametrization of the ruled surfaces which is called $\tilde{D}-scroll$. Then the normal vector field N of ruled surface $\tilde{D}-scroll$ is*

$$N=V_2. \tag{2.3}$$

Proof We can calculate that

$$\tilde{D}'=\left(\frac{k_2}{k_1}\right)'V_1.$$

For the surface

$$\varphi(s,u)=\alpha(s)+u\tilde{D}(s)$$

the vectors

$$\begin{aligned}\varphi_s &= \left(1+u\left(\frac{k_2}{k_1}\right)'\right)V_1,\\ \varphi_u &= \tilde{D}(s)=\left(\frac{k_2}{k_1}(s)V_1(s)+V_3(s)\right)\end{aligned}$$

are not a system of orthogonal vectors. Hence we will use the Gram–Schmidt orthogonalization. Let us take

$$e_1=\frac{\varphi_s}{\|\varphi_s\|}=\mp V_1,\ e_2=\frac{\varphi_u-\frac{\langle\varphi_s,\varphi_u\rangle}{\langle\varphi_s,\varphi_s\rangle}\varphi_s}{\|\varphi_u-\frac{\langle\varphi_s,\varphi_u\rangle}{\langle\varphi_s,\varphi_s\rangle}\varphi_s\|}=V_3$$

Since $\{e_1,e_2\}$ is a system of orthogonal vectors, normal vector field N is

$$N=e_1\wedge e_2=V_2.$$ □

Theorem 2.3 *The normal vector field of involute $\tilde{D}-scroll$ of the curve $\alpha$is*

$$N^*=\frac{-k_1V_1+k_2V_3}{(k_1^2+k_2^2)^{\frac{1}{2}}}\tag{2.5}$$

Proof We have already get the equation of the *involute* $\tilde{D}-scroll$ of the curve α. Also we know that the normal vector field N^* of any $\tilde{D}-scroll$ is

$$N^*=V_2^*.$$

So normal vector field N^* of the *involute* $\tilde{D}-scroll$ is

$$N^*=\frac{-k_1V_1+k_2V_3}{(k_1^2+k_2^2)^{\frac{1}{2}}}.$$ □

Lets examine the positions of the ruled surface $\tilde{D}-scroll$ and the *involute* $\tilde{D}-scroll$. Based on their normal vector fields.

Theorem 2.4 *The ruled surface $\tilde{D}-scroll$ and the involute $\tilde{D}-scroll$ of the curve α are perpendicular surfaces.*

Proof Using the orthogonality condition; $\langle N,N^*\rangle=0$,

$$\left\langle V_2,\frac{-k_1V_1+k_2V_3}{(k_1^2+k_2^2)^{\frac{1}{2}}}\right\rangle=0$$

it is easy to say that, the normal vector field N of $\tilde{D}-scroll$ of the curve α and the normal vector field $N^*$of *involute* $\tilde{D}-scroll$ of the curve α are perpendicular. then

$$N^* \perp N. \qquad \square$$

Theorem 2.5 *The tangent vector fields $V_1(s)$ and $V_1^*(s)$ are perpendicular, then the ruled surface $\tilde{D}-scrolls$ along to the curves α and the β are perpendicular surfaces.*

Helix is one of the fascinating curve in science and nature. A helix which lies on the cylinder is called cylindrical helix or general helix. A curve α with $k_2(s) \neq 0$ is called a cylindrical helix if the tangent lines of make a constant angle with a fixed direction. If the curve is a general helix, the ratio of the first curvature of the curve to the torsion of the curve must be constant. Further we call a curve a circular helix if both $k_2(s) \neq 0$ and $k_1(s)$ are constant.

Corollary 2.2 *If the curve α is a cylindrical helix, then the involute β of the curve α is a planar curve.*

Proof It has been known that the curve $\alpha(s)$ is a cylindrical helix if and only if $\left(\frac{k_1}{k_2}\right) = d$ is constant, then $\left(\frac{k_1}{k_2}\right)' = 0$, also $k_2^* = 0$. $\square$

Lemma 2.1([6]) *For a the ruled surface $\varphi(s,u) = \alpha(s) + u\eta(s)$ and its unit speed curve α, with $k_1 \neq 0$, the following are equivalent:*

(1) *The ruled surface is a non-singular surface;*
(2) *α is a cylindrical helix;*
(3) *The ruled surface of α is a cylindrical surface.*

Theorem 2.6 *The involute $\tilde{D}-scroll$ of the curve of α is a cylindrical surface, if*

$$\frac{\left(k_1^2+k_2^2\right)^{\frac{3}{2}}}{k_2^2\left(\frac{k_1}{k_2}\right)'} = constant. \tag{2.5}$$

Proof Let α not be a cylindrical helix , $k_1 \neq 0$, *involute* β is a helix. If *involute* β is a cylindrical helix, then $\frac{k_1^*}{k_2^*}$ is constant. Hence

$$\frac{k_1^*}{k_2^*} = -\frac{\left(k_1^2+k_2^2\right)^{\frac{3}{2}}}{k_2^2\left(\frac{k_1}{k_2}\right)'}$$

is constant. Where $\dfrac{k_1}{k_2} \neq constant$. Cause of the Lemma 2.1; if $-\frac{\left(k_1^2+k_2^2\right)^{\frac{3}{2}}}{k_2^2\left(\frac{k_1}{k_2}\right)'} = constant$, then *involute* $\tilde{D}-scroll$ is a cylindrical surface. $\square$

Theorem 2.7 *Let β be involute of α, if the curve α is a cylindrical helix, the angle between the modified Darboux vector field of the $involute-evolute$ pair (α,β) is a nonzero constant. It* *is the function of* $d=\frac{k_1}{k_2}$ *as in the following equality*

$$\left\langle \tilde{D}^*,\tilde{D}\right\rangle=\frac{\sqrt{d^2+1}}{d};\qquad d=\frac{k_1}{k_2}. \tag{2.6}$$

Proof Since

$$\left\langle \tilde{D}^*,\tilde{D}\right\rangle=\frac{\sqrt{k_1^2+k_2^2}}{k_1}$$

it is trivial. □

Theorem 2.8 *Let the involute curve of a cylindrical helix* α *be* $\beta(s)=\alpha(s)+(c-s)V_1(s)$, *then the involute* $\tilde{D}-scroll$ *of a cylindrical helix* α *with* $\frac{k_1}{k_2}=d$, *is*

$$\varphi^*(s,v)=\alpha+\left(c-s+\frac{v}{\sqrt{d^2+1}}\right)V_1+\frac{vd}{\sqrt{d^2+1}}V_3. \tag{2.7}$$

Proof If $\alpha(s)$ is a cylindrical helix, then $\left(\frac{k_1}{k_2}\right)'=0$. Using the equation $\tilde{D}^*$,

$$\varphi^*(s,v)=\alpha+\left(c-s+\frac{vk_2}{\sqrt{k_1^2+k_2^2}}\right)V_1+\frac{vk_1}{\sqrt{k_1^2+k_2^2}}V_3$$

is the *involute* $\tilde{D}-scroll$ of a cylindrical helix α. □

Example 2.1 Lets examine the $\tilde{D}-scroll$ of the a cylindrical helix

$$\alpha(s)=(a\cos ws,a\sin ws,bws)\ ,a>0,$$

with curvatures $k_1=w^2a$ and $k_2=w^2b$. Here $\frac{k_1}{k_2}=\frac{a}{b}=d$ and $w^2=\frac{1}{a^2+b^2}$ we have the parametrization of the $\tilde{D}-scroll$ of the cylindrical helix α

$$\varphi(s,u)=\alpha(s)+u\tilde{D}(s)$$
$$\varphi(s,u)=\alpha(s)+u\frac{k_2}{k_1}V_1(s)+uV_3(s)$$

$$\varphi(s,u)=\begin{pmatrix} a\cos ws-\sin ws\left(ubw-abuw^3\right),a\sin ws+\cos ws\left(buw-abuw^3\right), \\ bws+u\left(\frac{b^2}{a}w+a^2w^3\right)\end{pmatrix}$$

where

$$V_3=\left(abw^3\sin ws,-abw^3\cos ws,a^2w^3\right).$$

Example 2.2 The $\tilde{D}-scroll$ of the a cylindrical helix α,

$$\alpha(s)=\left(\cos\frac{s}{\sqrt{2}},\sin\frac{s}{\sqrt{2}},\frac{s}{\sqrt{2}}\right)\ ,a=1>0$$

with curvatures $k_1 = k_2 = \frac{1}{2}$, we have the parametrization of the $\tilde{D}-scroll$ along the cylindrical helix α

$$\begin{aligned}\varphi(s,u) &= \alpha(s) + u\frac{k_2}{k_1}(s)V_1(s) + uV_3(s)\\ &= \left(\cos\frac{s}{\sqrt{2}} - \frac{u}{2\sqrt{2}}\sin\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}} + \frac{u}{2\sqrt{2}}\cos\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} + u\frac{3}{2\sqrt{2}}\right).\end{aligned}$$

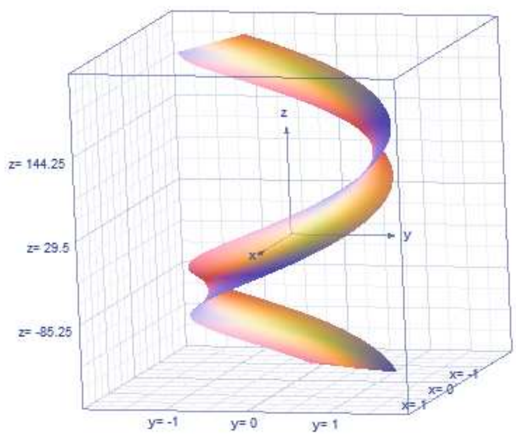

FIGURE 1 $\tilde{D}-$ scroll cylindrical helix

Example 2.3 The parametrization of the *involute* $\tilde{D} - scroll$ along the cylindrical helix $\alpha(s) = (a\cos ws, a\sin ws, bws)$, $a > 0$,with curvatures $k_1 = w^2 a$ and $k_2 = w^2 b$. Let find the *involute* $\tilde{D} - scroll$ along the cylindrical helix α have parametrization as with $\frac{k_1}{k_2} = \frac{a}{b} = d$, is

$$\varphi^*(s,v) = \alpha + \left(c - s + \frac{v}{\sqrt{d^2+1}}\right)V_1 + \frac{vd}{\sqrt{d^2+1}}V_3$$

$$\begin{aligned}\varphi^*(s,v) &= \alpha + \left(\lambda + \frac{vk_2}{\sqrt{k_1^2+k_2^2}}\right)V_1 + \frac{vk_1}{\sqrt{k_1^2+k_2^2}}V_3\\ &= \begin{bmatrix}\cos ws & -\sin ws & 0\\ \sin ws & \cos ws & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix} a\\ \left[\lambda aw + vabw^2\left(1 - aw^2\right)\right]\\ 0\end{bmatrix}\\ &+ \begin{bmatrix}0\\ 0\\ cbw + vw^2\left(b^2 + a^3w^2\right)\end{bmatrix}.\end{aligned}$$

Corollary 2.3 *The involute $\tilde{D}-scroll$ of the a cylindrical helix α can be produced by rigid motion.*

Example 2.4 The *involute* $\tilde{D}-scroll$ of the a cylindrical helix α,

$$\alpha(s) = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right) \ , a = 1 > 0$$

with curvatures $k_1 = k_2 = \frac{1}{2}$, we have the parametrization of the *involute* $\tilde{D}-scroll$ along the cylindrical helix α,

$$\varphi^*(s,v) = \left(\cos\frac{s}{\sqrt{2}} - \left[\frac{c-s}{\sqrt{2}} + \frac{v}{4}\right]\sin\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}} + \left[\frac{c-s}{\sqrt{2}} + \frac{v}{4}\right]\cos\frac{s}{\sqrt{2}}, \frac{c}{\sqrt{2}} + \frac{3v}{4}\right).$$

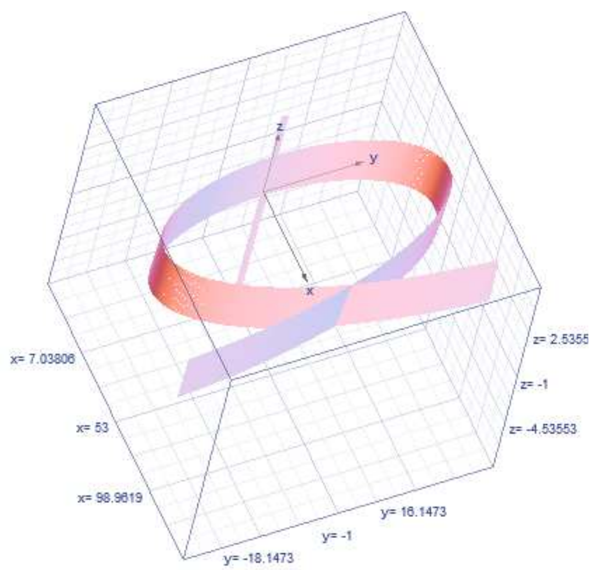

FIGURE 2 $\tilde{D}-$ scroll cylindrical helix

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On Prime Graph $PG_2(R)$ of a Ring

Sandeep S.Joshi

Department of Mathematics
D.N.C.V.P's Shirish Madhukarrao Chaudhari College, Jalgaon 425 001, India

Kishor F.Pawar

Department of Mathematics, School of Mathematical Sciences
Kavayitri Bahinabai Chaudhari North Maharashtra University, Jalgaon 425 001, India

E-mail: sandeep.s.joshi07@gmail.com, kfpawar@nmu.ac.in

Abstract: In the present paper we define a simple undirected graph $PG_2(R)$ with all the elements of a ring R as vertices, and two distinct vertices x, y are adjacent if and only if either $x \cdot y = 0$ or $y \cdot x = 0$ or $x + y \in Z(R)$, the set of all zero divisors of R (including zero). We have proved that $PG_2(\mathbb{Z}_n)$ is Eulerian for any odd positive integer n. Also we discuss the Planarity and girth of $PG_2(R)$ and some cases which gives the degree of all vertices in $PG_2(R)$, over a ring $\mathbb{Z}_n$, for $n \leq 100$.

Key Words: Ring, prime graph of a ring $PG_2(R)$, degree, planarity, girth.

AMS(2010): 05C25, 05C90, 05C99.

§1. Introduction

The study of graph theory for a commutative ring began when Beck in [1] introduced the notion of zero divisor of the graph. The graph $\Gamma_2(R)$ defined by R. Sen Gupta et al. [2] as: let R be a ring with unity and let $G = (V, E)$ be an undirected graph in which $V = R$ - $\{0\}$ and for any $a, b \in V$, $ab \in E$ if and only if $a \neq b$ and either $a \cdot b = 0$ or $b \cdot a = 0$ or $a + b$ is a zero divisor (including zero). Another graph structure associated to a ring called prime graph was introduced by Satyanarayana et al. [3]. Prime graph is defined as a graph whose vertices are all elements of the ring and any two distinct vertices $x, y \in R$ are adjacent if and only if $xRy = 0$ or $yRx = 0$. This graph is denoted by $PG(R)$. Pawar and Joshi in [5] gave a simple formulation for finding the degrees of vertices of prime graph $PG(R)$ as well as it's complement $(PG(R))^c$. Also the number of triangles in $PG(R)$ and $(PG(R))^c$ have been calculated using simple combinatorial approach. We introduced the prime graph $PG_1(R)$ of a ring and discussed all the results related to degree of vertices, Eulerianity, planarity and girth in [6]. Here, we introduced a new type of graph called $PG_2(R)$ as a generalization of [2].

In second section of this paper we give definition and some examples of $PG_2(R)$. In next section we try to find the degree of vertices in $PG_2(R)$ by distributing the vertex set $V(G)$ into

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two sets viz. the set of all zero-divisors and the set of all units and discussed some more cases which gives the degree of all vertices in $PG_2(\mathbb{Z}_n)$, for $n \leq 100$. In last section, we discussed the eulerianity, planarity and girth of $PG_2(R)$.

We refer to [3]-[4] for basic terminology and definitions.

§2. The Prime Graph $PG_2(R)$ of a Ring

Definition 2.1 *The prime graph $PG_2(R)$ is a graph with all the elements of a ring R as vertices, and any two distinct vertices x, y are adjacent if and only if $x \cdot y = 0$ or $y \cdot x = 0$ or $x + y \in Z(R)$, the set of all zero-divisors of R.*

Example 2.2 Consider $\mathbb{Z}_n$, the ring of integers modulo n.

(1) Let $R = \mathbb{Z}_2$. The vertex set $V(PG_2(\mathbb{Z}_2)) = \{0, 1\}$. Since $0R1 = 0$, the edge set $E(PG_2(\mathbb{Z}_2)) = \{01\}$ and the graph $PG_2(\mathbb{Z}_2)$ as shown in figure below.

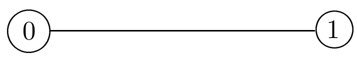

FIGURE 1. $PG_2(\mathbb{Z}_2)$

(2) Let $R = \mathbb{Z}_3$. The vertex set $V(PG_2(\mathbb{Z}_3)) = \{0, 1, 2\}$. Since $0R1 = 0, 0R2 = 0, 1+2 = 0$, the edge set $E(PG_2(\mathbb{Z}_3)) = \{01, 02, 12\}$ and the graph $PG_2(\mathbb{Z}_3)$ as shown in figure below.

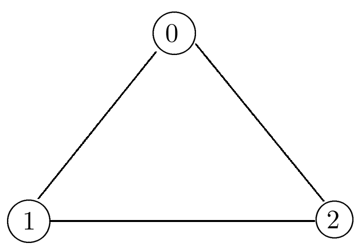

FIGURE 2. $PG_2(\mathbb{Z}_3)$

(3) Let $R = \mathbb{Z}_4$. The vertex set $V(PG_2(\mathbb{Z}_4)) = \{0, 1, 2, 3\}$, the edge set $E(PG_2(\mathbb{Z}_4)) = \{01, 02, 03, 13\}$ and the graph $PG_2(\mathbb{Z}_4)$ as shown in figure below-

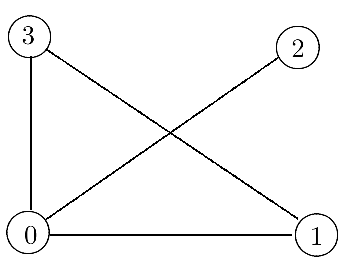

FIGURE 3. $PG_2(\mathbb{Z}_4)$

(4) Let $R = \mathbb{Z}_5$. The vertex set $V(PG_2(\mathbb{Z}_5)) = \{0, 1, 2, 3, 4\}$, the edge set $E(PG_2(\mathbb{Z}_5)) =$

$\{01, 02, 03, 04, 14, 23\}$ and the graph $PG_2(\mathbb{Z}_5)$ as shown in figure below.

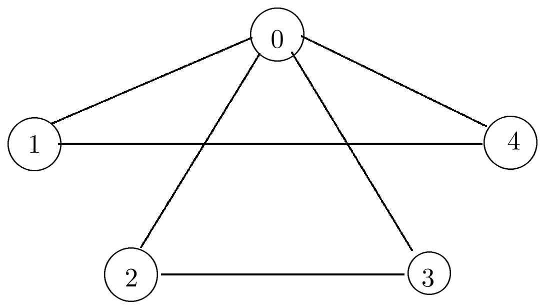

FIGURE 4. $PG_2(\mathbb{Z}_5)$

§3. Degree of Vertices in $PG_2(\mathbb{Z}_n)$

In this section, we find the degree of every vertex of $PG_2(\mathbb{Z}_n)$, for $n \leq 100$ by giving some illustrative examples.

Theorem 3.1 *$PG_2(\mathbb{Z}_n)$ is never complete graph unless $n = 2$ or 3.*

Proof From Figures 1 and 2 we conclude the theorem. □

Theorem 3.2 *$PG_2(\mathbb{Z}_{2^r})$, where $r \in \mathbb{N} - \{1\}$, has two components consisting of zero divisors and units of $(\mathbb{Z}_{2^r})$ respectively. The first is $K_{2^{r-1}}$ consists of all zero divisors and the other is $K_{2^{r-1}+1}$ consists of all the units and the element zero.*

Proof From Figure 3 we conclude the theorem. □

Theorem 3.3 *Let F be a finite field with $|F| = p^n, p \geq 3$ for some prime p and $n \in \mathbb{N}$, then $PG_2(F)$ is a union of $(p^n - 1)/2$ copies of K_3 in which the element zero is adjacent to all the vertices.*

Proof From Figure 4 we conclude the theorem. □

Example 3.4 Let $R = \mathbb{Z}_6$. The vertex set $V(PG_2(\mathbb{Z}_6)) = \{0, 1, 2, 3, 4, 5\}$, the edge set $E(PG_2(\mathbb{Z}_6)) = \{01, 02, 03, 04, 05, 12, 13, 15, 23, 24, 34, 35, 45\}$ and the graph $PG_2(\mathbb{Z}_6)$ as shown in figure below.

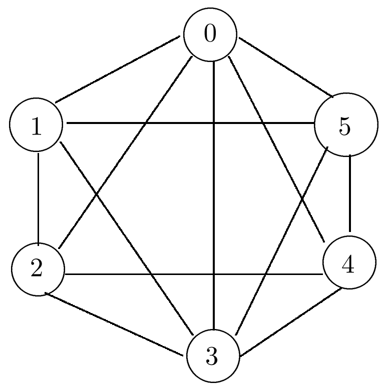

FIGURE 5. $PG_2(\mathbb{Z}_6)$

In $\mathbb{Z}_6$, zero-divisors $Z(\mathbb{Z}_6) = \{0, 2, 3, 4\}$, units $U(\mathbb{Z}_6) = \{1, 5\}$ and the value of $\phi(6) = 2$.

$$\begin{aligned}
deg(0) &= n - 1 = 6 - 1 = 5 \\
deg(2) &= n - \phi(n) = 6 - 2 = 4 \\
deg(3) &= 2q - 1 = 2 \cdot 3 - 1 = 6 - 1 = 5 \\
deg(4) &= n - \phi(n) = 6 - 2 = 4
\end{aligned}$$

and as n is even,

$$\begin{aligned}
deg(1) &= n - \phi(n) = 6 - 2 = 4 \\
deg(5) &= n - \phi(n) = 6 - 2 = 4.
\end{aligned}$$

From Example 3.4 we conclude the following three results.

Theorem 3.5 *For any $n \in \mathbb{N}$, the degree of vertex zero in $PG_2(\mathbb{Z}_n)$ is $n - 1$.*

Theorem 3.6 *Let u be the unit element in a ring $\mathbb{Z}_n$, for any $n \in \mathbb{N}$, the degree of u in $PG_2(\mathbb{Z}_n)$ is*

$$\begin{aligned}
deg(u) &= n - \phi(n), && \textit{if } n \textit{ is even} \\
&= n - \phi(n) + 1, && \textit{if } n \textit{ is odd.}
\end{aligned}$$

Theorem 3.7 *Let z be a non-zero zero-divisor in a ring $\mathbb{Z}_n$, for any $n \in \mathbb{N}$ and $n = p \cdot q$, where p and q are distinct primes. Then the degree of z in $PG_2(\mathbb{Z}_n)$ is*

(a) If $p = 2$, then

$$\begin{aligned}
deg(z) &= 2q - 1, && \text{if } z \text{ is multiple of } q \\
&= n - \phi(n), && \text{otherwise.}
\end{aligned}$$

(*b*) *If $p \neq 2$, then*

$$deg(z) = n - \phi(n) + (p-2), \quad \text{if } z \text{ is multiple of } p$$
$$= 2q + (p-3), \quad \text{if } z \text{ is multiple of } q.$$

Example 3.8 Let $R = \mathbb{Z}_9$. The vertex set $V(PG_2(\mathbb{Z}_9)) = \{0,1,2,3,4,5,6,7,8\}$, the edge set $E(PG_2(\mathbb{Z}_9)) = \{01,02,03,04,05,06,07,08,36,12,15,18,42,45,48,72,75,78\}$ and the graph $PG_2(\mathbb{Z}_9)$ as shown in figure below.

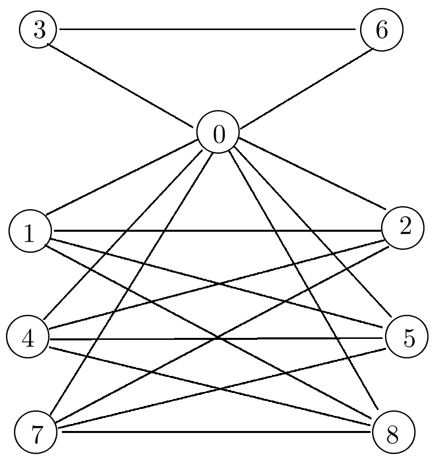

FIGURE 6. $PG_2(\mathbb{Z}_9)$

In $\mathbb{Z}_9$, zero-divisors $Z(\mathbb{Z}_9) = \{0,3,6\}$, units $U(\mathbb{Z}_9) = \{1,2,4,5,7,8\}$ and the value of $\phi(9) = 6$ and as n is odd,

$$deg(3,6) = 9 - \phi(9) - 1 = 9 - 6 - 1 = 2$$
$$deg(1,2,4,5,7,8) = 9 - \phi(9) + 1 = 9 - 6 + 1 = 4.$$

From Example 3.8 we conclude the following three results.

Theorem 3.9 *Let $n = p^r$, where p is an odd prime and $r \in \mathbb{N} - \{1\}$ then $PG_2(\mathbb{Z}_n)$ has $(p+1)/2$ components, one is $K_{p^{r-1}}$ consisting of the zero divisors and $(p-1)/2$ copies of $K_{p^{r-1},p^{r-1}} \bigcup \{0\}$ for the units and the element zero.*

Theorem 3.10 *Let z be a non-zero zero-divisor in a ring $\mathbb{Z}_n$, for any $n \in \mathbb{N}$ such that $z^2 \equiv 0 \ (mod\ n)$. Then the degree of z in $PG_2(\mathbb{Z}_n)$ is*

$$deg(z) = n - \phi(n) - 1.$$

Theorem 3.11 *Let u be the unit element and z be a non-zero zero-divisor in a ring $\mathbb{Z}_{p^2}$, for* *any prime p. Then from the Theorem 3.6 the degree of u is*

$$\begin{aligned} deg(u) &= n - \phi(n), & \text{if } n \text{ is even} \\ &= n - \phi(n) + 1, & \text{if } n \text{ is odd} \end{aligned}$$

and from the Theorem 3.10 the degree of z is

$$deg(z) = n - \phi(n) - 1.$$

Example 3.12 Let $R = \mathbb{Z}_{2^n p}$, for any $n \in \mathbb{N}$, where p is prime,

(a) If $p = 2$, then

(1) If $n = 1$, $R = \mathbb{Z}_4$, the non-zero zero-divisor is 2. Hence

$$deg(2) = 4 - \phi(4) - 1 = 4 - 2 - 1 = 1.$$

(2) If $n = 2$, $R = \mathbb{Z}_8$, the set of non-zero zero-divisors, $Z(\mathbb{Z}_8) - \{0\} = \{2, 4, 6\}$. So

$$deg(2, 4, 6) = 8 - \phi(8) - 1 = 8 - 4 - 1 = 3.$$

(3) If $n = 3$, $R = \mathbb{Z}_{16}$, the set of non-zero zero-divisors, $Z(\mathbb{Z}_{16}) - \{0\} = \{2, 4, 6, 8, 10, 12, 14\}$. Therefore

$$deg(2, 4, 6, 8, 10, 12, 14) = 16 - \phi(16) - 1 = 16 - 8 - 1 = 7.$$

Similarly, we find the degree of all non-zero zero-divisors in $R = \mathbb{Z}_{32}, \mathbb{Z}_{64}$ and so on. In general, we conclude that if $p = 2$, then

$$deg(z) = n - \phi(n) - 1.$$

(b) If $p \neq 2$, then

(1) If $n = 1$, $R = \mathbb{Z}_{2p}$ where $p = 3, 5, 7, \cdots$ then by Theorem 3.7

$$\begin{aligned} deg(z) &= n - \phi(n) + (p - 2), & \text{if } z \text{ is multiple of } p \\ &= 2q + (p - 3), & \text{if } z \text{ is multiple of } q. \end{aligned}$$

The results are same for $R = \mathbb{Z}_{10}, \mathbb{Z}_{14}$ and so on.

(2) If $n = 2$, $R = \mathbb{Z}_{4p}$, where $p = 3, 5, 7, \cdots$. Let $p = 3$, $R = \mathbb{Z}_{12}$, the set of non-zero zero-divisors, $Z(\mathbb{Z}_{12}) - \{0\} = \{2, 4, 6, 8, 10, 3, 9\}$ and $6^2 \equiv 0(mod\ 12)$. Hence

$$\begin{aligned} & deg(6) = 12 - \phi(12) - 1 = 12 - 4 - 1 = 7 \\ & deg(3, 9) = 12 - 4 - 1 = 7, & \text{if } z \text{ is multiple of } p \end{aligned}$$

$$\begin{aligned}
deg(2) &= 12 - 4 - 1 = 7, && \text{if } z \text{ is multiple of } 2^1\\
deg(8) &= 12 - 4 - 1 = 7, && \text{if } z \text{ is multiple of } 2^1\\
deg(10) &= 12 - 4 - 1 = 7, && \text{if } z \text{ is multiple of } 2^1\\
deg(4) &= 12 - 4 - 1 + 2 = 9, && \text{if } z \text{ is multiple of } 2^2.
\end{aligned}$$

The results are same for $R = \mathbb{Z}_{20}, \mathbb{Z}_{28}$ and so on.

(3) If $n = 3$, $R = \mathbb{Z}_{8p}$, where $p = 3, 5, 7, \cdots$. Let $p = 3$, $R = \mathbb{Z}_{24}$, the set of non-zero zero-divisor, $Z(\mathbb{Z}_{24}) - \{0\} = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 3, 9, 15, 21\}$. Therefore

$$\begin{aligned}
deg(6, 18) &= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } 2p\\
deg(12) &= n - \phi(n) - 1, && \text{if } z^2 \equiv 0 \ (mod\ n)\\
deg(3, 9, 15, 21) &= n - \phi(n) + (p - 2), && \text{if } z \text{ is multiple of } p\\
deg(2, 4, 10, 14, 16, 20, 22) &= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } 2^1\\
deg(8) &= n - \phi(n) - 1 + 2^{n-1}, && \text{if } z \text{ is multiple of } 2^n.
\end{aligned}$$

The results are same for $R = \mathbb{Z}_{40}, \mathbb{Z}_{56}$ and so on. In general, we conclude that if $p \neq 2$, then

$$\begin{aligned}
deg(z) &= n - \phi(n) + (p - 2), && \text{if } z \text{ is multiple of } p\\
&= n - \phi(n) - 1, && \text{if } z^2 \equiv 0 \ (mod\ n)\\
&= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } 2p\\
&= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } 2, 2^2, ., 2^{n-1}\\
&= n - \phi(n) - 1 + 2^{n-1}, && \text{if } z \text{ is multiple of } 2^n.
\end{aligned}$$

From Example 3.12 we conclude the following theorem.

Theorem 3.13 *Let z be a non-zero zero-divisor in a ring $\mathbb{Z}_{2^n p}$, for any $n \in \mathbb{N}$, where p is prime*

(*a*) *If $p = 2$, then*

$$deg(z) = n - \phi(n) - 1.$$

(*b*) *If $p \neq 2$, then*

$$\begin{aligned}
deg(z) &= n - \phi(n) + (p - 2), && \text{if } z \text{ is multiple of } p\\
&= n - \phi(n) - 1, && \text{if } z^2 \equiv 0 \ (mod\ n)\\
&= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } 2p\\
&= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } 2, 2^2, \cdots, 2^{n-1}\\
&= n - \phi(n) - 1 + 2^{n-1}, && \text{if } z \text{ is multiple of } 2^n.
\end{aligned}$$

Example 3.14 Let $R = \mathbb{Z}_{2^n p^2}$, for any $n \in \mathbb{N}$, where p is odd prime.

(a) If $n = 1$, then $R = \mathbb{Z}_{2p^2}$, where $p = 3, 5, 7, \cdots$.

(1) Let $p = 3$, $R = \mathbb{Z}_{18}$ and 6^2, $12^2 \equiv 0(mod\ 18)$. Hence

$$\begin{aligned}
deg(6,12) &= n - \phi(n) - 1, && \text{if } z^2 \equiv 0\ (mod\ n) \\
deg(3,15) &= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } p \\
deg(9) &= n - \phi(n) - 1 + p(p-1), && \text{if } z \text{ is multiple of } p^2 \\
deg(2,4,8,10,14,16) &= n - \phi(n), && \text{if } z \text{ is multiple of } 2^1.
\end{aligned}$$

(2) Let $p = 5$, $R = \mathbb{Z}_{50}$ and 10^2, 20^2, 30^2, $40^2 \equiv 0(mod\ 50)$. So

$$\begin{aligned}
deg(10,20,30,40) &= n - \phi(n) - 1, && \text{if } z^2 \equiv 0\ (mod\ n) \\
deg(5,15,35,45) &= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } p \\
deg(25) &= n - \phi(n) - 1 + p(p-1), && \text{if } z \text{ is multiple of } p^2 \\
deg(2,4,6,\cdots,48) &= n - \phi(n), && \text{if } z \text{ is multiple of } 2^1.
\end{aligned}$$

The results are same for $R = \mathbb{Z}_{98}, \mathbb{Z}_{242}$ and so on. In general, we conclude that if $n = 1$, then

$$\begin{aligned}
deg(z) &= n - \phi(n) - 1 + p(p-1), && \text{if } z \text{ is multiple of } p^2 \\
&= n - \phi(n) - 1, && \text{if } z^2 \equiv 0\ (mod\ n) \\
&= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } p \\
&= n - \phi(n), && \text{if } z \text{ is multiple of } 2.
\end{aligned}$$

(b) If $n \neq 1$,

(1) If $n = 2$, $R = \mathbb{Z}_{4p^2}$, where $p = 3, 5, 7, \cdots$, then $R = \mathbb{Z}_{36}$, $\mathbb{Z}_{100}$ and so on. If $n = 3$, $R = \mathbb{Z}_{8p^2}$, where $p = 3, 5, 7, \cdots$, then $R = \mathbb{Z}_{72}$, $\mathbb{Z}_{200}$ and so on. Therefore, we conclude the result as

$$\begin{aligned}
deg(z) &= n - \phi(n) - 1 + p(p-1), && \text{if } z \text{ is multiple of } p^2 \\
&= n - \phi(n) - 1, && \text{if } z^2 \equiv 0\ (mod\ n) \\
&= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } p \\
&= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } 2, 2^2, \cdots, 2^{n-1} \\
&= n - \phi(n) - 1 + 2^{n-1}, && \text{if } z \text{ is multiple of } 2^n.
\end{aligned}$$

From Example 3.14 we conclude the following theorem.

Theorem 3.15 *Let z be a non-zero zero-divisor in a ring $\mathbb{Z}_{2^n p^2}$, for any $n \in \mathbb{N}$, where p is odd prime*

(a) *If $n = 1$, then*

$$\begin{aligned} deg(z) &= n - \phi(n) - 1 + p(p-1), && \text{if } z \text{ is multiple of } p^2 \\ &= n - \phi(n) - 1, && \text{if } z^2 \equiv 0 \ (mod\ n) \\ &= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } p \\ &= n - \phi(n), && \text{if } z \text{ is multiple of } 2. \end{aligned}$$

(b) *If $n \neq 1$, then*

$$\begin{aligned} deg(z) &= n - \phi(n) - 1 + p(p-1), && \text{if } z \text{ is multiple of } p^2 \\ &= n - \phi(n) - 1, && \text{if } z^2 \equiv 0 \ (mod\ n) \\ &= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } p \\ &= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } 2, 2^2, \cdots, 2^{n-1} \\ &= n - \phi(n) - 1 + 2^{n-1}, && \text{if } z \text{ is multiple of } 2^n. \end{aligned}$$

Example 3.16 Let $R = \mathbb{Z}_{2^n pq}$, for any $n \in \mathbb{N}$, where p and q are distinct odd primes and $p < q$. Then

(a) Let $n = 1$,

(1) $R = \mathbb{Z}_{2pq}$, where $p = 3$ and $q = 5, 7, 11, \cdots$. We find the degree of all non-zero zero-divisors in $R = \mathbb{Z}_{30}, \mathbb{Z}_{42}, \mathbb{Z}_{66}, \mathbb{Z}_{78}$ and so on.

(2) $R = \mathbb{Z}_{2pq}$, where $p = 5$ and $q = 7, 11, 13, \cdots$. We find the degree of all non-zero zero-divisors in $R = \mathbb{Z}_{70}, \mathbb{Z}_{110}$ and so on.

(b) Let $n \neq 1$.

(1) If $n = 2$, $R = \mathbb{Z}_{4pq}$, where $p = 3$ and $q = 5, 7, 11, \cdots$ then we find the degree of all non-zero zero-divisors in $R = \mathbb{Z}_{60}, \mathbb{Z}_{84}$ and so on.

(2) If $n = 3$, $R = \mathbb{Z}_{8pq}$, where $p = 3$ and $q = 5, 7, 11, \cdots$ then we find the degree of all non-zero zero-divisors in $R = \mathbb{Z}_{120}, \mathbb{Z}_{168}$ and so on.

From Example 3.16 and previous discussion we conclude results following.

Theorem 3.17 *Let z be a non-zero zero-divisor in a ring $\mathbb{Z}_{2^n pq}$, for any $n \in \mathbb{N}$, where p and q are distinct odd primes and $p < q$.*

(a) *If $n = 1$, then*

$$\begin{aligned} deg(z) &= n - \phi(n), && \text{if } z \text{ is multiple of } 2 \\ &= n - \phi(n) + p - 2, && \text{if } z \text{ is multiple of } p \ or \ 2p \\ &= n - \phi(n) + q - 2, && \text{if } z \text{ is multiple of } q \ or \ 2q \\ &= 2pq - 1, && \text{if } z \text{ is multiple of } pq. \end{aligned}$$

(*b*) *If* $n \neq 1$, *then*

$$\begin{aligned} deg(z) &= n - \phi(n) - 1, && \text{if } z^2 \equiv 0 \ (mod\ n) \\ &= n - \phi(n) + pq - (p+q), && \text{if } z \text{ is multiple of } pq \\ &= n - \phi(n) + p - 2, && \text{if } z \text{ is multiple of } p \\ &= n - \phi(n) + q - 2, && \text{if } z \text{ is multiple of } q \\ &= n - \phi(n) - 1 + 2^{n-1}, && \text{if } z \text{ is multiple of } 2^n \\ &= n - \phi(n) - 1 + 2^n, && \text{if } z \text{ is multiple of } 2^n p \\ &= n - \phi(n)/2 - 1, && \text{if } z \text{ is multiple of } 2^n q \\ &= n - \phi(n) - 1, && \text{otherwise.} \end{aligned}$$

We are also discussed some more cases in continuation to Theorem 3.5− Theorem 3.17 which calculates the degree of vertices in $PG_2(\mathbb{Z}_n)$, for $n \leq 100$.

Case 1. (*a*) Let z be a non-zero zero-divisor in a ring $\mathbb{Z}_n$, $n = 3pq$ where $p = 3$, $q = 5, 7, 11, 13, \cdots$. Then

$$\begin{aligned} deg(z) &= n - \phi(n) - 1, && \text{if } z^2 \equiv 0 \ (mod\ n) \\ &= n - \phi(n) + q - 2, && \text{if } z \text{ is multiple of } q \\ &= n - \phi(n) + 2p - 1, && \text{if } z \text{ is multiple of } 3p \\ &= n - \phi(n) - 1, && \text{otherwise.} \end{aligned}$$

(*b*) In this case when $p = q = 3$, then $deg(z) = n - \phi(n) - 1$.

Case 2. Let z be a non-zero zero-divisor in a ring $\mathbb{Z}_n$, $n = 3p^2$, $p = 3, 5, 7, \cdots$. Then

$$\begin{aligned} deg(z) &= n - \phi(n) - 1, && \text{if } z^2 \equiv 0 \ (mod\ n) \\ &= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } p \ and \ 3p \\ &= n - \phi(n) + 1, && \text{if } z \text{ is multiple of } 3 \\ &= n - \phi(n) - 1 + p(p-1), && \text{if } z \text{ is multiple of } p^2. \end{aligned}$$

Case 3. Let z be a non-zero zero-divisor in a ring $\mathbb{Z}_n$, $n = 2p^3$, $p = 3, 5, 7, \cdots$, $p > 2$. Then

$$\begin{aligned} deg(z) &= n - \phi(n) - 1, && \text{if } z^2 \equiv 0 \ (mod\ n) \\ &= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } p \ and \ p^2 \\ &= n - \phi(n), && \text{if } z \text{ is multiple of } 2 \\ &= n - \phi(n) - p, && \text{if } z \text{ is multiple of } 2p \\ &= n - \phi(n) - p + 1, && \text{if } z \text{ is multiple of } 2p^2 \\ &= n - \phi(n) - 1 + 2p^2, && \text{if } z \text{ is multiple of } p^3. \end{aligned}$$

Case 4. Let z be a non-zero zero-divisor in a ring $\mathbb{Z}_n$, $n = p^4$, $p = 2, 3, 5, 7, \cdots$. Then

$$deg(z) = n - \phi(n) - 1.$$

Case 5. Let z be a non-zero zero-divisor in a ring $\mathbb{Z}_n$, $n = 2p^2q$, $p = 3$, $q = 5, 7, 11, \cdots$. Then

$$\begin{aligned}
deg(z) &= n - \phi(n) - 1, && \text{if } z^2 \equiv 0 \ (mod\ n)\\
&= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } p,\ 2p \ and \ pq\\
&= n - \phi(n) + 2p - 1, && \text{if } z \text{ is multiple of } 2p^2 \text{ and } p^2\\
&= n - \phi(n) + 2q + 1, && \text{if } z \text{ is multiple of } 2q\\
&= n - \phi(n) + q - 2, && \text{if } z \text{ is multiple of } q\\
&= n - 1, && \text{if } z \text{ is multiple of } p^2q\\
&= n - \phi(n), && \text{if } z \text{ is multiple of } 2.
\end{aligned}$$

§4. Eulerianity, Planarity and Girth of $PG_2(\mathbb{Z}_n)$

In this section, we proved that $PG_2(\mathbb{Z}_n)$ is Eulerian for any odd positive integer n and is planar if and only if $n = 4, 6$ or n is a prime number. Also, we found that the girth of $PG_2(\mathbb{Z}_n)$ is 3, for $n \neq 2$.

Theorem 4.1 *$PG_2(\mathbb{Z}_n)$ is Eulerian, when n is odd positive integer.*

Proof Let n be even, so from Theorem 3.5, we have that $deg(0) = n - 1$, which is an odd number, so not Eulerian. Again if n is odd, then by Theorems $3.6 - 3.11$ and from the above discussion, degree of every vertex in $PG_2(\mathbb{Z}_n)$ is an even number. Hence, $PG_2(\mathbb{Z}_n)$ is Eulerian, when n is odd positive integer. □

Theorem 4.2 *$PG_2(\mathbb{Z}_n)$ is planar if and only if $n = 4, 6$ or n is a prime.*

Proof We discuss different cases for planarity of $PG_2(\mathbb{Z}_n)$.

Case 1. For $n = 2$, $PG_2(\mathbb{Z}_2)$ is a complete graph K_2. Hence it is a planar graph.

Case 2. For $n = 3$, $PG_2(\mathbb{Z}_3)$ is complete graph K_3. Therefore it is a planar graph.

Case 3. If n is prime and $n > 3$, $PG_2(\mathbb{Z}_n)$ is a union of copies of K_3 in which again zero is a common vertex. So, the graph is planar when n is prime.

Case 4. If $n = 4$, $PG_2(\mathbb{Z}_4)$ has two components consisting of zero divisors and units of $\mathbb{Z}_2^2$. The first is K_2 and the other is K_3 in which zero is again a common vertex, hence planar.

Case 5. If $n = 6$, $PG_2(\mathbb{Z}_6)$ is union of eight copies of K_3 hence planar.

Case 6. If $n = 8$, the graph $PG_2(\mathbb{Z}_8)$ contains a subgraph K_5. So, it cannot be a planar graph.

Case 7. Let $n = 2^m$, $m > 2$ contains K_5 and hence cannot be planar.

Case 8. Let $p \geq 3$, $PG_2(\mathbb{Z}_p^m)$, where $m > 1$ contains $K_{3,3}$, hence it cannot be planar.

Case 9. Let n be even. If $n = 10$, then the subgraph induced by the vertices $\{0, 2, 4, 6, 8\}$ forms K_5 and for $n = 12$, the subgraph induced by the vertices $\{0, 2, 4, 6, 8\}$ forms again K_5. So, the subgraph of $PG_2(\mathbb{Z}_n)$ where n is even forms K_5 and hence the graph is not planar.

Case 10. Let n be odd. If $n = 15$ then the subgraph induced by $\{0, 3, 6, 9,\ 12\}$ forms K_5 and for $n = 21$ the subgraph induced by $\{0, 3, 6, 9,\ 12\}$ forms again K_5. So, the subgraph of $PG_2(\mathbb{Z}_n)$, where n is odd forms a subgraph K_5 and hence the graph is nonplanar. Hence the result. □

Theorem 4.3 *The girth, $gr(PG_2(\mathbb{Z}_n))$ is 3, for $n \geq 3$.*

Proof We know that $PG_2(\mathbb{Z}_2)$ is a complete graph K_2, hence girth of $PG_2(\mathbb{Z}_2)$ is ∞. Now, let $n \geq 3$, then in $PG_2(\mathbb{Z}_n)$ always 3-cycle exist and hence $gr((PG_2(\mathbb{Z}_n)) = 3$, for $n \geq 3$. □

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Non-Holonomic Frame for
A Finsler Space with Some Special (α, β)-Metric

C.K.Yadav and M.K.Gupta

(Department of Pure & Applied Mathematics, Guru Ghasidas Vishwavidyalaya, Bilaspur (C.G.), India)

E-mail: chiranjeev86@gmail.com, mkgiaps@gmail.com

Abstract: In this paper, we study the Finsler deformation of some special (α, β)-metric. We determine a nonholonomic Finsler frame for Finsler space with (α, β)-metric $F = \alpha e^{\frac{\beta}{\alpha}}$ and Randers change of Matsumoto metric.

Key Words: Finsler space, (α, β)-metric, Randers metric, Matsumoto metric, exponential metric, non-holonomic Finsler frame.

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§1. Introduction

The concept of non-holonomic space which is more general than a Riemannian space and generalized the parallelism of Levi-Civita and geodesic curves in that space is introduced by G. Vranceanu [22]. A non-holonomic region as a space with a non-holonomic dynamical system was considered by Z. Horak [13] with another aspect. The non-holonomic space in a space of line elements with an affine connection was first conferred by T. Hosokawa [9]. The theory of non-holonomic system in Finsler space was introduced by Y. Katsurada [15].

The concept of non-holonomic frame is a deformation arising from the consideration of a charged particle moving in external electromagnetic field in the studied of a unified formalism that uses a non-holonomic frame on space-time [11, 12]. Further, the gauge transformation is studied as a non-holonomic frame on the tangent bundle of a four-dimensional base manifold [3, 4]. The geometry which arises from these consideration gives a more unified approach to gravitation and gauge symmetries. In these papers, the common Finsler idea used by the physicist is the existence of non-holonomic frame on the vertical subbundle of a base manifold M. In [1, 2], P. L. Antonelli and I. Bucataru, determined such a non-holonomic frame for the two important classes of Finsler spaces that are dual in the sense of Randers and Kropina spaces. Since Randers and Kropina spaces are the Finsler space with (α, β)-metric, is a member of the bigger class of Finsler space. It appears a natural question that how many Finsler spaces with (α, β) metrics have such a non-holonomic frame [7]? Yes, there is a number of Finsler space with (α, β) metrics. Some auther's which discusses the non-holonomic frame for (α, β)

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metrics are [8, 20, 21]. In the present work, we determine the non-holonomic Finsler frame for the exponential (α, β) metric and Randers change of Matsumoto metric.

§2. Preliminaries

The physicists R. G. Beil in [3, 4] and R. R. Holland in [11, 12] are using non-holonomic Finsler frames to develop unified field theories. For a Finsler space with (α, β)-metric a non-holonomic frame is a product of two non-holonomic frames, each of these being determined by Finsler deformation.

Let U be an open set of TM and

$$V_i : u \in U \mapsto V_i(u) \in V_u TM, \quad i \in \{1, 2, \cdots, n\}$$

be a vertical frame over U. If $V_i(u) = V_i^j(u) \frac{\partial}{\partial y^j}|_u$ are the entries of a invertible matrix for all $u \in U$. Denote by $\tilde{V}_i^j(u)$ the inverse of this matrix, i.e.,

$$V_j^i \tilde{V}_k^j = \delta_k^i, \quad \tilde{V}_j^i V_k^j = \delta_k^i.$$

We call V_j^i a non-holonomic Finsler frame.

An important class of Finsler space with (α, β)-metrics are given in [18]. The first Finsler space with (α, β)-metric was introduced in the forties by G. Randers and known as Randers space [19].

Definition 2.1 *A Finsler space $F^n = (M, F(x, y))$ is called with (α, β)-metric if there exists a two homogenius function L of two variables such that the Finsler metric $F : TM \to \Re$ is given by*

$$F^2(x, y) = L(\alpha(x, y), \beta(x, y))$$

where $\alpha(x, y) = \sqrt{a_{ij} y^i y^j}$, where a_{ij} is a Riemannian metric on M and $\beta(x, y) = b_i(x) y^i$, is a 1-form on M.

Example 2.2 If $L(\alpha, \beta) = (\alpha + \beta)^2$, then the Finsler space with metric

$$F(x, y) = \sqrt{a_{ij} y^i y^j} + b_i(x) y^i$$

is called a Randers space.

Example 2.3 If $L(\alpha, \beta) = \frac{\alpha^4}{\beta^2}$, then the Finsler space with metric

$$F(x, y) = \frac{a_{ij} y^i y^j}{|b_i(x) y^i|}$$

is called a Kropina space.

The Randers space and Kropina space play an important role in Finsler geometry and are dual in sense of [10]. For a Finsler space with (α, β)-metric $F^2(x, y) = L(\alpha(x, y), \beta(x, y))$, the

Finsler invariants are [17]

$$\begin{cases} \rho = \frac{1}{2\alpha}\frac{\partial L}{\partial \alpha}, \ \ \rho_0 = \frac{1}{2}\frac{\partial^2 L}{\partial \beta^2}, \ \ \rho_{-1} = \frac{1}{2\alpha}\frac{\partial^2 L}{\partial\alpha\partial\beta}, \\ \rho_{-2} = \frac{1}{2\alpha^2}\left(\frac{\partial^2 L}{\partial\alpha^2} - \frac{1}{\alpha}\frac{\partial L}{\partial \alpha}\right), \end{cases} \tag{1}$$

For a Finsler space with (α,β)-metric, we have

$$\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0. \tag{2}$$

With respect to above notation (i.e. Finsler invariants), the metric tensor g_{ij} of a Finsler space with (α,β)-metric is given by [18]

$$g_{ij} = \rho a_{ij}(x) + \rho_0 b_i(x)b_j(x) + \rho_{-1}(b_i(x)y_j + b_j(x)y_i) + \rho_{-2}y_iy_j. \tag{3}$$

We can be arranged the metric tensor g_{ij} of Finsler space into the form

$$\begin{aligned} g_{ij} &= \rho a_{ij}(x) + \tfrac{1}{\rho_{-2}}(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j) \\ &+ \tfrac{1}{\rho_{-2}}(\rho_0\rho_{-2} - \rho_{-1}^2)b_ib_j. \end{aligned} \tag{4}$$

From the equation (4), we can see that the metric tensor g_{ij} is the result of the two deformations

$$\begin{cases} a_{ij} \mapsto h_{ij} = \rho a_{ij}(x) + \frac{1}{\rho_{-2}}(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j), \\ h_{ij} \mapsto g_{ij} = h_{ij} + \frac{1}{\rho_{-2}}(\rho_0\rho_{-2} - \rho_{-1}^2)b_ib_j. \end{cases} \tag{5}$$

The non-holonomic Finsler frame corresponding to the first deformation of equation (5) is according to Theorem 7.9.1 in reference [7], is given by

$$X_j^i = \sqrt{\rho}\,\delta_j^i - \frac{1}{A^2}\left(\sqrt{\rho} \pm \sqrt{\rho + \frac{A^2}{\rho_{-2}}}\right)(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b_j + \rho_{-2}y_j), \tag{6}$$

where $A^2 = a_{ij}(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b_j + \rho_{-2}y_j) = \rho_{-1}^2 b^2 + \beta\rho_{-1}\rho_{-2}$. The metric tensor a_{ij} and h_{ij} are related by:

$$h_{ij} = X_i^k X_j^l a_{kl}. \tag{7}$$

Similarly, a non-holonomic frame Finsler frame corresponding to the second deformation of equation (5) is according to Theorem 7.9.1 in reference [7] given by

$$Y_j^i = \delta_j^i - \frac{1}{B^2}\left(1 \pm \sqrt{1 + \frac{\rho_{-2}B^2}{\rho_0\rho_{-2} - \rho_{-1}^2}}\right)b^ib_j, \tag{8}$$

where $B^2 = h_{ij}b^ib_j = \rho b^2 + \frac{1}{\rho_{-2}}(\rho_{-1}b^2 + \rho_{-2}\beta^2$. The metric temsor h_{ij} and g_{ij} are related by

$$g_{mn} = Y_m^i Y_n^j h_{ij}. \tag{9}$$

From (7) and (9), we have $V_m^k = X_i^k Y_m^i$, with X_i^k and Y_m^i are given by (6) and (8), are the non-holonomic Finsler frame of the Finsler space with (α,β) metric.

§3. Non-Holonomic Frame for Finsler Space with $F = \alpha e^{\frac{\beta}{\alpha}}$ - Metric

We consider a (α,β)- metric given as

$$F = \alpha e^{\frac{\beta}{\alpha}}, \tag{10}$$

in a Finsler space. For the fundamental function $L = \alpha^2 e^{\frac{2\beta}{\alpha}}$, the Finsler invariants (1) are given by:

$$\begin{cases} \rho = \frac{(\alpha-\beta)}{\alpha} e^{\frac{2\beta}{\alpha}}, \quad \rho_0 = 2e^{\frac{2\beta}{\alpha}}, \quad \rho_{-1} = \frac{(\alpha-2\beta)}{\alpha^2} e^{\frac{2\beta}{\alpha}}, \\ \rho_{-2} = \frac{\beta(2\beta-\alpha)}{\alpha^4} e^{\frac{2\beta}{\alpha}}, \end{cases} \tag{11}$$

and

$$\begin{cases} A^2 = \frac{(\alpha-2\beta)(\alpha^3 b^2 - 2\alpha^2\beta b^2 + 2\beta^3 - \alpha^2\beta^2)}{\alpha^6} e^{\frac{4\beta}{\alpha}}, \\ B^2 = e^{\frac{2\beta}{\alpha}} \left[\frac{b^2(\alpha-\beta)}{\alpha} - \frac{(\alpha-2\beta)}{\beta} \left(b^2 - \frac{\beta}{\alpha^2} \right)^2 \right]. \end{cases} \tag{12}$$

The Finsler invariants satisfies the relation $\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0$. Then by using equation (6) with respect to first deformation of (5), we get

$$\begin{aligned} X_j^i \quad = \quad & \sqrt{\frac{(\alpha-\beta)e^{\frac{2\beta}{\alpha}}}{\alpha}} \delta_j^i - \frac{\alpha^2}{(\alpha^3 b^2 - 2\alpha^2\beta b^2 + 2\beta^3 - \alpha^2\beta^2)} \\ & \times \left[\sqrt{\frac{(\alpha-\beta)e^{\frac{2\beta}{\alpha}}}{\alpha}} \pm \sqrt{\frac{e^{\frac{2\beta}{\alpha}} (\alpha^2\beta - \alpha\beta^2 - \alpha^3 b^2 + 2\alpha^2\beta b^2 - 2\beta^3 + \alpha^2\beta^2)}{\alpha^2\beta}} \right] \\ & \times \left(b^i - \frac{\beta}{\alpha^2} y^i \right) \left(b_j - \frac{\beta}{\alpha^2} y_j \right), \end{aligned} \tag{13}$$

and also using equation (6) with respect to second deformation of (5), we get

$$\begin{aligned} Y_j^i = \delta_j^i \; - & \frac{\alpha^4\beta}{e^{\frac{2\beta}{\alpha}} \{(\alpha^4\beta - \alpha^3\beta^2) b^2 - (\alpha-2\beta)(\alpha^2 b^2 - \beta)^2\}} \\ & \times \left[1 \pm \sqrt{1 + \frac{\alpha^4\beta - \alpha^3\beta^2 - (\alpha-2\beta)(\alpha^2 b^2 - \beta)^2}{\alpha^4}} \right] b^i b_j. \end{aligned} \tag{14}$$

By using X_k^i and Y_j^k, we obtain the non-holonomic Finsler frame as follows,

$$\begin{aligned} V_j^i \quad = \quad & X_k^i Y_j^k = \sqrt{\frac{(\alpha-\beta)e^{\frac{2\beta}{\alpha}}}{\alpha}} \, \delta_j^i - [1 \pm D] E b^i b_j \delta_k^i \sqrt{\frac{(\alpha-\beta)e^{\frac{2\beta}{\alpha}}}{\alpha}} \\ & - \frac{\alpha^2 C[1 - E(1 \pm D) b^i b_j]}{(\alpha^3 b^2 - 2\alpha^2\beta b^2 + 2\beta^3 - \alpha^2\beta^2)} \left(b^i - \frac{\beta}{\alpha^2} y^i \right) \left(b_j - \frac{\beta}{\alpha^2} y_j \right), \end{aligned} \tag{15}$$

where

$$C \quad = \quad \sqrt{\frac{(\alpha-\beta)e^{\frac{2\beta}{\alpha}}}{\alpha}} \pm \sqrt{\frac{e^{\frac{2\beta}{\alpha}} (\alpha^2\beta - \alpha\beta^2 - \alpha^3 b^2 + 2\alpha^2\beta b^2 - 2\beta^3 + \alpha^2\beta^2)}{\alpha^2\beta}},$$

$$
\begin{aligned}
D &= 1 \pm \sqrt{1 + \frac{\alpha^4\beta - \alpha^3\beta^2 - (\alpha - 2\beta)(\alpha^2 b^2 - \beta)^2}{\alpha^4}},\\
E &= \frac{\alpha^4\beta}{e^{\frac{2\beta}{\alpha}}\{(\alpha^4\beta - \alpha^3\beta^2)b^2 - (\alpha - 2\beta)(\alpha^2 b^2 - \beta)^2\}}.
\end{aligned}
$$

Theorem 3.1 *Consider a Finsler space $F^n = (M, F)$ with $L = (\alpha e^{\frac{\alpha}{\beta}})^2$, for which the condition (2) is true. Then $V^i_j = X^i_k Y^k_j$ is a non-holonomic Finsler frame given in equation (15), where X^i_k and Y^k_j are given by (13) and (14) respectively.*

§4. Non-Holonomic Frame for Finsler Space with Randers Change of Matsumoto Metric

We consider a Randers change of Matsumoto metric as given by

$$
F = \frac{\alpha^2}{\alpha + \beta} + \beta, \tag{16}
$$

in a Finsler space. For the fundamental function $L = \left(\frac{\alpha^2}{\alpha+\beta} + \beta\right)^2$, the Finsler invariants (1) are given by

$$
\begin{cases}
\rho = \frac{(\alpha^3 - \alpha^2\beta - 3\alpha\beta^2 + 2\beta^3)}{(\alpha-\beta)^3},\\[2ex]
\rho_0 = \frac{6\alpha^4 + \beta^4 - 6\alpha^3\beta + 6\alpha^2\beta^2 - 4\alpha\beta^3}{(\alpha-\beta)^4},\\[2ex]
\rho_{-1} = \frac{(2\alpha^3 - 8\alpha^2\beta + 3\alpha\beta^2)}{(\alpha-\beta)^4},\\[2ex]
\rho_{-2} = \frac{8\alpha^2\beta^2 - 2\alpha^3\beta - 3\alpha\beta^3)}{\alpha^2(\alpha-\beta)^4},
\end{cases} \tag{17}
$$

and

$$
\begin{cases}
A^2 = \frac{b^2 S(\alpha,\beta) - T(\alpha,\beta)}{(\alpha-\beta)^8},\\
B^2 = \frac{\{b^2(2\alpha^4 - 8\alpha^3\beta + 3\alpha^2\beta^2) + (8\alpha\beta^3 - 2\alpha^2\beta^2 - 3\beta^4)\}^2}{\alpha^2(\alpha-\beta)^8}\\
\qquad + \frac{b^2 U(\alpha,\beta)}{(\alpha-\beta)^3},
\end{cases} \tag{18}
$$

where

$$
\begin{aligned}
&S(\alpha,\beta) = 4\alpha^6 - 32\alpha^5\beta + 76\alpha^4\beta^2 - 48\alpha^3\beta^3 + 9\alpha^2\beta^4,\\
&T(\alpha,\beta) = 4\alpha^4\beta^2 - -32\alpha^3\beta^3 + 76\alpha^2\beta^4 - 48\alpha\beta^5 + 9\beta^6,\\
&U(\alpha,\beta) = (\alpha^3 - 2\alpha^2\beta - \alpha\beta^2 + 2\alpha\beta - 4\beta^2 - 2\beta^3).
\end{aligned}
$$

The Finsler invariants satisfies the relation $\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0$. Then by using equation (6) with respect to first deformation of (5), we get

$$
\begin{aligned}
X^i_j = {} & \sqrt{\frac{(\alpha^3 - \alpha^2\beta - 3\alpha\beta^2 + 2\beta^3)}{(\alpha-\beta)^3}}\,\delta^i_j - \frac{(2\alpha^8 - 8\alpha^5\beta + 3\alpha^4\beta)^2}{b^2 S(\alpha,\beta) - T(\alpha,\beta)}\\
& \times \left[\sqrt{\frac{(\alpha^3 - \alpha^2\beta - 3\alpha\beta^2 + 2\beta^3)}{(\alpha-\beta)^3}} \pm \sqrt{\frac{(\alpha^3 - \alpha^2\beta - 3\alpha\beta^2 + 2\beta^3)}{(\alpha-\beta)^3} + \frac{\alpha(b^2 S(\alpha,\beta) - T(\alpha,\beta))}{\beta(\alpha-\beta)^4(8\alpha\beta - 2\alpha^2 - 3\beta^2)}}\right]\\
& \times \left(b^i - \frac{\beta}{\alpha^2}y^i\right)\left(b_j - \frac{\beta}{\alpha^2}y_j\right),
\end{aligned} \tag{19}
$$

and also using equation (6) with respect to second deformation of (5), we get

$$Y^i_j = \delta^i_j \; - \frac{\alpha^2(\alpha-\beta)^8}{V} \times \left[1 \pm \sqrt{1 + \frac{V\beta}{\alpha^2 W}} \right] b^i b_j. \tag{20}$$

Where

$$V = \alpha^2(\alpha-\beta)^5 b^2 U(\alpha,\beta) + \{b^2(2\alpha^4 - 8\alpha^3\beta + 3\alpha^2\beta^2) \\ + (8\alpha\beta^3 - 2\alpha^2\beta^2 - 3\beta^4)\}^2,$$

$$W = (2\alpha^6 - 8\alpha^5\beta + 6\alpha^4\beta + 3\alpha^4\beta^2 - 6\alpha^3\beta^2 + 6\alpha^2\beta^4 - 4\alpha\beta^5 + \beta^5).$$

By using X^i_k and Y^k_j, we obtain the non-holonomic Finsler frame as follows.

Theorem 4.1 *Consider a Finsler space $F^n = (M, F)$ with $L = (\frac{\alpha^2}{\alpha+\beta} + \beta)^2$ for which the condition (2) is true. Then $V^i_j = X^i_k Y^k_j$ is a non-holonomic Finsler frame, where X^i_k and Y^k_j are given by (19) and (20) respectively.*

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R_1 Concepts in Fuzzy Bitopological Spaces with Quasi-Coincidence Sense

Ruhul Amin

(Department of Mathematics, Faculty of Science, Begum Rokeya University, Rangpur-5400, Bangladesh)

Sahadat Hossain

(Department of Mathematics, Faculty of Science, University of Rajshahi, Rajshahi-6205, Bangladesh)

E-mail: ruhulbru1611@gmail.com, sahadat@ru.ac.bd

Abstract: In this paper, we have defined some new notions of R_1-separation in fuzzy bitopological spaces using quasi-coincidence sense. We have discuss the relations among our and other such notions. We have observed that all these notions satisfy good extension property. We have shown that these notions are preserved under the one-one, onto and FP-continuous mapping. Moreover, we have obtained some other properties of this new concept. Initial and final topologies are studied here.

Key Words: Quasi-coincidence, fuzzy bitopological spaces, fuzzy pairwise R_1 separations.

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§1. Introduction

The concept of R_1-property in classical topology first defined by Yang [23]. In fuzzy topology, the concept of fuzzy R_1 spaces was first introduced by Hutton and Relly [12] in 1980. Since then much attention has been paid to define such notion by many fuzzy topologist e.g., by Ali [4], Hossain and Ali [11], Caldas [9], Roy and Mukherjee [21], Keskin and Nori [16], Srivastava and Ali [3] and Petricevic [20]. In 2012, Ali And Azom [3] introduced some other definitions of fuzzy R_1-axioms in fuzzy topological spaces. In 1990, Kandil [13] introduced the concept of fuzzy bitopological spaces and in 1991, Kandil [13] first defined R_1-property in fuzzy bitopological spaces. After then Abu Safiya [1, 2], Kandil [14] and Nouh [19] defined several type of R_1-properties.

In this paper, we introduce four notions of R_1–property in fuzzy bitopological spaces by using quasi-coincident sense. We show that all these notions satisfy good extension property. Also hereditary is satisfied by these concepts. We have observed that all these concepts are preserved under one-one, onto and continuous mappings. Finally, we have showed that initial and final fuzzy bitopological spaces satisfy R_1-property.

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§2. Preliminaries

In this section, we recall some known definitions and results useful in the sequel. For details, we refer to [1]–[10].

We give some elementary concepts and results which will be used in the sequel. Throughout this paper, X will be a nonempty set, $I = [0, 1]$, $I_0 = (0, 1]$, $I_1 = [0, 1)$ and FP(resp P) stands for fuzzy pairwise (resp pairwise). The class of all fuzzy sets on a universe X will be denoted by I^X and fuzzy sets on X will be denoted by u, v, w, etc. Crisp subset of X will be denoted by capital letters U, V, W etc. In this paper (X, t) and (X, s, t) will be denoted fuzzy topological space and fuzzy bitopological space respectively. $x_r q u$ denotes x_r is quasi-coincident with u and $x_r \bar{q} u$ denotes that x_r is not quasi-coincident with u throughout this paper.

We shall follow [5] for the definitions of fuzzy singleton, quasi-coincident, fuzzy topology, image of fuzzy set, the inverse images of a fuzzy set, fuzzy continuous mapping good extension property.

Definition 2.1([13]) *A fuzzy singleton x_r is said to be quasi-coincident with a fuzzy set μ, denoted by $x_r q\mu$ iff $r + \mu(x) > 1$. If x_r is not quasi-coincident with μ, we write $x_r \bar{q}\mu$.*

Definition 2.2([22]) *Let f be a real valued function on a topological space. If $\{x : f(x) > \alpha\}$ is open for every real $\alpha \in I_1$, then f is called lower semi continuous function.*

C. L.Chang [10] have defined fuzzy topology and fuzzy continuous mapping.

Definition 2.3([10]) *A function f from a fuzzy topological space (X, t) into a fuzzy topological space (Y, s) is called fuzzy continuous if and only if for every $u \in s$, $f^{-1}(u) \in t$.*

Definition 2.4([11]) *A fuzzy topological space (X, t) is called*

(a) $FR_1(i)$ iff for each pair of fuzzy singletons x_r, y_s in X with $x \neq y$, whenever there exists a fuzzy set $\gamma \in t$ with $\gamma(x) \neq \gamma(y)$, then $\exists\, \mu, \lambda \in t$ such that $x_r q\mu$, $y_s q\lambda$ and $\mu\bar{q}\lambda$;

(b) $FR_1(ii)$ iff for each pair of fuzzy singletons x_r, y_s in X with $x \neq y$, whenever there exists a fuzzy set $\gamma \in t$ with $\gamma(x) \neq \gamma(y)$, then $\exists\, \mu, \lambda \in t$ such that $x_r \in \mu$, $y_s \in \lambda$ and $\mu \cap \lambda = 0$;

(c) $FR_1(iii)$ iff for each pair of fuzzy singletons x_r, y_s in X with $x \neq y$, whenever there exists a fuzzy set $\gamma \in t$ with $\gamma(x) \neq \gamma(y)$, then $\exists\, \mu, \lambda \in t$ such that $x_r \in \mu$, $y_s \in \lambda$ and $\mu \subseteq \lambda^c$;

(d) $FR_1(iv)$ iff for each pair of fuzzy singletons x_r, y_s in X with $x \neq y$, whenever there exists a fuzzy set $\gamma \in t$ with $\gamma(x) \neq \gamma(y)$, then $\exists\, \mu, \lambda \in t$ such that $x_r q\mu$, $y_s q\lambda$ and $\mu \cap \lambda = 0$.

Definition 2.5([15]) *Let X be any non empty set and S and T be any two general topologies on X then the triple (X, S, T) is called a bitopological space.*

Definition 2.6([13]) *A fuzzy bitopological space (fbts, in short) is a triple (X, s, t) where s and t are arbitrary fuzzy topologies on X.*

In previous works we have introduced the following definitions and discussed many related concepts among them.

Definition 2.7([13]) *A fuzzy bitopological space* (X, t_1, t_2) *is called* FPR_0 *if and only if* $x_t \bar{q} t_i.cl(y_r)$ *implies* $y_s \bar{q} t_j.cl(x_t)\,(i, j \in \{1, 2\},\, i \neq j)$.

Definition 2.8([2]) *A fbts* (X, t_1, t_2) *is said to be* PFR_0 *if and only if for any distinct fuzzy points* p *and* q *in* X*, whenever there exists* $\mu \in t_i$ *such that* $p \in \mu$ *and* $q \cap \mu = 0$*, then there exists* $\gamma \in t_j$ *such that* $p \cap \gamma = 0$ *and* $q \in \gamma\,(i, j = 1, 2, i \neq j)$.

Kelly defines bitopological space in his classical paper [15] as a bitopological space (X, S, T) is called pairwise-R_0 (PR_0, in short) if for all $x, y \in X$, $x \neq y$, whenever $\exists U \in S$ with $x \in U$, $y \notin U$, then $\exists V \in T$ such that $y \in V$, $x \notin V$.

In previous works [6], [7], we introduced the following definitions and discussed many related concepts among them.

Definition 2.9([6]) *A fbts* (X, s, t) *is called* FPT_0*-space iff for every pair of fuzzy singletons* x_p, y_r *in* X *with* $x \neq y$*, there exist fuzzy set* $\mu \in s \cup t$ *such that* $(x_p q \mu, y_r \cap \mu = 0)$ *or* $(y_r q \mu, x_p \cap \mu = 0)$.

Definition 2.10([7]) *A fbts* (X, s, t) *is called* FPT_2 *iff for every pair of fuzzy singletons* x_r, y_s *in* X *with* $x \neq y$*, there exist fuzzy sets* $\mu \in s$, $\lambda \in t$ *such that* $x_r q \mu$, $y_s q \lambda$ *and* $\mu \cap \lambda = 0$.

§3. Main Results with Proofs

Definition 3.1 *A fbts* (X, s, t) *is called*

(a) $FPR_1(i)$ *iff for each pair of fuzzy singletons* x_r, y_s *in* X *with* $x \neq y$*, whenever there exists a fuzzy set* $\gamma \in s \cup t$ *with* $\gamma(x) \neq \gamma(y)$*, then* $\exists\, \mu \in s$, $\lambda \in t$ *such that* $x_r q \mu$, $y_s q \lambda$ *and* $\mu \bar{q} \lambda$;

(b) $FPR_1(ii)$ *iff for each pair of fuzzy singletons* x_r, y_s *in* X *with* $x \neq y$*, whenever there exists a fuzzy set* $\gamma \in s \cup t$ *with* $\gamma(x) \neq \gamma(y)$*, then* $\exists\, \mu \in s$, $\lambda \in t$ *such that* $x_r \in \mu$, $y_s \in \lambda$ *and* $\mu \cap \lambda = 0$;

(c) $FPR_1(iii)$ *iff for each pair of fuzzy singletons* x_r, y_s *in* X *with* $x \neq y$*, whenever there exists a fuzzy set* $\gamma \in s \cup t$ *with* $\gamma(x) \neq \gamma(y)$*, then* $\exists\, \mu \in s$, $\lambda \in t$ *such that* $x_r \in \mu$, $y_s \in \lambda$ *and* $\mu \subseteq \lambda^c$;

(d) $FPR_1(iv)$ *iff for each pair of fuzzy singletons* x_r, y_s *in* X *with* $x \neq y$*, whenever there exists a fuzzy set* $\gamma \in s \cup t$ *with* $\gamma(x) \neq \gamma(y)$*, then* $\exists\, \mu \in s$, $\lambda \in t$ *such that* $x_r q \mu$, $y_s q \lambda$ *and* $\mu \cap \lambda = 0$.

In general it is true that union of fuzzy topologies is not a topology. But if union of two fuzzy topologies is again a topology then we have the following theorem.

Theorem 3.1 *Let* (X, s, t) *be a fuzzy bitopological space and* $(X, s \cup t)$ *be a fuzzy topological space, then*

$$(X, s, t) \text{ is } FPR_1(j) \Rightarrow (X, s \cup t) \text{ is } FR_1(j),$$

where $j = i, ii, iii, iv$.

Proof First suppose that $(X,\, s,\, t)$ is $FPR_1(i)$. We have to prove that $(X,\, s \cup t)$ is $FR_1(i)$. Let $x_r,\, y_s$ be two distinct fuzzy singletons in X and $\gamma \in s \cup t$ with $\gamma(x) \neq \gamma(y)$. Since $(X,\, s,\, t)$ is $FPR_1(i)$, then there exist $\mu \in s,\, \lambda \in t$ such that

$$x_r q\mu,\, y_s q\lambda \quad \text{and} \quad \mu\bar{q}\lambda.$$

But it follows that $\lambda \in s \cup t$ with $\gamma(x) \neq \gamma(y)$ and $\mu \in s \cup t,\, \lambda \in s \cup t$ such that

$$x_r q\mu,\, y_s q\lambda \quad \text{and} \quad \mu\bar{q}\lambda.$$

Hence the topological space $(X,\, s \cup t)$ is $FR_1(i)$. □

For non-implications, we have the the following counter example that will serve the purpose.

Example 3.1 Let $X = \{x,\, y\}$ and s be the discrete fuzzy topology on X. Again t be the indiscrete fuzzy topology on X. Then $(X,\, s \cup t)$ is $FR_1(i)$, $FR_1(ii)$, $FR_1(iii)$ and $FR_1(iv)$. On the other hand, $(X,\, s,\, t)$ is none of the $FPR_1(i)$, $FPR_1(ii)$, $FPR_1(iii)$ and $FPR_1(iv)$.

Remark 3.1 *Let $(X,\, s)$ and $(X,\, t)$ be two fuzzy topological space and $(X,\, s,\, t)$ be its corresponding bitopological space. Then "$(X,\, s,\, t)$ is $FPR_1(j)$" does not imply $(X,\, s)$ and $(X,\, t)$ are $FR_1(j)$ in general, where $j = i,\, ii,\, iii,\, iv$.*

Example 3.2 Let $X = \{x,\, y\}$ and s be the fuzzy topology on X generated by $\{x_1,\, x_{0.6}\} \cup \{\text{constants}\}$. Again t be the fuzzy topology on X generated by $\{y_1\} \cup \{\text{constants}\}$. Then $(X,\, s,\, t)$ is $FPR_1(i)$, $FPR_1(ii)$, $FPR_1(iii)$, and $FPR_1(iv)$. On the other hand, $(X,\, s)$ and $(X,\, t)$ are none of the $FR_1(i)$, $FR_1(ii)$, $FR_1(iii)$ and $FR_1(iv)$.

Remark 3.2 *Let $(X,\, s)$ and $(X,\, t)$ be two fuzzy topological space and $(X,\, s,\, t)$ be its corresponding bitopological space. Then "$(X,\, s)$ and $(X,\, t)$ are both $FR_1(j)$" does not imply $(X,\, s,\, t)$ is $FPR_1(j)$ in general, where $j = i,\, ii,\, iii,\, iv$.*

Example 3.3 Let $X = \{x,\, y\}$ and s be the fuzzy topology on X generated by $\{x_1,\, y_1\} \cup \{\text{constants}\}$. Again t be the fuzzy topology on X generated by $\{\text{constants}\}$. Then it is clear that $(X,\, s)$ and $(X,\, t)$ are both $FR_1(i)$, $FR_1(ii)$, $FR_1(iii)$ and $FR_1(iv)$. But on the other hand, the fuzzy bitopological space $(X,\, s,\, t)$ is none of $FPR_1(i)$, $FPR_1(ii)$, $FPR_1(iii)$, and $FPR_1(iv)$.

Theorem 3.2 *Let $(X,\, s,\, t)$ be an fbts. Then the following are equivalent:*

(i) $(X,\, s,\, t)$ *is* FPT_2;

(ii) $(X,\, s,\, t)$ *is* FPT_0 *and* FPR_1.

Proof $(ii) \Rightarrow (i)$: Let $x_r,\, y_s$ be two fuzzy singletons in X with $x \neq y$. Since $(X,\, s,\, t)$ is FPT_0, there exists a fuzzy set $\mu \in s \cup t$ such that

$$x_r q \quad \text{and} \quad \mu \cap y_s = 0.$$

This implies that $\mu(x) \neq \mu(y)$. Again since $\mu(x) \neq \mu(y)$ and $(X,\, s,\, t)$ is FPR_1, there exist

$v \in s$, $w \in t$ such that

$$x_r q v,\, y_s q w \quad \text{and} \quad v \cap w = 0.$$

Hence (X, s, t) is FPT_2.

$(i) \Rightarrow (ii)$: $FPT_2 \Rightarrow FPR_1$ is obvious. We have to show that $FPT_2 \Rightarrow FPT_0$. Let $x_r, y_s \in S(X)$ with $x \neq y$. Since (X, s, t) is FPT_2, there exist a fuzzy sets $u \in s$, $v \in t$ such that

$$x_r q u,\, y_s q v \quad \text{and} \quad u \cap v = 0.$$

To show that (X, s, t) is FPT_0, it is enough to show that $x_r \cap v = 0$. Suppose that

$$x_r \cap v \neq 0.$$

This implies that $v(x) > 0$. Since $u \cap v = 0$, we have

$$u(x) = 0, \quad \text{that is, } x_r \bar{q} u$$

which is a contradiction. Hence $x_r \cap v = 0$. □

In the following theorem now we discuss about the good extension property of FPR_1 concepts given earlier. All the properties $FPR_1(i)$, $FPR_1(ii)$, $FPR_1(iii)$ and $FPR_1(v)$ are good extension of PR_1.

Theorem 3.3 *Let (X, S, T) be a bitopological space. Then (X, S, T) is $PR_1 \Leftrightarrow (X, \omega(S), \omega(T))$ is $FPR_1(j)$, for $j = i, ii, iii, iv$.*

Proof Let (X, S, T) be PR_1 space. Suppose $x_r, y_s \in S(X)$, with $x \neq y$ and $\gamma \in \omega(S) \cup \omega(T)$ with $\gamma(x) \neq \gamma(y)$. Then we have

$$\gamma(x) < \gamma(y) \quad \text{or} \quad \gamma(x) > \gamma(y).$$

Suppose $\gamma(x) < \gamma(y)$. Then $\gamma(x) < r < \gamma(y)$ for some $r \in I_0$. So, it is clear that

$$x \notin \gamma^{-1}(r, 1],\, y \in \gamma^{-1}(r, 1] \quad \text{and} \quad \gamma^{-1}(r, 1] \in S \cup T.$$

Since (X, S, T) is PR_1 space, then there exist $U \in S$, $V \in T$ such that

$$x \in U,\, y \in V \quad \text{and} \quad U \cap V = \phi.$$

So, by definition of lower semi-continuous, we get $1_U \in \omega(S)$ and $1_V \in \omega(S)$. Now, we have

$$1_U(x) = 1,\, 1_V(y) = 1 \quad \text{and} \quad 1_{U \cap V} = 0.$$

We know that $1_{U \cap V} = 0$ implies $1_U \cap 1_V = 0$. Therefore $x_r q 1_U$, $y_s q 1_V$ and $1_U \cap 1_V = 0$. Hence $(X, \omega(S), \omega(T))$ is $FPR_1(iv)$.

Conversely, suppose that $(X, \omega(S), \omega(T))$ is FPR_1. Let $x, y \in X$, $x \neq y$ and $M \in S \cup T$

with

$$x \in M,\, y \notin M \quad \text{or} \quad x \notin M,\, y \in M.$$

Suppose $x \in M$, $y \notin M$. But from definition of lower semi-continuous function, we have

$$1_M \in \omega(S) \cup \omega(T) \quad \text{and} \quad 1_M(x) = 1, 1_M(y) = 0.$$

So, $1_M(x) \neq 1_M(y)$. Since $(X, \omega(S), \omega(T))$ is $FPR_1(iv)$, then there exist $\mu \in \omega(S), \lambda \in \omega(T)$ such that

$$x_1 q\mu,\, y_1 q\lambda \quad \text{and} \quad \mu \cap \lambda = 0.$$

Now $x_1 q\mu$, $y_1 q\lambda$ implies that

$$\mu(x) > 0, \lambda(y) > 0.$$

So, $x \in \mu^{-1}(0,1]$, $y \in \lambda^{-1}(0,1]$.

To show that $\mu^{-1}(0,1] \cap \lambda^{-1}(0,1] = \phi$, suppose that $\mu^{-1}(0,1] \cap \lambda^{-1}(0,1] \neq \phi$. Then there exits $z \in \mu^{-1}(0,1] \cap \lambda^{-1}(0,1]$ such that

$$\mu(z) > 0,\, \lambda(z) > 0.$$

Consequently $(\mu \cap \lambda)(z) \neq 0$ which contradicts the fact that $\mu \cap \lambda = 0$. Hence (X, S, T) is PR_1. Other proofs are similar. □

We discuss the hereditary and productive properties of $FPR_1(j)$, for $j = i, ii, iii, iv, v$ in the following two theorems respectively.

Theorem 3.4 *Let (X, s, t) be a fuzzy bitopological space, $A \subseteq X$ and $S_A = \{u/A : u \in s\}$, $t_A = \{v/A : v \in t\}$. Then,*

(a) *(X, s, t) is $FPR_1(i) \Longrightarrow (A, s_A, t_A)$ is $FPR_1(i)$;*
(b) *(X, s, t) is $FPR_1(ii) \Longrightarrow (A, s_A, t_A)$ is $FPR_1(ii)$;*
(c) *(X, s, t) is $FPR_1(iii) \Longrightarrow (A, s_A, t_A)$ is $FPR_1(iii)$;*
(d) *(X, s, t) is $FPR_1(iv) \Longrightarrow (A, s_A, t_A)$ is $FPR_1(iv)$.*

Proof (a) First suppose that (X, s, t) is $FPR_1(i)$. We have to prove that (A, s_A, t_A) is $FPR_1(i)$. Let x_r, y_s be two distinct fuzzy singletons in A and $\gamma \in s_A \cup t_A$ with $\gamma(x) \neq \gamma(y)$. Then γ can be written as $\gamma = \sigma/A$, where $\sigma \in s \cup t$ with $\sigma(x) \neq \sigma(y)$. Since (X, s, t) is $FPR_1(i)$, then there exist fuzzy sets $\mu \in s$, $\lambda \in t$ such that

$$x_r q\mu,\, y_s q\lambda \quad \text{and} \quad \mu\bar{q}\lambda.$$

Now $\mu/A \in t_A$, $\lambda/A \in t_A$ for every $\mu \in s$, $\lambda \in t$ respectively. So

$$x_r q(\mu/A),\, y_s q(\lambda/A) \quad \text{and} \quad (\mu/A)\bar{q}(\lambda/A).$$

Hence the fuzzy subspace bitopological space (A, s_A, t_A) is $FPR_1(i)$. Proofs of others are similar. □

In the following two theorems, we observe the preservations of $FPR_1(j)$, $j = i, ii, iii, iv$ properties under continuous, one-one and open mappings.

Definition 3.2([18]) *A function f from a fuzzy bitopological space (X, s, t) into a fuzzy bitopological space (Y, s_1, t_1) is called FP-continuous if and only if $f : (X, s) \to (Y, s_1)$ and $f : (X, t) \to (Y, t_1)$ are both fuzzy continuous.*

Theorem 3.5 *Let $(X,\, s,\, t)$ and $(Y,\, s_1,\, t_1)$ be two fuzzy bitopological spaces and $f : X \to Y$ be bijective, FP-continuous and FP-open map, then*

$$(X,\, s,\, t) \text{ is } FPR_1(j) \Longrightarrow (Y,\, s_1,\, t_1) \text{ is } FPR_1(j),$$

where $j = i, ii, iii, iv$.

Proof Suppose $(X,\, s,\, t)$ is $FPR_1(iv)$. We shall prove that $(Y,\, s_1,\, t_1)$ is $FPR_1(iv)$. Let $a_r,\, b_p \in S(Y)$ with $a \neq b$ and $\gamma \in s_1 \cup t_1$ with $\gamma(x) \neq \gamma(y)$. Since f is bijective, then there exist $c_r,\, d_p \in S(X)$ such that

$$f(c) = a,\ f(d) = b \quad \text{and} \quad c \neq d.$$

Again $f^{-1}(\gamma) \in s \cup t$ as f is FP-continuous. We have

$$f^{-1}(\gamma)(c) = \gamma(f(c)) = \gamma(a),$$

$$f^{-1}(\gamma)(d) = \gamma(f(d)) = \gamma(b).$$

So $f^{-1}(\gamma)(c) \neq f^{-1}(\gamma)(d)$ as $\gamma(x) \neq \gamma(y)$.

Since $(X,\, s,\, t)$ is $FPR_1(iv)$, then there exist $\mu \in s$, $\lambda \in t$ such that

$$c_r q \mu,\ d_p q \lambda \quad \text{and} \quad \mu \cap \lambda = 0.$$

Then $c_r q \mu$, $d_p q \lambda$ implies that

$$\mu(c) + r > 1 \quad \text{and} \quad \lambda(d) + p > 1.$$

Now we have

$$f(\mu)(a) = f(\mu)(f(c)) = \sup \mu(c) = \mu(c)$$

and

$$f(\lambda)(b) = f(\lambda)(f(d)) = \sup \lambda(d) = \lambda(d)$$

because f is bijective. So we have

$$f(\mu)(a) + r = \mu(c) + r > 1 \quad \text{and} \quad f(\lambda)(b) + p > 1.$$

Therefore

$$a_r q f(\mu) \quad \text{and} \quad b_p q f(\lambda).$$

Again we have

$$f(\mu \cap \lambda)(y) = \{\sup(\mu \cap \lambda)(x) : f(x) = y\} = (\mu \cap \lambda)(x) = 0$$

as $\mu \cap \lambda = 0$. Also $f(\mu \cap \lambda) = 0 \Rightarrow f(\mu) \cap f(\lambda) = 0$. Since f is FP-open, then $f(\mu) \in s_1,\ f(\lambda) \in t_1$. Therefore, there exist $f(\mu) \in s_1,\ f(\lambda) \in t_1$ such that

$$a_r q f(\mu),\ b_p q f(\lambda) \ \text{ and } \ f(\mu) \cap f(\lambda) = 0.$$

Hence $(Y,\, s_1,\, t_1)$ is $FPR_1(iv)$. □

Theorem 3.6 *Let* $(X,\, s,\, t)$ *and* $(Y,\, s_1,\, t_1)$ *be two fuzzy bitopological spaces and* $f : X \to Y$ *be* FP*-continuous,* FP*-open and injective map, then*

$$(Y,\, s_1,\, t_1) \text{ is } FPR_1(j) \Longrightarrow (X,\, s,\, t) \text{ is } FPR_1(j)$$

where $j = i,\, ii,\, iii,\, iv$.

Proof Suppose that $(Y,\, s_1,\, t_1)$ is $FPR_1(iv)$. Let $x_r,\, y_p \in S(X)$, $x \neq y$ and $\gamma \in s \cup t$ with $\gamma(x) \neq \gamma(y)$. Since f is injective, then $f(x) \neq f(y)$. Also $f(\gamma) \in s_1 \cup t_1$ as f is FP-open.

We know that

$$f(\gamma)(f(x)) = \sup \gamma(x) = \gamma(x)$$

and

$$f(\gamma)(f(y)) = \sup \gamma(y) = \gamma(y).$$

Then we have

$$f(\gamma)(f(x)) \neq f(\gamma)(f(y)).$$

Since (Y, s_1, t_1) is $FPR_1(iv)$, then $\exists \mu \in s_1, \lambda \in t_1$ such that

$$\mu(f(x)) + r > 1,\ \lambda(f(y)) + p > 1 \quad \text{and} \quad \lambda \cap \mu = 0,$$

which implies that

$$f^{-1}(\mu)(x) + r > 1,\ f^{-1}(\lambda)(y) + p > 1$$

and $f^{-1}(\mu \cap \lambda) = 0$ implies that

$$f^{-1}(\mu) \cap f^{-1}(\lambda) = 0.$$

Since f is FP-continuous, then $f^{-1}(\mu) \in s,\ f^{-1}(\lambda) \in t$. So, there exist $f^{-1}(\mu) \in s,\ f^{-1}(\lambda) \in t$ such that

$$x_r q f^{-1}(\mu),\ y_p q f^{-1}(\lambda) \ \text{ and } \ f^{-1}(\mu) \cap f^{-1}(\lambda) = 0.$$

Therefore $(X,\, s,\, t)$ is $FPR_1(iv)$. Other proofs are similar. □

In previous a work [5], we have introduced the following definitions and discussed many related concepts among them.

Definition 3.3([5]) *The initial fuzzy bitopology on a set X for the family of fbts $\{(X_i, s_i, t_i)\}_{i\in J}$ and the family of functions $\{f_i : X \longrightarrow (X_i, s_i\, t_i)\}_{i\in J}$ is smallest fuzzy topology on X making each f_i fuzzy continuous.*

Definition 3.4([5]) *The final fuzzy topology on a set X for the family of fts $\{(X_i, s_i, t_i)\}_{i\in J}$ and the family of functions $\{f_i : (X_i, s_i, t_i) \longrightarrow X\}_{i\in J}$ is finest fuzzy topology on X making each f_i fuzzy continuous.*

Theorem 3.7 *If $\{(X_i, s_i, t_i)\}_{i\in J}$ is family of $FPR_1(iv)$ fbts and $\{f_i : X \to (X_i, s_i, t_i)\}_{i\in J}$, a family of functions, then the initial fuzzy bitopology on X for the family $\{f_i\}_{i\in J}$ is $FPR_1(iv)$.*

Proof Let s and t be the initial fuzzy topologies on X. Let $x, y \in X$ with $x \neq y$ and let a fuzzy set $w \in s \cup t$ with $w(x) \neq w(y)$. So, there exists $r \in (0, 1)$ such that

$$w(x) < r < w(y).$$

Let x_r and y_r be two fuzzy points of X. For any $\alpha \in (0, r)$, consider the fuzzy point y_α. Then $y_\alpha \in w$ and so it is possible to find a basic fuzzy s-open set, say

$$f_{i_1}^{-1}(u_{i_1}^\alpha) \cap f_{i_2}^{-1}(u_{i_2}^\alpha) \cap \ldots \cap f_{i_n}^{-1}(u_{i_n}^\alpha),\ u_{i_k}^\alpha\ (1 \leq k \leq n)$$

being s_{i_k}–open fuzzy set such that

$$y_\alpha \in \inf f_{i_k}^{-1}(u_{i_k}^\alpha) \subset w \tag{1}$$

So for all $\alpha \in (0, r)$,

$$\alpha < \inf f_{i_k}^{-1}(u_{i_k}^\alpha)(y) \leq w(y)$$

or

$$\alpha < \inf u_{i_k}^\alpha(f_{i_k}(y))\ \ (\text{for all } \alpha \in (0,\, r)).$$

Thus,

$$r = \sup inf u_{i_k}^\alpha(f_{i_k}(y)).$$

Now as $\forall \alpha \in (0, r)$,

$$u_{i_k}^\alpha(f_{i_k}(y)) \leq \sup u_{i_k}^\alpha(f_{i_k}(y)),$$

we have

$$\inf u_{i_k}^\alpha(f_{i_k}(y)) \leq \inf \sup\ u_{i_k}^\alpha(f_{i_k}(y)).$$

Hence

$$r = \sup \inf u_{i_k}^\alpha(f_{i_k}(y)) \leq \inf \sup u_{i_k}^\alpha(f_{i_k}(y)).$$

This implies that

$$\sup u_{i_k}^\alpha(f_{i_k}(y)) > r$$

for all k, $1 \leq k \leq n$. In particular,

$$\sup u_{i_1}^{\alpha}(f_{i_1}(y)) > r.$$

Now let $u_1 = \sup\ u_{i_1}^{\alpha}$. Then $u_1 \in s_{i_1} \cup t_{i_1}$ and $u_1(f_{i_1}(y)) > r$. Also as $w(x) < r$, from (1), we have

$$u_{i_1}^{\alpha}(f_{i_1}(x)) < r\, \forall \alpha \in (0, r).$$

Thus $u_1(f_{i_1}(x)) = r$. Hence $u_1(f_{i_1}(x)) \neq u_1(f_{i_1}(y))$.

Since $(X_{i_1}, s_{i_1}, t_{i_1})$ is $FPR_1(iv)$, then for every two distinct fuzzy points $(f_{i_1}(x))_r$, $(f_{i_1}(y))_r$ of X_{i_1}, there exist fuzzy sets $v_1 \in s_{i_1}$, $u_1 \in t_{i_1}$ such that

$$(f_{i_1}(x))_r q v_1,\ (f_{i_1}(y))_r q u_1 \quad \text{and} \quad u_1 \cap v_1 = 0.$$

Let $v_r = f_{i_1}^{-1}(v_1)$ and $u_r = f_{i_1}^{-1}(u_1)$. We have to show that $x_r q v_r$. For this, since $(f_{i_1}(x))_r q v_1$ we have

$$v_1(f_{i_1}(x)) + r > 1, \ \text{ that is } f_{i_1}^{-1}(v_1)(x) + r > 1,$$

i.e., $v_r(x) + r > 1$. Hence, it is true for $x_r q v_r$. Similarly, it is also true for $y_r q u_r$.

Now, we have to show that $u_r \cap v_r = 0$. Suppose $u_r \cap v_r \neq 0$, then there exists $z \in X$ with $u_r(f_{i_1}(z)) > 0$ and $v_r(f_{i_1}(z)) > 0$. Notice that $v_r(z) = f_{i_1}^{-1}(v_1)(z) = v_1(f_{i_1}(z)) > 0$ and similarly, $u_1(f_{i_1}(z)) > 0$ contradict that $u_1 \cap v_1 = 0$. Hence (X, s, t) is must FPR_1. □

Theorem 3.8 *If $\{(X_i, s_i, t_i)\}_{i \in J}$ is family of $FPR_1(iv)$ fbts and $\{f_i : (X_i, s_i, t_i) \to X\}_{i \in J}$, a family of FP-open and bijective functions, then the final fuzzy bitopology on X for the family $\{f_i\}_{i \in J}$ is $FPR_1(iv)$.*

Proof Let s and t be the final fuzzy topologies on X. Let $x, y \in X$ with $x \neq y$ and let a fuzzy set $w \in s \cup t$ with $w(x) \neq w(y)$. So, there exists $r \in (0, 1)$ such that

$$w(x) < r < w(y).$$

Let x_r and y_r be two distinct fuzzy points of X. For any $\alpha \in (0, r)$, consider the fuzzy point y_α. Then $y_\alpha \in w$ and so it is possible to find a basic fuzzy s–open set, say

$$f_{i_1}(u_{i_1}^{\alpha}) \bigcap f_{i_2}(u_{i_2}^{\alpha}) \bigcap \cdots \bigcap f_{i_n}(u_{i_n}^{\alpha}), \quad u_{i_k}^{\alpha}, \, (1 \leq k \leq n)$$

being s_{i_k}–open fuzzy set such that

$$y_\alpha \in \inf f_{i_k}(u_{i_k}^{\alpha}) \subset u.$$

But $\forall \alpha \in (0, r)$,

$$\alpha < \inf f_{i_k}(u_{i_k}^{\alpha})(y) \leq u(y)$$

or

$$r = \sup \inf f_{i_k}(u_{i_k}^{\alpha})(y).$$

But as $\forall \alpha \in (0, r)$,

$$f_{i_k}(u_{i_k}^{\alpha})(y) \leq \sup f_{i_k}(u_{i_k}^{\alpha})(y).$$

We have $\forall \alpha \in (0, r)$,

$$\inf f_{i_k}(u_{i_k}^{\alpha})(y) \leq \inf \sup f_{i_k}(u_{i_k}^{\alpha})(y).$$

Hence

$$r = \sup \inf f_{i_k}(u_{i_k}^{\alpha})(y) \leq \inf \sup f_{i_k}(u_{i_k}^{\alpha})(y).$$

This implies that

$$\sup f_{i_k}(u_{i_k}^{\alpha})(y) > r, \quad k(1 \leq k \leq n)$$

or

$$\sup(u_{i_k}^{\alpha})(y_{i_k}) > r,$$

where $f_{i_k}(y_{i_k}) = y$, since f_{i_k} is bijective. In particular

$$\sup(u_{i_1}^{\alpha})(y_{i_1}) > r.$$

Now let $u_1 = \sup u_{i_1}^{\alpha}$. Then $u_1 \in s_{i_1} \cup t_{i_1}$ and $u_1(y_{i_1}) > r$. Also as $w(x) < r$, from (1), we get

$$f_{i_1}(u_{i_1}^{\alpha})(x) < r \quad \forall \alpha \in (0, r).$$

Thus

$$\sup f_{i_1}(u_{i_1}^{\alpha})(x) = r, \quad \forall \alpha \in (0, r).$$

or

$$\sup(u_{i_1}^{\alpha})(x_{i_1}) = r$$

where $f_{i_1}(x_{i_1}) = x$, since f_{i_1} is bijective. Hence $u_1(x_{i_1}) = r$. Therefore

$$u_1(x_{i_1}) \neq u_1(y_{i_1}).$$

Since $(X_{i_1}, s_{i_1}, t_{i_1})$ is $FPR_1(iv)$, then for every two distinct fuzzy points $(x_{i_1})_r$, $(y_{i_1})_r$ of X_{i_1}, there exist fuzzy sets $v_1 \in s_{i_1}$, $u_1 \in t_{i_1}$ such that

$$(x_{i_1})_r q v_1, \ (y_{i_1})_r q u_1 \text{ and } u_1 \cap v_1 = 0.$$

Let $v_r = f_{i_1}(v_1)$ and $u_r = f_{i_1}(u_1)$. Now we have to show that $(x_{i_1})_r q v_r$. For this, since $(x_{i_1})_r q v_1$ that is, $v_1(x_{i_1}) + r > 1$, we have

$$v_r(x_{i_1}) = f_{i_1}(v_1)(x_{i_1}) = v_1(x_{i_1}) > 1 - r.$$

So, $v_r(x_{i_1}) + r > 1$. Hence $x_{i_1} q v_r$. Similarly, $y_r q u_r$.

Now, to show that $u_r \cap v_r = 0$, suppose $u_r \cap v_r \neq 0$, then there exists $z \in X$ with $u_r(z) > 0$

and $v_r(z) > 0$. Notice that $v_r(z) = f_{i_1}(v_1)(z) = v_1(z_{i_1}) > 0$, where $f_{i_1}(z_{i_1}) = z$, as f_{i_1} is bijective. Similarly, we can prove that $u_1(z_{i_1}) > 0$ contradict that $u_1 \cap v_1 = 0$. Hence (X, s, t) is must $FPR_1(iv)$. □

§4. Conclusion

The main result of this paper is introducing some new concepts of fuzzy pairwise R_1 bitopological spaces. We discuss some features of these concepts and present their good extension, hereditary. Initial and final topologies introduced in FPR_1 spaces are interesting result.

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Separation Axioms (T_1) on Fuzzy Bitopological Spaces in Quasi-Coincidence Sense

Saikh Shahjahan Miah[1], M. R. Amin[2] and Md. Fazlul Hoque[3]

1. Department of Civil Engineering, Faculty of Science and Engineering, Pundra University of Science and Technology, Bogura-5800, Bangladesh
2. Department of Mathematics, Faculty of Science, Begum Rokeya University, Rangpur-5404, Bangladesh
3. Department of Mathematics, Faculty of Science, Pabna University of Science and Technology, Pabna-6600, Bangladesh

E-mail: skhshahjahan@gmail.com, ruhulbru1611@gmail.com, fazlul_math@yahoo.co.in

Abstract: In this paper, we introduce some new definitions of T_1 separation on fuzzy bitopological space in quasi-coincidence sense and establish relations among them and their counterparts. We show that the notions satisfy good extension, hereditary, productive and projective properties. We present their one-one, onto, fuzzy open and fuzzy continuous mappings. In addition, we also discuss the initial and final fuzzy bitopological spaces in quasi-coincidence sense.

Key Words: Fuzzy bitopological space, quasi-coincidence, fuzzy T_1 bitopological space, good extension, mapping, initial and final fuzzy topology.

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§1. Introduction

The fuzzy set was first explored in [36] and this concept extended to fuzzy topological spaces in [4]. Much research has been done to extend the theory of fuzzy topological spaces in various directions; in particular, fuzzy normality [11, 23], fuzzy uniformity [12], fuzzy regularity [1], fuzzy topological representation [5], separations on fuzzy topological spaces [2, 9, 20, 22], fuzzy topological groups [6], fuzzy bitopological spaces [2, 3, 10, 14, 26], product of fuzzy topological spaces [13], strong-separation and strong countability on fuzzy topological spaces [31], supra fuzzy topological spaces [7, 8, 18] and infra fuzzy topological spaces [28, 33]. One of the important topics in fuzzy mathematics is fuzzy bitopological space with separation axioms, which continuously attracted significant international attention.

The research for fuzzy bitopological spaces started in early nineties [14]. The fuzzy bitopological spaces with separation axioms has become attractive as these spaces possess many desirable properties and can be found throughout various areas in fuzzy topologies. Recent progress has been made constructing separation axioms on fuzzy bitopological spaces in [14,

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27]. One most studied in separation axioms on fuzzy bitopological spaces is T_1 separation [27].

The purpose of this paper is to further contribute to the development of fuzzy bitopological spaces, especially on fuzzy T_1 bitopological spaces in quasi-coincidence sense. In this paper, we define fuzzy T_1 bitopological space in quasi-coincidence sense [19, 21, 27]. We show that the definitions of the T_1 separation satisfy the good extension property. We also present the hereditary, order preserving, productive, and projective properties of these new concepts. In addition, we discuss the initial and final fuzzy bitopologies of the T_1 separation.

§2. Basic Notions and Preliminary Results

In this section, we review some concepts, which will be needed in the sequel. In this paper, X and Y are always presented non-empty sets.

Definition 2.1([36]) *A function u from X into the unit interval I is called a fuzzy set in X. For every $x \in X, u(x) \in I$ is called the grade of membership of x in u. Some authors say that u is a fuzzy subset of X instead of saying that u is a fuzzy set in X. The class of all fuzzy sets from X into the closed unit interval I is denoted by I^X.*

Definition 2.2([24]) *A fuzzy set u in X is called a fuzzy singleton if and only if $u(x) = r, 0 < r \leq 1$, for a certain $x \in X$ and $u(y) = 0$ for all points y of X except x. The fuzzy singleton is denoted by x_r and x is its support. The class of all fuzzy singletons in X will be denoted by $S(X)$. If $u \in I^X$ and $x_r \in S(X)$, then we say that $x_r \in u$ if and only if $r \leq u(x)$*

Definition 2.3([35]) *A fuzzy set u in X is called a fuzzy point if and only if $u(x) = r, 0 < r < 1$, for a certain $x \in X$ and $u(y) = 0$ for all points y of X except x. The fuzzy point is denoted by x_r and x is its support.*

Definition 2.4([14]) *A fuzzy singleton x_r is said to be quasi-coincidence with u, denoted by $x_r q u$ if and only if $u(x) + r > 1$. If x_r is not quasi-coincidence with u, we write $x_r \bar{q} u$ and defined as $u(x) + r \leq 1$.*

Definition 2.5([4]) Let f be a mapping from a set X into a set Y and v be a fuzzy subset of Y. Then the inverse of v written as $f^{-1}(v)$ is a fuzzy subset of X defined by $f^{-1}(v)(x) = v(f(x))$, for $x \in X$.

Definition 2.6([25]) The function $f : (X, t) \to (Y, s)$ is called fuzzy continuous if and only if for every $v \in s, f^{-1}(v) \in t$, the function f is called fuzzy homeomorphic if and only if f is bijective and both f and f^{-1} are fuzzy continuous.

Definition 2.7([17]) The function $f : (X, t) \to (Y, s)$ is called fuzzy open if and only if for every open fuzzy set u in $(X, t), f(u)$ is open fuzzy set in (Y, s).

Definition 2.8([29]) Let f be a real valued function on a topological space. If $\{x : f(x) > \alpha\}$ is open for every real α, then f is called lower semi continuous function.

Definition 2.9([4]) A fuzzy topology t on X is a collection of members of I^X which is closed under arbitrary suprema and finite infima and which contains constant fuzzy sets 1 and 0. The pair (X,t) is called a fuzzy topological space (in short fts) and members of t are called $t-open$ fuzzy sets. A fuzzy set μ is called a $t-$ closed (or simply closed) fuzzy set if $1-\mu \in t$.

Definition 2.10([30]) *A bitopological space (X,S,T) is called pairwise$-T_1(PT_1$ in short) if for all $x,y \in X, x \neq y$, there exist $U \in S, V \in T$ such that $x \in U, y \notin U$ and $x \notin V, y \in V$.*

A fuzzy bitopological property P is called hereditary if each subspace of a fuzzy bitopological space with property P, also has property P.

Definition 2.11([34]) *Let $\{(X_i,s_i,t_i): i \in \Lambda\}$ is a family of fuzzy bitopological spaces. Then the space $(\prod X_i, \prod s_i, \prod t_i)$ is called the product fuzzy bitopological space of the family $\{(X_i,s_i,t_i): i \in \Lambda\}$, where $\prod s_i$ and $\prod t_i$ denote the usual product fuzzy topologies of the families $\{\prod s_i : i \in \Lambda\}$ and $\{\prod t_i : i \in \Lambda\}$ of the fuzzy topologies respectively on X.*

A fuzzy bitopological property P is called productive if the product of fuzzy bitopological spaces of a family of fuzzy bitopological space, each having property P, has property P.

A fuzzy bitopological property P is called projective if for a family of fuzzy bitopological space $\{(X_i,s_i,t_i): i \in \Lambda\}$, the product fuzzy bitopological space $(\prod X_i, \prod s_i, \prod t_i)$ has property P implies that each coordinate space has property P.

Definition 2.12([15]) Let (X,T) be an ordinary topological space. The set of all lower semi continuous functions from (X,T) into the closed unit interval I equipped with the usual topology constitutive a fuzzy topology associated with (X,T) and is denoted by $(X,\omega(T))$.

Definition 2.13([16]) The initial fuzzy topology on a set X for the family of fuzzy topological spaces $\{(X_i,t_i)_{i\in\Lambda}\}$ and the family of functions $\{f_i : X \to (X_i,t_i)\}_{i\in\Lambda}$ is the smallest fuzzy topology on X making each f_i fuzzy continuous. It is easily seen that it is generated by the family $\{f_i^{-1}(u_i) : u_i \in t_i\}_{i\in\Lambda}$.

Definition 2.14([16]) The final fuzzy topology on a set X for the family of fuzzy topological spaces $\{(X_i,t_i)_{i\in\Lambda}\}$ and the family of functions $\{f_i : (X_i,t_i) \to X\}_{i\in\Lambda}$ is the finest fuzzy topology on X making each f_i fuzzy continuous.

Definition 2.15([26]) A function f from a fuzzy bitopological space (X,s,t) into a fuzzy bitopological space (Y,s_1,t_1) is called fuzzy $FP-$continuous if and only if $f:(X,s) \to (Y,s_1)$ and $f:(X,t) \to (Y,t_1)$ are both fuzzy continuous.

Theorem 2.1([3]) A bijective mapping from an fts (X,t) to an fts (Y,s) preserves the value of a fuzzy singleton (fuzzy point).
Note: Preimage of any fuzzy singleton (fuzzy point) under bijective mapping preserves its value.

§3. Fuzzy T_1 Bitopological Space

In this section, we present some new notions on fuzzy T_1 bitopological spaces and their relevant results. We also discuss existing some well-known properties using these new concepts and establish relationships between these new notions and the relevant existing notions.

Definition 3.1 *A fuzzy bitopological space (X, s, t) is called*

(a) $FPT_1(i)$ if and only if for any pair $x_m, y_n \in S(X)$ with $x \neq y$, there exist $u, v \in s \cup t$ such that $x_m qu, y_n \bar{q} u$ and $y_n qv, x_m \bar{q} v$;

(b) $FPT_1(ii)$ if and only if for any pair $x_m, y_n \in S(X)$ with $x \neq y$, there exist $u, v \in s \cup t$ such that $x_m qu, y_n \cap u = 0$ and $y_n qv, x_m \cap v = 0$;

(c) $FPT_1(iii)$ if and only if for any pair of fuzzy points x_m, y_n in X with $x \neq y$, there exist $u, v \in s \cup t$ such that $x_m \in u, y_n \bar{q} u$ and $y_n \in v, x_m \bar{q} v$;

(d) $FPT_1(iv)$ if and only if for any pair of distinct fuzzy points p, q in X, there exists a fuzzy set $u, v \in s \cup t$ such that $p \in u, q \cap u = 0$ or $q \in u, p \cap u = 0$;

(e) $FPT_1(v)$ if and only if for all $x, y \in X$ with $x \neq y$, there exist $u, v \in s \cup t$ such that $u(x) = 1, u(y) = 0$ and $v(y) = 1, v(x) = 0$.

Here it is mentioned that $FPT_1(iii)$ and $FPT_1(iv)$ are according to Sufiya et al.[32], $FPT_1(i)$ is according to A A Nouh [27], and $FPT_1(v)$ is according to M. Srivastava and R. Srivastava [30].

The examples of definitions of $FPT_1(i)$ and $FPT_1(ii)$ are as follows:

Example 3.1 Let $X = \{x, y\}$, $u, v \in I^X$ with $u(x) = 1$, $u(y) = 0$ and $v(y) = 1$, $v(x) = 0$ and t be the fuzzy topology on X generated by $\{0, u, v, 1\}$ and s be the fuzzy topology on X generated by $\{constants\}$. Also, let $x_m, y_n \in S(X)$ with $x \neq y$, then $u(x) + m > 1$ and $u(y) + s \leq 1$ for $m, n \in (0, 1]$. Thus $x_m qu, y_n \bar{q} u$. Similarly, $y_n qv, x_m \bar{q} v$. Hence (X, s, t) is $FPT_1(i)$ as $u, v \in s \cup t$. Also, as $u(y) = 0, y_n \cap u = 0$ and similarly $x_m \cap v = 0$. Therefore, (X, s, t) is $FPT_1(ii)$.

Theorem 3.1 *Let (X, s, t) be a fuzzy bitopological space and $(X, s \cup t)$ be a fuzzy topological space. If (X, s, t) is FPT_1 then $(X, s \cup t)$ is fuzzy T_1 topological space.*

Proof Let (X, s, t) be FPT_1. Since $s \subseteq s \cup t$ and $t \subseteq s \cup t$, it follows immediately that $(X, s \cup t)$ is FT_1. ◻

Theorem 3.2 *If the fuzzy topological space (X, s) and (X, t) are both fuzzy $T_1(j)$ topological spaces, then their corresponding fuzzy bitopological space (X, s, t) is $FPT_1(j)$, for $j = i, ii$. But the converse is not true in general.*

Proof Let (X, s) and (X, t) are both $FT_1(j)$. Then their corresponding fuzzy bitopological space (X, s, t) is $FPT_1(j)$, for $j = i, iii$ as $s \subseteq s \cup t$ and $t \subseteq s \cup t$. To prove (X, s, t) is $FPT_1(j)$ does not imply (X, s) and (X, t) are both $FT_1(j)$, for $j = i, ii$, the following is its a counter example. ◻

Example 3.2 Let $X = \{x, y\}$, $u, v \in I^X$ and t be the fuzzy topology on X generated by $\{u, v\} \cup \{constants\}$, with $u(x) = 1$, $u(y) = 0$ and $v(y) = 1$, $v(x) = 0$. Also, let s be the fuzzy topology on X generated by $\{constants\}$. Then, for any $0 < m \leq 1$ and $0 < n \leq 1, u(x) + m > 1$ and $u(y) + n \leq 1$, which imply that $x_m qu, y_n \bar{q} u$. Similarly, $y_n qv, x_m \bar{q} v$. Also, $u(y) = 0 \Rightarrow y_n \cap u = 0$ and similarly $x_m \cap v = 0$. As $u, v \subseteq s \cup t, (X, s, t)$ is $FPT_1(j)$ but (X, s) is not $FT_1(j)$, for $j = i, ii$.

Theorem 3.3 *If a fuzzy bitopological space (X, s, t) is $FPT_1(j)$ then (X, s, t) is $FPT_0(j)$, for $j = i, ii, iii, iv, v$.*

Proof The proof is obvious. □

Theorem 3.4 *For a fuzzy bitopological space (X, s, t) the implications in Figure 1 are true.*

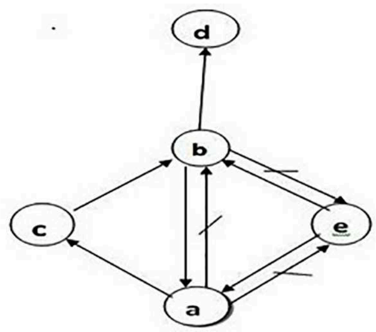

Figure 1

Proof $(c) \Rightarrow (b)$: Let (X, s, t) be $FPT_1(iii)$ and x_m, y_n be fuzzy singletons in X with $x \neq y$. Also let $r > 1 - m$ for $0 < m < 1$. Since (X, s, t) is $FPT_1(iii)$, there exist fuzzy sets $u, v \in s \cup t$ such that $x_r \in u, y_1 \bar{q} u$ and $y_1 \in v, x_r \bar{q} v$, where x_r and y_1 are distinct fuzzy points in X. Now, $x_r \in u \Rightarrow u(x) \geq r > 1 - m \Rightarrow u(x) + m > 1 \Rightarrow u(x) + m > 1$ for $0 < m \leq 1$ also. $\Rightarrow x_m q u$ when $x_m \in S(X)$ and $y_1 \bar{q} u \Rightarrow u(y) + 1 \leq 1 \Rightarrow u(y) \leq 1 - 1 = 0 \Rightarrow u(y) = 0 \Rightarrow y_n \cap u = 0$ for $0 < n \leq 1$.

Similarly, it is easy to prove that $y_n q v$ and $x_m \cap v = 0$. It follows that for any fuzzy singletons x_m, y_n in X with $x \neq y$ there exist $u, v \in s \cup t$ such that $x_m q u, y_n \cap u = 0$ and $y_n q v, x_m \cap v = 0$. Thus (X, s, t) is $FPT_1(ii)$.

$(b) \Rightarrow (d)$: Let x_m, y_n be distinct fuzzy points in X and $0 < r \leq 1, 0 < s \leq 1$ with $r \leq 1 - m, s \leq 1 - n$. Since (X, s, t) is $FPT_1(ii)$, there exist $u, v \in s \cup t$ such that $x_r q u, y_s \cap u = 0$ and $y_s q v, x_r \cap v = 0$.

Now, $x_r q u \Rightarrow u(x) + r > 1 \Rightarrow u(x) > 1 - r \geq m \Rightarrow u(x) \geq m \Rightarrow x_m \in u$ and $y_s \cap u = 0 \Rightarrow u(y) = 0 \Rightarrow y_n \cap u = 0$. Similarly, we can prove that $y_n \in v$ and $x_m \cap v = 0$.

It follows that for any distinct fuzzy points x_m, y_n in X with $x \neq y$ there exist $u, v \in s \cup t$ such that $x_m \in u, y_n \cap u = 0$ and $y_n \in v, x_m \cap v = 0$. Thus (X, s, t) is $FPT_1(iv)$.

$(b) \Rightarrow (a)$: Let (X, s, t) be $FPT_1(ii)$ and x_m, y_n be fuzzy singletons in X with $x \neq y$. Since (X, s, t) is $FPT_1(ii)$, there exist fuzzy sets $u, v \in s \cup t$ such that $x_m q u, y_n \cap u = 0$ and $y_n q v, x_m \cap v = 0$. To prove (X, s, t) is $FPT_1(i)$, it is only needed to prove that $y_n \bar{q} u$ and $x_m \bar{q} v$.

Now, $y_n \cap u = 0 \Rightarrow u(y) = 0 \Rightarrow u(y) + n \leq 1 \Rightarrow y_n \bar{q} u$ and similarly $x_m \bar{q} v$. Thus (X, s, t) is $FPT_1(i)$. To show $(a) \not\Rightarrow (b)$, we give a counter example in Example 3.3.

Example 3.3 Let $X = \{x, y\}$, $u, v \in I^X$ be given by $u(x) = 1$, $u(y) = 0.1$, $v(y) = 1$, $v(x) = 0.1$. Let us consider the fuzzy topology $s \cup t$ on X generated by $\{0, u, v, 1\}$. For $0 < m \leq 1, 0 < n < 0.9, u(x) + m > 1 \Rightarrow x_m q u$ and $u(y) + n \leq 1 \Rightarrow y_n \bar{q} u$. Similarly, $y_n q v$ and $x_m \bar{q} v$. Thus (X, s, t) is $FPT_1(i)$. But $u(y) \neq 0 \Rightarrow y_n \cap u \neq 0$. Also, $v(x) \neq 0 \Rightarrow x_m \cap v \neq 0$. Thus (X, s, t)

is not $FPT_1(ii)$.

$(e) \Rightarrow (a)$: Let (X, s, t) be $FPT_1(v)$ and x_m, y_n be fuzzy singletons in X with $x \neq y$. Since (X, s, t) is $FPT_1(v)$, there exist fuzzy sets $u, v \in s \cup t$ such that $u(x) = 1, u(y) = 0$ and $v(y) = 1, v(x) = 0$. Now, $u(x) = 1 \Rightarrow u(x) + m > 1 \Rightarrow x_m qu$ and $u(y) = 0 \Rightarrow u(y) + n \leq 1 \Rightarrow y_n \bar{q} u$. Similarly, it is easy to prove that $y_n qv$ and $x_m \bar{q} v$. Thus (X, s, t) is $FPT_1(i)$. To show $(a) \not\Rightarrow (e)$, we give a counter example in Example 3.4

Example 3.4 Let $X = \{x, y\}$, $u, v \in I^X$ be given by $u(x) = 1 - \gamma$, $u(y) = 0$, $v(y) = 1 - \delta$, $v(x) = 0$, where $\gamma = m/2, \delta = n/2$ for $m, n \in (0, 1]$. Let the fuzzy topology $s \cup t$ on X generated by $\{0, u, v, 1\} \cup \{constants\}$.

Now, $u(x) = 1 - \gamma \Rightarrow u(x) = 1 - m/2 \Rightarrow u(x) + m/2 = 1 \Rightarrow u(x) + m > 1 \Rightarrow x_m qu$ and $u(y) = 0 \Rightarrow u(y) + n \leq 1 \Rightarrow y_n \bar{q} u$. In the similar way, $y_n qv$ and $x_m \bar{q} v$. Thus (X, s, t) is $FPT_1(i)$. But $u(x) \neq 1$ and $v(y) \neq 1$. Thus (X, s, t) is not $FPT_1(v)$.

$(a) \Rightarrow (c)$: As $(b) \Rightarrow (d)$ we can say that $(a) \Rightarrow (c)$.

$(e) \Rightarrow (b)$: Let (X, s, t) be $FPT_1(v)$ and x_m, y_n be fuzzy singletons in X with $x \neq y$. Since (X, s, t) is $FPT_1(v)$, there exist fuzzy sets $u, v \in s \cup t$ such that $u(x) = 1, u(y) = 0$ and $v(y) = 1, v(x) = 0$. Now, $u(x) = 1 \Rightarrow u(x) + m > 1 \Rightarrow x_m qu$ and $u(y) = 0 \Rightarrow y_n \cap u = 0$. Similarly, we can show that $y_n qv$ and $x_m \cap v = 0$. Thus (X, s, t) is $FPT_1(ii)$. A counter example in Example 3.5 shows that $(b) \not\Rightarrow (e)$.

Example 3.5 Let $X = \{x, y\}$, $u, v \in I^X$ be given by $u(x) = 1 - \gamma$, $u(y) = 0$, $v(y) = 1 - \delta$, $v(x) = 0$, where $\gamma = m/2, \delta = n/2$ for $m, n \in (0, 1]$. Let the fuzzy topology $s \cup t$ on X generated by $\{0, u, v, 1\} \cup \{constants\}$.

Now, $u(x) = 1 - \gamma \Rightarrow u(x) = 1 - m/2 \Rightarrow u(x) + m/2 = 1 \Rightarrow u(x) + m > 1 \Rightarrow x_m qu$ and $u(y) = 0 \Rightarrow y_n \cap u = 0$. In the similar way, $y_n qv$ and $x_m \cap v = 0$. Thus (X, s, t) is $FPT_1(ii)$. But $u(x) \neq 1$ and $v(y) \neq 1$. Thus (X, s, t) is not $FPT_1(v)$. Thus proof is completed. □

Theorem 3.5 *Let (X, S, T) be a bitopological space. Then (X, S, T) is PT_1 if and only if $(X, \omega(S), \omega(T))$ is $FPT_1(j)$, where $j = i, ii, iii, iv, v$.*

Proof Let (X, S, T) be a PT_1 topological space. We shall prove that $(X, \omega(S), \omega(T))$ is $FPT_1(ii)$. Let x, y in X with $x \neq y$. Since (X, S, T) be a PT_1 topological space hence there exists $U, V \in S \cup T$ such that $x \in U, y \notin U$ and $y \in V, x \notin V$. From the definition of lower semi continuous function, $1_U, 1_V \in (\omega(S) \cup \omega(T))$, i.e., $1_U \in \omega(S)$ or $1_U \in \omega(T)$. Then $1_U(x) = 1 \Rightarrow 1_U(x) + m > 1 \Rightarrow x_m q 1_U$ and $1_U(y) = 0 \Rightarrow y_n \cap 1_U = 0$.

Similarly, we can prove that $y_n q 1_V$ and $x_m \cap 1_V = 0$. Hence $(X, \omega(S), \omega(T))$ is $FPT_1(ii)$.

Conversely, let $(X, \omega(S), \omega(T))$ is $FPT_1(ii)$. It is required to prove that (X, S, T) be a PT_1 topological space. Let x, y in X with $x \neq y$. Since $(X, \omega(S), \omega(T))$ is $FPT_1(ii)$, we have for any fuzzy singletons x_m, y_n in X, there exist $u, v \in \omega(S) \cup \omega(T)$ such that $x_m qu, y_n \cap u = 0$ and $y_n qv, x_m \cap v = 0$.

Now, $x_m qu \Rightarrow u(x) + m > 1 \Rightarrow u(x) > 1 - m = \alpha \Rightarrow x \in u^{-1}(\alpha, 1]$ And $y_n \cap u = 0 \Rightarrow u(y) = 0 \Rightarrow u(y) + n \leq 1 \Rightarrow u(y) \leq 1 - n = \alpha \Rightarrow u(y) \leq \alpha \Rightarrow y \notin u^{-1}(\alpha, 1]$. Similarly, we can prove that $y \in v^{-1}(\alpha, 1]$ and $x \notin v^{-1}(\alpha, 1]$. Also, $u^{-1}(\alpha, 1], v^{-1}(\alpha, 1] \in S \cup T$. Hence (X, S, T)

be a PT_1 topological space. Proof for $j = i, iii, iv, v$ is similar to above. □

§4. Hereditary, Productive and Projective Properties

In this section, we describe the hereditary, productive and projective properties on our given concepts. The first theorem is on hereditary property and the second one is on productive and projective properties.

Theorem 4.1 *If (X, s, t) be a fuzzy bitopological space and $A \subseteq X, s_A = \{u/A : u \in s\}, t_A = \{v/A : v \in t\}$ and (X, s, t) is $FPT_1(j)$ then (A, s_A, t_A) is $FPT_1(j)$, where $j = i, ii, iii, iv, v$.*

Proof We first prove this theorem for $j = ii$ and remaining are similar. Let (X, s, t) is $FPT_1(ii)$ and x_m, y_n are fuzzy singletons in A with $x \neq y$. Since $A \subseteq X, x_m, y_n$ are also fuzzy singletons in X. Also since (X, s, t) is $FPT_1(ii)$, there exist $u, v \in s \cup t$ such that $x_m q u, y_n \cap u = 0$ and $y_n q v, x_m \cap v = 0$. For $A \subseteq X$, we have $u/A, v/A \in s_A \cup t_A$.

Now, $x_m q u \Rightarrow u(x) + m > 1, x \in X \Rightarrow u/A(x) + m > 1, x \in A \subseteq X \Rightarrow x_m q u/A$ and $y_n \cap u = 0 \Rightarrow u(y) = 0, y \in X \Rightarrow u/A(y) = 0, y \in A \subseteq X \Rightarrow y_n \cap u/A = 0$. Similarly, we can show that $y_n q v/A, x_m \cap v/A = 0$. Therefore, (A, s_A, t_A) is $FPT_1(ii)$. □

Theorem 4.2 *If $\{(X_i, s_i, t_i) : i \in \Lambda\}$ is a family of fuzzy bitopological spaces then the product fuzzy bitopological space $(\prod X_i, \prod s_i, \prod t_i) = (X, s, t)$ is $FPT_1(j)$ if and only if each coordinate space (X_i, s_i, t_i) is $FPT_1(j)$, where $j = i, ii, iii, iv, v$.*

Proof Let for all $i \in \Lambda, (X_i, s_i, t_i)$ is $FPT_1(ii)$ space. We have to prove that (X, s, t) is $FPT_1(ii)$. Let x_m, y_n be fuzzy singletons in X with $x \neq y$. Then $(x_i)_m, (y_i)_n$ are fuzzy singletons with $x_i \neq y_i$ for some $i \in \Lambda$. Since(X_i, s_i, t_i) is $FPT_1(ii)$, there exist $u_i, v_i \in s_i \cup t_i$ such that $(x_i)_m q u_i, (y_i)_n \cap u_i = 0$ and $(y_i)_n q v_i, (x_i)_m \cap v_i = 0$. But we have $\pi_i(x) = x_i$ and $\pi_i(y) = y_i$.

Now, $(x_i)_m q u_i \Rightarrow u_i(x_i) + m > 1 \Rightarrow u_i(\pi_i(x)) + m > 1 \Rightarrow (u_i \circ \pi_i)(x) + m > 1 \Rightarrow x_m q(u_i \circ \pi_i)$ and $(y_i)_n \cap u_i = 0 \Rightarrow u_i(y_i) = 0 \Rightarrow u_i(\pi_i(y)) = 0 \Rightarrow (u_i \circ \pi_i)(y) = 0 \Rightarrow y_n \cap (u_i \circ \pi_i) = 0$. Similarly, we can show that $y_n q(v_i \circ \pi_i), x_m \cap (v_i \circ \pi_i) = 0$. Hence (X, s, t) is $FPT_1(ii)$.

Conversely, let the product fuzzy bitopological space (X, s, t) is $FPT_1(ii)$. It is required to prove that for all $i \in \Lambda, (X_i, s_i, t_i)$ is $FPT_1(ii)$ space. Let a_i be a fixed element in X_i. Let $A_i = \{x \in X = \Pi_{i\in\Lambda} X_i : x_j = a_j \text{ for some } i \neq j\}$. Then A_i is a subset of X, and hence (A_i, s_{A_i}, t_{A_i}) is a subspace of (X, s, t). Since (X, s, t) is $FPT_1(ii)$, so (A_i, s_{A_i}, t_{A_i}) is $FPT_1(ii)$. Again, A_i is homeomorphic image of X_i. Therefore, for all $i \in \Lambda, (X_i, s_i, t_i)$ is $FPT_1(ii)$. Similarly, one can prove the others. □

§5. Mappings in Fuzzy T_1 Bitopological Space

We discuss in this section about order preserving property of the notions under one-one, onto, fuzzy open and fuzzy continuous mappings.

Theorem 5.1 *Suppose (X, s, t) and (Y, s_1, t_1) are two fuzzy bitopological spaces and $f : X \to Y$*

is bijective and fuzzy open map. If (X, s, t) *is* $FPT_1(j)$ *then* (Y, s_1, t_1) *is* $FPT_1(j)$*, where* $j = i, ii, iii, iv, v$.

Proof Let (X, s, t) is $FPT_1(ii)$ and x'_m, y'_n be fuzzy singletons in Y with $x' \neq y'$. Since f is onto then there exist $x, y \in X$ with $f(x) = x', f(y) = y'$ and x_m, y_n are fuzzy points in X with $x \neq y$ as f is one-one. Again, (X, s, t) is $FPT_1(ii)$, there exist $u, v \in s \cup t$ such that $x_m qu, y_n \cap u = 0$ and $y_n qv, x_m \cap v = 0$.

Now, $x_m qu \Rightarrow u(x) + m > 1$ and $y_n \cap u = 0 \Rightarrow u(y) = 0$. Again, $f(u)(x') = \{\sup u(x) : f(x) = x'\} \Rightarrow f(u)(x') = u(x)$ for some x and $f(u)(y') = \{\sup u(y) : f(y) = y'\} \Rightarrow f(u)(y') = u(y)$ for some y. Also, since f is a fuzzy open hence $f(u) \in s_1 \cup t_1$ as $u \in s \cup t$.

Again, $u(x) + m > 1 \Rightarrow (f(u))(x') + m > 1 \Rightarrow x'_m qf(u)$ and $u(y) = 0 \Rightarrow f(u)(y') = 0 \Rightarrow y'_n \cap f(u) = 0$. Similarly, it is easy to show that $y'_n qf(v), x'_m \cap f(v) = 0$. Thus, (Y, s_1, t_1) is $FPT_1(ii)$. Similarly, one can prove the others. □

Theorem 5.2 *Suppose* (X, s, t) *and* (Y, s_1, t_1) *are two fuzzy bitopological spaces and* $f : X \to Y$ *is one-one and fuzzy* $FP-$*continuous map. If* (Y, s_1, t_1) *is* $FPT_1(j)$*, then* (X, s, t) *is* $FPT_1(j)$*, where* $j = i, ii, iii, iv, v$.

Proof Let (Y, s_1, t_1) is $FPT_1(ii)$ and x_m, y_n be fuzzy singletons in X with $x \neq y$. Then $(f(x))_m, (f(y))_n$ are fuzzy singletons in Y with $f(x) \neq f(y)$ as f is one-one. Also, since (Y, s_1, t_1) is $FPT_1(ii)$, there exist $u, v \in s_1 \cup t_1$ such that $(f(x))_m qu, (f(y))_n \cap u = 0$ and $(f(y))_n qv, (f(x))_m \cap v = 0$.

Now, $(f(x))_m qu \Rightarrow u(f(x)) + m > 1 \Rightarrow f^{-1}(u(x)) + m > 1 \Rightarrow (f^{-1}(u))(x) + m > 1 \Rightarrow x_m q(f^{-1}(u))$ and $(f(y))_n \cap u = 0 \Rightarrow u(f(y)) = 0 \Rightarrow f^{-1}(u(y)) = 0 \Rightarrow (f^{-1}(u))(y) = 0 \Rightarrow y_n \cap (f^{-1}(u)) = 0$. Since f is fuzzy continuous and $u \in s_1 \cup t_1$ hence $f^{-1}(u) \in s \cup t$. In the same way, it is easy to prove that $y_n \cap q(f^{-1}(v))$ and $x_m \cap (f^{-1}(v)) = 0$. Therefore, (X, s, t) is $FPT_1(ii)$. The proof of other properties is similar to above. □

§6. Initial and Final Fuzzy T_1 Bitopological Space

We define and discuss the initial and final fuzzy bitopologies in this section.

Definition 6.1 *The initial fuzzy bitopology on a set* X *for the family of fuzzy bitopological spaces* $\{(X_i, s_i, t_i)\}_{i\in\Lambda}$ *and the family of functions* $\{f_i : X \to (X_i, s_i \cup t_i)\}_{i\in\Lambda}$ *is the smallest fuzzy bitopology on* X *making each* f_i *fuzzy continuous. It is easily seen that it is generated by the family* $\{f_i^{-1}(u_i) : u_i \in s_i \cup t_i\}_{i\in\Lambda}$.

Definition 6.2 *The final fuzzy bitopology on a set* X *for the family of fuzzy bitopological spaces* $\{(X_i, s_i, t_i)\}_{i\in\Lambda}$ *and the family of functions* $\{f_i : (X_i, s_i \cup t_i) \to X\}_{i\in\Lambda}$ *is the finest fuzzy bitopology on* X *making each* f_i *fuzzy continuous.*

Theorem 6.1 *If* $\{(X_i, s_i, t_i)\}_{i\in\Lambda}$ *is a family of* $FPT_1(j)$ *fts and* $\{f_i : X \to (X_i, s_i \cup t_i)\}_{i\in\Lambda}$ *, a family of one-one and fuzzy continuous functions, then the initial fuzzy bitopology on* X *for the family* $\{f_i\}_{i\in\Lambda}$ *is* $FPT_1(j)$*, for* $j = i, ii, iii, iv, v$.

Proof We shall prove the above theorem for $j = ii$ and the remaining is similar. Let t, s be the initial fuzzy topologies on X for the family $\{f_i\}_{i\in\Lambda}$. Let x_r, y_s be fuzzy singletons in X with $x \neq y$. Then $f_i(x), f_i(y) \in X_i$ and $f_i(x) \neq f_i(y)$ as f_i is one-one. Scince (X_i, s_i, t_i) is $FPT_1(ii)$, then for any two distinct fuzzy singletons $(f_i(x))_r, (f_i(y))_s$ in X_i, there exist fuzzy sets $u_i, v_i \in s_i \cup t_i$ such that $(f_i(x))_r q u_i, (f_i(y))_s \cap u_i = 0$ and $(f_i(y))_s q v_i, (f_i(x))_r \cap v_i = 0$.

Now, $(f_i(x))_r q u_i \Rightarrow u_i(f_i(x)) + r > 1 \Rightarrow f_i^{-1}(u_i)(x) + r > 1$. This is true for every $i \in \Lambda$. So, $\inf f_i^{-1}(u_i)(x) + r > 1$ and $(f_i(y))_s \cap u_i = 0 \Rightarrow u_i(f_i(y)) = 0 \Rightarrow f_i^{-1}(u_i)(y) = 0$. This is true for every $i \in \Lambda$. So, $\inf f_i^{-1}(u_i)(y) = 0$. Let $u = \inf f_i^{-1}(u_i)$. Then $u \in s_i \cup t_i$ as f_i is fuzzy continuous. So $u(x) + r > 1$ and $u(y) = 0$. Hence $x_r q u$ and $y_s \cap u = 0$. Similarly, we can prove that $y_s q v$ and $x_r \cap v = 0$. Therefore, (X, s, t) is $FPT_1(ii)$. □

Theorem 6.2 *If $\{(X_i, s_i, t_i)\}_{i\in\Lambda}$ is a family of $FPT_1(j)$ fts and $\{f_i : (X_i, s_i \cup t_i) \to X\}_{i\in\Lambda}$, a family of fuzzy open and bijective function, then the final fuzzy topology on X for the family $\{f_i\}_{i\in\Lambda}$ is $FPT_1(j)$, for $j = i, ii, iii, iv, v$.*

Proof We shall prove the above theorem for $j = ii$ and the remaining is similar. Let s, t be the final fuzzy topologies on X for the family $\{f_i\}_{i\in\Lambda}$. Let x_r, y_s be fuzzy singletons in X with $x \neq y$. Then $f_i^{-1}(x), f_i^{-1}(y) \in X_i$ and $f_i^{-1}(x) \neq f_i^{-1}(y)$ as f_i is bijective. Since (X_i, s_i, t_i) is $FPT_1(ii)$, then for any two distinct fuzzy singletons $(f_i^{-1}(x))_r, (f_i^{-1}(y))_s$ in X_i, there exist fuzzy sets $u_i, v_i \in s_i \cup t_i$ such that $(f_i^{-1}(x))_r q u_i, (f_i^{-1}(y))_s \cap u_i = 0$ and $(f_i^{-1}(y))_s q v_i, (f_i^{-1}(x))_r \cap v_i = 0$.

Now, $(f_i^{-1}(x))_r q u_i \Rightarrow u_i(f_i^{-1}(x)) + r > 1 \Rightarrow f_i(u_i)(x) + r > 1$. This is true for every $i \in \Lambda$. So, $\inf f_i(u_i)(x) + r > 1$ and $(f_i^{-1}(y))_s \cap u_i = 0 \Rightarrow u_i(f_i^{-1}(y)) = 0 \Rightarrow f_i(u_i)(y) = 0$. This is true for every $i \in \Lambda$. So, $\inf f_i(u_i)(y) = 0$. Let $u = \inf f_i(u_i)$. Then $u \in s_i \cup t_i$ as f_i is fuzzy open. So, $u(x) + r > 1$ and $u(y) = 0$. Hence $x_r q u$ and $y_s \cap u = 0$. Similarly, we can prove that $y_s q v$ and $x_r \cap v = 0$. Therefore, (X, s, t) is $FPT_1(ii)$. □

§7. Conclusion

One of the main results of this paper is introducing some new definitions of fuzzy T_1 bitopological spaces in sense of quasi-coincidence. We present their good extension, hereditary, productive and projective properties. We compare the results with other existing notions and their counterparts' examples [27, 30, 32]. These concepts would be interesting to more expansion on fuzzy bitopological spaces [30] and extending to general fuzzy topological space [4].

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On Radicals for Ternary Semirings

Swapnil P. Wani

University Institute of Chemical Technology
Kavayitri Bahinabai Chaudhari North Maharashtra University, Jalgaon - 425001, India

Kishor F. Pawar

Department of Mathematics, School of Mathematical Sciences
Kavayitri Bahinabai Chaudhari North Maharashtra University, Jalgaon - 425 001, India

E-mail: swapwani@gmail.com, kfpawar@nmu.ac.in

Abstract: The notion of radical theory of rings was introduced by Kurosh [1] in 1953. In the present paper, the concepts of radical theory of rings and semirings are generalized for ternary semirings. Later, the notions like semisimple class, upper radical and Hoehnke radical for class of ternary semirings are introduced. Also proved some consequences of semisimple and upper radicals.

Key Words: Ternary semiring, radical class, semisimple class, upper radical, Hoehnke radical.

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§1. Introduction

The idea of a ternary algebraic system was first invented in 1924 by Prüfer in [5]. In 1971, Lister [8]investigated the notion of ternary ring and studied some properties of a ternary ring. The concept of semiring was first introduced in [6] by Vandiver in 1934. Later, the notion of a ternary semiring which generalizes the notion of ternary ring and semiring was introduced by Dutta and Kar [7] in 2003. Pawar and Deore in [2]-[4] generalizes concepts of radical classes for a class of semirings. The present paper extends the notions of radical theory of rings and semirings to a ternary semiring. The concept of radical class with few examples and results are introduced in Section 3. Section 4 introduces the notions of semisimple class, upper radical and their properties and relationship. In Section 5, the notion of Hoehnke radical for class of ternary semirings is introduced.

§2. Preliminary Definitions

Definition 2.1([7]) *A non-empty set S together with a binary operation, called addition and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if S is an*

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additive commutative semigroup satisfying the following conditions:

(i)(*Associative Law*) $(abc)de = a(bcd)e = ab(cde)$;

(ii)(*Right Distributive Law*) $(a + b)cd = acd + bcd$;

(iii)(*Lateral Distributive Law*) $a(b + c)d = abd + acd$;

(iv)(*Left Distributive Law*) $ab(c + d) = abc + abd$

for all $a, b, c, d, e \in T$.

Example 2.2([7]) Let Z_0^- be the set of all negative integers with zero. Then with the usual binary addition and ternary multiplication, Z_0^- forms a ternary semiring.

Definition 2.3([7]) *An additive subsemigroup I of a ternary semiring S is called ideal of S if $SSI \subseteq I, SIS \subseteq I$ and $ISS \subseteq I$. An ideal I of a ternary semiring S is called k-ideal (subtractive) if for $a \in I, a + b \in I, b \in S$ imply $b \in I$. We denote $I \triangleleft S$, a ternary semiring ideal in S.*

Definition 2.4([7]) *A ternary semiring S is said to be regular if for each element a in S there exists an element x in S such that $a = axa$. If the element x is unique and satisfies $x = xax$, then S is called an inverse ternary semiring. x is called the inverse of a.*

Definition 2.5([7]) *Let S be a ternary semiring and M be an ideal of S. Then M is called maximal (largest) ideal of S if $M \neq S$ and there does not exist any other ideal I of S such that $M \subset I \subset S$.*

§3. Radical Class

In this section, the radical class for ternary semiring is defined on the lines of Kurosh [1]. Also discussed some properties and theorems on radical classes for ternary semirings on the line of [2] and [4].

Definition 3.1 *A class $\mathcal{R}$ of ternary semirings is called radical class if*

(a) *$\mathcal{R}$ is homomorphically closed;*

(b) *Every ternary semiring $S \in \mathbb{U}$, where $\mathbb{U}$ is the universal class of ternary semirings, contains a largest $\mathcal{R}$-k-ideal, $\mathcal{R}(S)$;*

(c) *If $S \in \mathbb{U}$, then $S/\mathcal{R}(S)$ is $\mathcal{R}$-semisimple. i.e. $\mathcal{R}(S/\mathcal{R}(S)) = 0$.*

Proposition 3.2 *Assuming conditions (a) and (b) on a class $\mathcal{R}$ of ternary semirings, condition (c) is equivalent to*

(c') *If I is a k-ideal of the ternary semiring S and if both $I, S/I \in \mathcal{R}$, then $S \in \mathcal{R}$.*

Proof Let us consider that (c) holds and that both $I, S/I \in \mathcal{R}$. Then $I \subseteq \mathcal{R}(S)$ by condition (b) and $\phi : S/\mathcal{R}(S) \longmapsto (S/I)/(\mathcal{R}(S)/I)$ is isomorphic, which implies that $S/\mathcal{R}(S) \in \mathcal{R}$. But $0 = \mathcal{R}(S/\mathcal{R}(S)) = S/\mathcal{R}(S)$. Therefore, $S = \mathcal{R}(S)$ is in $\mathcal{R}$ and hence (c') hold.

Conversely, assume that condition (c') holds and that $\mathcal{R}(S/\mathcal{R}(S)) \neq 0$. Now, $\mathcal{R}(S/\mathcal{R}(S)) = K/\mathcal{R}(S)$ for some k-ideal K of S. Since both $\mathcal{R}(S)$ and $K/\mathcal{R}(S)$ are in $\mathcal{R}$, by (c') K is in $\mathcal{R}$. So, $K \subseteq \mathcal{R}(S)$ and $K/\mathcal{R}(S) = 0$, a contradiction. Thus (c) holds. □

The class $\mathcal{R}$ with the condition (c') is said to be closed under extensions.

Proposition 3.3 *Assuming conditions* (a) *and* (c') *on a class* $\mathcal{R}$ *of ternary semirings, condition* (b) *is equivalent to*

(b') *If* $I_1 \subset I_2 \subset \cdots \subset I_\lambda \subset \cdots$ *is an ascending chain of* k*-ideals of a ternary semiring* S *and if each* $I_\lambda \in \mathcal{R}$*, then* $\bigcup I_\lambda \in \mathcal{R}$.

Proof Consider that (b) holds and let $K = \bigcup I_\lambda$. Thus $K = \mathcal{R}(K)$ is in $\mathcal{R}$ and hence (b') holds.

Conversely, suppose that (b') holds. Then by applying the Zorn's lemma, we obtain a maximal (largest) $\mathcal{R}$-k-ideal K of S. If J is any $\mathcal{R}$-k-ideal of S, then $\phi : (K+J)/J \longmapsto K/(K \cap J)$ is isomorphic. Thus both J and $(K+J)/J$ are in $\mathcal{R}$ and by (c'), $K+J$ is in $\mathcal{R}$. Thus $\mathcal{R}(S) = K$ is in $\mathcal{R}$ and hence (b) holds. □

The class $\mathcal{R}$ with the condition (b') is said to has the inductive property.

Theorem 3.4 *A class* $\mathcal{R}$ *of ternary semirings is called radical class if*

(a) $\mathcal{R}$ *is homomorphically closed;*

(b') $\mathcal{R}$ *has the inductive property;*

(c') $\mathcal{R}$ *is closed under extensions.*

Theorem 3.5 *For any class* $\mathcal{R}$ *of ternary semirings, the following conditions are equivalent:*

(I) $\mathcal{R}$ *is radical class;*

(II) $(R1)$ *If* $S \in \mathcal{R}$*, then for every* $S \longmapsto T \neq 0$ *there is a* k*-ideal* I *in* T *such that* $0 \neq I \in \mathcal{R}$*;*

$(R2)$ *If* $S \in \mathbb{U}$ *and for every* $S \longmapsto T \neq 0$ *there is a* k*-ideal* I *in* T *such that* $0 \neq I \in \mathcal{R}$*, then* $S \in \mathcal{R}$*;*

(III) $\mathcal{R}$ *satisfies condition* $(R1)$*, has the inductive property and is closed under extensions.*

Proof $(I) \Longrightarrow (III)$: It is immediate from Theorem 3.4.

$(III) \Longrightarrow (II)$: Let S be a ternary semiring such that for every $S \longmapsto T \neq 0$ there is a k-ideal I in T such that $0 \neq I \in \mathcal{R}$ and that $S \notin \mathcal{R}$. By inductive property and applying Zorn's lemma, we obtain a maximal k-ideal $J \in S$ with respect to being in $\mathcal{R}$. Since $S \notin \mathcal{R}, S/J \neq 0$ holds. Then there exists an k-ideal I/J of S/J such that $0 \neq I/J \in \mathcal{R}$ which implies $I \in \mathcal{R}$. But this contradicts the maximality of J and thus we have $(R2)$ and hence (II).

$(II) \Longrightarrow (I)$: Its immediate from $(R2)$ that $\mathcal{R}$ is homomorphically closed. Let $I_1 \subset I_2 \subset \cdots \subset I_\lambda \subset \cdots$ is an ascending chain of k-ideals of a ternary semiring S such that each $I_\lambda \in \mathcal{R}$. Let $(\bigcup I_\lambda)/J$ be any factor ternary semiring of $\bigcup I_\lambda$. Then there exists an index λ such that $I_\lambda \nsubseteq J$ and thus $0 \neq (I_\lambda + J)/J$ is in $(\bigcup I_\lambda)/J$. Also $(I_\lambda + J)/J$ is isomorphic to $I_\lambda/(I_\lambda \cap J)$ which is in $\mathcal{R}$. Thus, by $(R2)$ we have $\bigcup I_\lambda \in \mathcal{R}$ and that $\mathcal{R}$ has the inductive property. Now, consider J and S/J both in $\mathcal{R}$. Let S/K be any non-zero factor ternary semiring of S. In this case when $J \subseteq K$, $0 \neq (S/K)$ is isomorphic to $(S/J)/(K/J)$ and this is in $\mathcal{R}$. In this case when $J \nsubseteq K$, $0 \neq (J+K)/K$ is in S/K and $(J+K)/K$ is isomorphic to $J/(J \cap K)$ and this is in $\mathcal{R}$. Thus, in both cases S/K has a non-zero k-ideal in $\mathcal{R}$ and by $(R2)$, S itself is in $\mathcal{R}$. Therefore $\mathcal{R}$ is closed under extensions and hence (I). □

Example 3.6 (1) *Nil Radical.* The class

$$\mathcal{N} = \{S \mid \forall\, a \in S\ \exists\, n > 1,\ n \text{ depending on } a,\ \text{such that } a^n = 0\}$$

(i.e. the class of nil ternary semirings) is a radical class called the Nil-radical class.

(2) *Von-Neumann Radical.* A ternary semiring S is said to be Von-Neumann regular if for every $a \in S,\ a = aba,\ \forall b \in S$ or $a \in aSa$. The class

$$\mathcal{V} = \{S \mid S \text{ is Von-Neumann regular}\} = \{a \in S, a = aba,\ \forall\, b \in S\}$$

is a radical class.

§4. Semisimple Class and Upper Radical Class

In this section, the semisimple, hereditary and regular class for ternary semirings are defined on the lines of Kurosh [1]. Also discussed some properties and theorems on semisimple class for ternary semiring.

Definition 4.1 *A class $\mathcal{R}$ of ternary semirings is called hereditary if I is ideal of a ternary semiring S and $S \in \mathcal{R}$, then $I \in \mathcal{R}$.*

Definition 4.2 *A class $\mathcal{R}$ of ternary semirings is called regular if $S \in \mathcal{R}$ and I is non-zero ideal of a ternary semiring S, then there is a non-zero homomorphic image of I in $\mathcal{R}$.*

Remark 4.3 *In particular, every hereditary class is clearly regular.*

Definition 4.4 *A class $\mathcal{S}$ of ternary semirings is called semisimple class if*

$(S1)$ *If $S \in \mathcal{S}$, then every non-zero ideal of S has a non-zero homomorphic image in $\mathcal{S}$*
$(S2)$ *If every non-zero ideal of S has a non-zero homomorphic image in $\mathcal{S}$, then $S \in \mathcal{S}$.*

Proposition 4.5 *If $\mathcal{R}$ is a radical class of ternary semirings, then it admits a semisimple class $\mathbb{S}_{\mathcal{R}} = \{S \in \mathbb{U} : \mathcal{R}(S) = 0\}$.*

Proof Let $S \in \mathbb{S}_{\mathcal{R}}$ and I be any non-zero ideal of S such that I has no non-zero homomorphic image in $\mathbb{S}_{\mathcal{R}}$. As $\mathcal{R}$ is radical class, $\mathcal{R}(I/\mathcal{R}(I)) = 0$ and this implies $I/\mathcal{R}(I) \in \mathbb{S}_{\mathcal{R}}$. Thus $I/\mathcal{R}(I) = 0$ and $I = \mathcal{R}(I) \in \mathcal{R}$. Then $0 \neq I \subseteq \mathcal{R}(S)$, which is contradicting to $\mathcal{R}(S) = 0$ and $(S1)$ holds for $\mathbb{S}_{\mathcal{R}}$. Now, if $S \notin \mathbb{S}_{\mathcal{R}}$ then $\mathcal{R}(S) \neq 0$. Since $\mathcal{R}$ is homomorphically closed, no non-zero homomorphic image of $\mathcal{R}(S)$ is in $\mathbb{S}_{\mathcal{R}}$. Thus, a contrapositive form of $(S2)$ holds for $\mathbb{S}_{\mathcal{R}}$. □

The operator $\mathbb{S}$ is called the semisimple operator.

Theorem 4.6 *If $\mathcal{R}$ is a regular class of ternary semirings then the class*

$$\mathcal{U}_{\mathcal{R}} = \{S : \ S \text{ has no nonzero homomorphic image in } \mathcal{R}\}$$

is a radical class, $\mathcal{R} \cap \mathcal{U}_{\mathcal{R}} = 0$ and $\mathcal{U}_{\mathcal{R}}$ is the largest radical having zero intersection with $\mathcal{R}$.

Proof Let S has a non-zero homomorphic image T such that T has no non-zero ideal in $\mathcal{U}_{\mathcal{R}}$, then $S \notin \mathcal{U}_{\mathcal{R}}$. If such a T exists, then $T \notin \mathcal{U}_{\mathcal{R}}$ and T must have a non-zero homomorphic image V in $\mathcal{R}$ and which is also a non-zero homomorphic image of S in $\mathcal{R}$. Therefore $S \notin \mathcal{U}_{\mathcal{R}}$ and a contrapositive form of $(R1)$ holds for $\mathcal{U}_{\mathcal{R}}$.

Now, assume that $S \notin \mathcal{U}_{\mathcal{R}}$, then S has a non-zero homomorphic image T in $\mathcal{R}$. Since T is regular, every non-zero ideal of T has a non-zero homomorphic image in $\mathcal{R}$. Thus, a contrapositive form of $(R2)$ holds for $\mathcal{U}_{\mathcal{R}}$. Hence, by Theorem 3.5, $\mathcal{U}_{\mathcal{R}}$ is a radical class. □

The operator $\mathcal{U}$ is called the upper radical operator and $\mathcal{U}_{\mathcal{R}}$ is called the upper radical of the class $\mathcal{R}$.

Theorem 4.7 *For any semisimple class $\mathcal{S}$ and radical class $\mathcal{R}$ we have $\mathbb{S}\mathcal{U}_{\mathcal{S}} = \mathcal{S}$ and $\mathcal{U}\mathbb{S}_{\mathcal{R}} = \mathcal{R}$.*

Proof Let $S \in \mathcal{S}$ Then by using $(S1)$ and definition of upper radical we have $S \in \mathbb{S}\mathcal{U}_{\mathcal{S}}$. Also, by using $(S2)$ and definition of upper radical we have $\mathbb{S}\mathcal{U}_{\mathcal{S}} \subseteq \mathcal{S}$. Hence $\mathbb{S}\mathcal{U}_{\mathcal{S}} = \mathcal{S}$.

Similarly, using $(R1)$ and $(R2)$ we have $\mathcal{U}\mathbb{S}_{\mathcal{R}} = \mathcal{R}$. □

Theorem 4.8 *Every semisimple class $\mathcal{S}$ is closed under extensions.*

Proof Let I is a k-ideal of the ternary semiring S such that both $I, S/I \in \mathcal{S}$. Then $(\mathcal{U}_{\mathcal{S}}(S)+I)/I$ is isomorphic to $\mathcal{U}_{\mathcal{S}}(S)/(\mathcal{U}_{\mathcal{S}}(S) \cap I)$ and this is in $\mathcal{U}_{\mathcal{S}}$. Also $(\mathcal{U}_{\mathcal{S}}(S)+I)/I \triangleleft S/I \in \mathcal{S} = \mathbb{S}\mathcal{U}_{\mathcal{S}}$. Thus $(\mathcal{U}_{\mathcal{S}}(S)+I)/I$ must be 0 and so $\mathcal{U}_{\mathcal{S}}(S) \subseteq I$.

Now $\mathcal{U}_{\mathcal{S}}(S) \triangleleft S$, also $\mathcal{U}_{\mathcal{S}}(S) \triangleleft I$. Since $\mathcal{U}_{\mathcal{S}}(S) \in \mathcal{U}_{\mathcal{S}}$, we have $\mathcal{U}_{\mathcal{S}}(S) = \mathcal{U}_{\mathcal{S}}(I) = 0$. Therefore $S \in \mathbb{S}\mathcal{U}_{\mathcal{S}} = \mathcal{S}$ and hence the semisimple class $\mathcal{S}$ is closed under extensions. □

Theorem 4.9 *The classes $\mathcal{R}$ and $\mathcal{S}$ are corresponding radical and semisimple classes if and only if*

- (i) *$S \in \mathcal{R}$ and $S \longrightarrow T \neq 0$ imply $T \notin \mathcal{S}$, that is, $\mathcal{R} \subseteq \mathcal{U}_{\mathcal{S}}$;*
- (ii) *$S \in \mathcal{S}$ and a non-zero k-ideal T of S imply $T \notin \mathcal{R}$, that is, $\mathcal{S} \subseteq \mathbb{S}_{\mathcal{R}}$;*
- (iii) *Every ternary semiring $S \in \mathbb{U}$ has an k-ideal I such that $I \in \mathcal{R}$ and $S/I \in \mathcal{S}$.*

Proof If $\mathcal{R}$ and $\mathcal{S}$ are corresponding radical and semisimple classes, then the if part is obvious (to get (iii) just take $T = \mathcal{R}(S)$). Conversely, suppose that the classes $\mathcal{R}$ and $\mathcal{S}$ satisfying these thrre conditions.

Now, let a ternary semiring $S \in \mathcal{U}_{\mathcal{S}}$. Then by (iii), S has an k-ideal $T \in \mathcal{R}$ such that $S/T \in \mathcal{S}$ and this implies that $S/T = 0$. Thus $S = T \in \mathcal{R}$ holds and proving $\mathcal{U}_{\mathcal{S}} \subseteq \mathcal{R}$. And by using (i) we have $\mathcal{R} = \mathcal{U}_{\mathcal{S}}$. Similarly, we have $\mathcal{S} = \mathbb{S}_{\mathcal{R}}$. Since, $\mathcal{S} = \mathbb{S}_{\mathcal{R}} = \mathbb{S}\mathcal{U}_{\mathcal{S}}$, also $\mathcal{S} \subseteq \mathbb{S}\mathcal{U}_{\mathcal{S}}$ holds and this is the regularity of the class $\mathcal{S}$. Hence $\mathcal{R} = \mathcal{U}_{\mathcal{S}}$ is a radical class and

$\mathcal{S} = \mathbb{S}\mathcal{U}_{\mathcal{S}} = \mathbb{S}_{\mathcal{R}}$ the corresponding semisimple class. □

Proposition 4.10 *A semisimple class $\mathcal{S}$ is hereditary if and only if the corresponding radical class $\mathcal{R} = \mathcal{U}_{\mathcal{S}}$ satisfies*

($S3$) $\mathcal{R}(I) \subseteq \mathcal{R}(S)$*, for every k-ideal I of S.*

Proof Let ($S3$) holds, then for any $S \in \mathcal{S}$ and any k-ideal I of S we have $\mathcal{R}(I) \subseteq \mathcal{R}(S) = 0$. Thus $I \in \mathcal{S}$ and hence $\mathcal{S}$ is hereditary.

Conversely, assume that a semisimple class $\mathcal{S}$ is hereditary. Then for any k-ideal I of S we have $(\mathcal{R}(I) + \mathcal{R}(S))/\mathcal{R}(S) \triangleleft (I + \mathcal{R}(S))/\mathcal{R}(S) \triangleleft S/\mathcal{R}(S) \in \mathcal{S}$. Since $\mathcal{S}$ is hereditary, $(I+\mathcal{R}(S))/\mathcal{R}(S) \in \mathcal{S}$ and $(\mathcal{R}(I)+\mathcal{R}(S))/\mathcal{R}(S) \in \mathcal{S}$. But this implies that $\mathcal{R}(I)/(\mathcal{R}(I) \cap \mathcal{R}(S)) \cong (\mathcal{R}(I) + \mathcal{R}(S))/\mathcal{R}(S) \in \mathcal{R} \cap \mathcal{S} = 0$. Hence $\mathcal{R}(I) \subseteq \mathcal{R}(S)$. □

§5. Hoehnke Radical

With an axiomatic point of view an assignment $\mathcal{R} : S \longrightarrow \mathcal{R}(S)$ designating a certain k-ideal $\mathcal{R}(S)$ to every ternary semiring S is called a Hoehnke radical if:

(i) $f(\mathcal{R}(S)) \subseteq \mathcal{R}(f(S))$, for every homomorphism $f : S \longrightarrow \mathcal{R}(S)$;

(ii) If $S \in \mathbb{U}$, then $S/\mathcal{R}(S)$ is $\mathcal{R}$-semisimple. i.e. $\mathcal{R}(S/\mathcal{R}(S)) = 0$.

A Hoehnke radical $\mathcal{R}$ may satisfy also the following conditions:

(iii) $\mathcal{R}$ is complete: if I is a k-ideal of S and $\mathcal{R}(I) = I$ then $I \subseteq \mathcal{R}(S)$;

(iv) $\mathcal{R}$ is idempotent: if $\mathcal{R}(\mathcal{R}(S)) = \mathcal{R}(S)$, for every ternary semiring S.

Proposition 5.1 *If $\mathcal{R}$ is a radical class then the assignment $\mathcal{R} : S \longrightarrow \mathcal{R}(S)$ is a complete, idempotent Hoehnke radical. Conversely, if $\mathcal{R}$ is a complete, idempotent Hoehnke radical, then there is a radical class $\wp$ such that $\mathcal{R}(S) = \wp(S)$ for every ternary semiring S. Moreover, $\wp = \{S/\mathcal{R}(S) = S\}$.*

Proof (i) and (ii) is obvious. Since $\mathcal{R}(S)$ is a largest $\mathcal{R}$-k-ideal of S. So, for any k-ideal I of S, $\mathcal{R}(I) = I$ implies that $I \subseteq \mathcal{R}(S)$. Also, for every ternary semiring S,

$$\mathcal{R}(\mathcal{R}(S)) = \mathcal{R}(S).$$

This proves (iii) and (iv).

Conversely, suppose that $\mathcal{R}$ is a complete, idempotent Hoehnke radical, and let define the class $\wp$ by $\wp = \{S/\mathcal{R}(S) = S\}$. If $S \in \wp$ and $f : S \longrightarrow T$ is a surjective homomorphism, then by (i),

$$T = f(S) = f(\mathcal{R}(S)) \subseteq \mathcal{R}(f(S)) = \mathcal{R}(T).$$

So $T \in \wp$. Thus (a) holds for $\wp$.

If I is any k-ideal of S and $\mathcal{R}(I) = I$ then $I \subseteq \wp(S)$ and by (iii), $I \subseteq \mathcal{R}(\wp(S))$, therefore $\wp(S) = \mathcal{R}(\wp(S))$ which implies that $\wp(S)$ is a largest $\wp$-k-ideal of S. Thus (b) holds for $\wp$.

Now, $\wp(S) = \mathcal{R}(\wp(S))$ and (iii) implies that $\wp(S) \subseteq \mathcal{R}(S)$. But by (iv), $\mathcal{R}(S) \subseteq \wp(S)$. Thus $\mathcal{R}(S) = \wp(S)$ for every ternary semiring S. Therefore

$$\wp(S/\wp(S)) = \mathcal{R}(S/\mathcal{R}(S)) = 0$$

and (c) holds for $\wp$. □

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Computation of Misbalance Type Degree Indices of Certain Classes of Derived-Regular Graph

V. Lokesha[1], K. Zeba Yasmeen[1], B. Chaluvaraju[2] and T. Deepika[1]

1. Department of Studies in Mathematics, Vijayanagara Sri Krishnadevaraya University, Ballari, India
2. Department of Mathematics, Bangalore University, Jnana Bharathi Campus, Bangalore-560 056, India

E-mail: v.lokesha@gmail.com, zebasif44@gmail.com, bchaluvaraju@gmail.com, sastry.deepi@gmail.com

Abstract: A topological index can be considered as transformation of chemical structure into real number which can be used for correlation with Physical properties in Quantitative Structure Activity Relationship ($QSAR$) and Quantitative Structure Property Relationship ($QSPR$) studies. Adriatic indices are part of topological indices they were scrutinized on the testing sets provided by the International Academy of Mathematical Chemistry ($IAMC$) and it has been shown that they have good predictive properties in many cases. In this article, we compute some Adriatic indices of certain classes of derived-regular graph.

Key Words: Topological indices, line graph, subdivision graph, edge-semi total graph, vertex-semi total graph, Smarandachely vertex-semitotal graph, Smarandachely edge-semitotal graph, total graph, jump graph and para-line graph.

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§1. Introduction

A molecular graph is a graph in which the vertices correspond to the atoms and the edges to the bonds of a molecule. A single number that can be computed from the molecular graph, and used to characterize some property of the underlying molecule is said to be a topological index or molecular structure descriptor. Numerous such descriptors have been considered in theoretical chemistry, and have found application in various areas of chemistry, physics, mathematics, informatics, biology. Recently [17], D. Vukicevic revealed the set of 148 discrete Adriatic indices. They ever analyzed on the testing sets provided by the International Academy of Mathematical Chemistry and it had been shown that they have good predictive properties in many cases.

The graphs considered here are finite, undirected, without loops and multiple edges. Let $G = (V, E)$ be a connected graph with $|V(G)| = n$ vertices and $|E(G)| = m$ edges. The degree d_u of a vertex u is the number of vertices adjacent to u. The edge connecting the vertices u

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and v will be denoted by uv. For other definitions and notations, the reader may refer to [3].

Definitions 1.1 *Let $\alpha_m(G)$ be the misbalance type index where $m \in \{-\frac{1}{2}, \frac{1}{2}, -1, 1\}$, then it is defined as*

$$\alpha_m(G) = \sum_{uv \in E(G)} |d_u^m - d_v^m| . \tag{1}$$

Now,

- The $m = -\frac{1}{2}$ corresponds to misbalnce irdeg index is defined as

$$\alpha_{-\frac{1}{2}}(G) = \sum_{uv \in E(G)} \left| \frac{1}{\sqrt{d_u}} - \frac{1}{\sqrt{d_v}} \right| . \tag{2}$$

- The $m = \frac{1}{2}$ corresponds to misbalnce rodeg index is defined by

$$\alpha_{\frac{1}{2}}(G) = \sum_{uv \in E(G)} \left| \sqrt{d_u} - \sqrt{d_v} \right| . \tag{3}$$

- The $m = -1$ corresponds to misbalnce indeg index is defined to be

$$\alpha_{-1}(G) = \sum_{uv \in E(G)} |d_u - d_v| . \tag{4}$$

- The $m = 1$ corresponds to misbalnce deg index defined by

$$\alpha_1(G) = \sum_{uv \in E(G)} |d_u - d_v| . \tag{5}$$

The misbalance haddeg index-MHD is defined by

$$MHD = \sum_{uv \in E(G)} \left| \frac{1}{2^{d_u}} - \frac{1}{2^{d_u}} \right| . \tag{6}$$

These are the significant predictor of enthalpy of vaporisation and of standard enthalpy of vaporisation for octane isomers for more information the reader can see [17]. In forthcoming sections, we established misbalance degree based adriatic indices of regular and complete bipartite graph using some operators such as line, subdivision, semi-total(vertex and edge) graph, total, jump and para-line graphs.

§2. **Line Graph**

In this section, we established misbalance type degree based adriatic indices of line graph of regular and complete bipartite graph.

The line graph $L(G)$ of a graph G is that graph whose vertices can be put in one-to-one correspondence with the edges of G in such a way that two vertices of $L(G)$ are adjacent whenever the corresponding edges of G are adjacent. For more details, the reader refers to [1].

Corollary 2.1 *Let G be a r- regular graph with $n \geq 2$ vertices. Then, $\alpha_m(L(G)) = 0, \forall m \in \{-\frac{1}{2}, \frac{1}{2}, -1, 1\}$ iff the equality holds for $MHD(L(G))$.*

Corollary 2.2 *Let $K_{r,s}$ be a complete bipartite graph with $1 \leq r \leq s$ vertices. Then, $\alpha_m(L(K_{r,s})) = 0, \forall m \in \{-\frac{1}{2}, \frac{1}{2}, -1, 1\}$ iff the equality holds for $MHD(L(K_{r,s}))$.*

§3. Subdivision Graph

In this section, we established misbalance type degree based adriatic indices of subdivision graph of regular and complete bipartite graph.

The subdivision graph $S(G)$ is the graph obtained from G by replacing each of its edges by a path of length two, or equivalently, by inserting an additional vertex into each edge of G with vertex set $V(G) \cup E(G)$. For more details, refer to [10].

Theorem 3.1 *Let G be a r- regular graph with $n \geq 2$ vertices. Then*

$$\alpha_m(S(G)) = \begin{cases} nr\left|\frac{\sqrt{r}-\sqrt{2}}{\sqrt{2r}}\right| & \text{when } m = -\frac{1}{2}; \\ nr\left|\sqrt{2}-\sqrt{r}\right| & \text{when } m = \frac{1}{2}; \\ n\left|\frac{r-2}{2}\right| & \text{when } m = -1; \\ nr|2-r| & \text{when } m = 1, \end{cases}$$

$$MHD(S(G)) = nr\left|2^{-2} - 2^{-r}\right|.$$

Proof Let G be a r- regular graph with $n \geq 2$ vertices. By algebraic method, the cardinality for vertex and edge set is $n + \frac{nr}{2}$ and nr respectively. The edge set as follows $E_1 = \{uv \in E(S(G)) : d_{S(G)}(u) = 2, d_{S(G)}(v) = r\}$; Then by deploying these cardinalities for the definition of misbalance type degree indices the required results are obtained. □

Theorem 3.2 *Let $K_{r,s}$ be a complete bipartite graph with $1 \leq r \leq s$ vertices. Then*

$$\alpha_m(S(K_{r,s})) = \begin{cases} rs\left[\left|\frac{\sqrt{r}-\sqrt{2}}{\sqrt{2r}}\right| + \left|\frac{\sqrt{s}-\sqrt{2}}{\sqrt{2s}}\right|\right] & \text{when } m = -\frac{1}{2}; \\ rs\left[\left|\sqrt{2}-\sqrt{r}\right| + \left|\sqrt{2}-\sqrt{s}\right|\right] & \text{when } m = \frac{1}{2}; \\ rs\left[\left|\frac{r-2}{2r}\right| + \left|\frac{s-2}{2s}\right|\right] & \text{when } m = -1; \\ rs[|2-r| + |2-s|] & \text{when } m = 1; \end{cases}$$

$$MHD(S(K_{r,s})) = rs\left|2^{-2} - 2^{-r}\right| + \left|2^{-2} - 2^{-s}\right|.$$

Proof Let $K_{r,s}$ be complete bipartite graph with $(r+s)$ vertices. By algebraic method, the cardinality for vertex and edge set is $r + s + rs$ and $2rs$ respectively.

The two partitions of the edge set $E(S(K_{r,s}))$ as follows:

$E_1 = \{uv \in E(S(K_{r,s})) : d_{S(K_{r,s})}(u) = 2, d_{S(K_{r,s})}(v) = r\}$,

$E_2 = \{uv \in E(S(K_{r,s})) : d_{S(K_{r,s})}(u) = 2, d_{S(K_{r,s})}(v) = s\}$.

The cardinality for edge set E_1 and E_2 is rs. Then by deploying these cardinalities for the definition of misbalance type degree indices the required results are obtained. □

§4. Vertex-Semitotal Graph

In this section, we established misbalance type degree based adriatic indices of vertex-semitotal graph of regular and complete bipartite graph.

The vertex-semitotal graph $T_1(G)$ with vertex set $V(G) \cup E(G)$ and edge set $E(S(G)) \cup E(G)$ is the graph obtained from G by adding a new vertex corresponding to each edge of G and by joining each new vertex to the end vertices of the edge corresponding to it. Generally, a Smarandachely vertex-semitotal graph $T^{S1}_{E_1}(G)$ on edge set $E_1 \subset E(G)$ is such a graph with vertex set $V(G) \cup E_1(G)$ and edge set $E_1(S(G)) \cup E(G)$. Clearly, $T^{S1}_{E_1}(G) = T_1(G)$ if $E_1 = E(G)$.

Theorem 4.1 *Let G be a r- regular graph with $n \geq 2$ vertices. Then*

$$\alpha_m(T_1(G)) = \begin{cases} nr\left|\frac{\sqrt{r}-1}{\sqrt{2r}}\right| & when\ m = -\frac{1}{2}; \\ nr\left|\sqrt{2}\,[1-\sqrt{r}]\right| & when\ m = \frac{1}{2}; \\ n\left|\frac{r-1}{r}\right| & when\ m = -1; \\ nr|2[1-r]| & when\ m = 1, \end{cases}$$

$$MHD(T_1(G)) = nr|2^{-2} - 2^{-2r}|.$$

Proof Let G be a r- regular graph with $n \geq 2$ vertices. By algebraic method, the cardinality for vertex and edge set is $\frac{nr}{2} + n$ and $\frac{3nr}{2}$ respectively.

The two partitions of the edge set $E(T_1(G))$ as follows:

$E_1 = \{uv \in E(T_1(G)) : d_{T_1(G)}(u) = 2, d_{T_1(G)}(v) = 2r\}$,
$E_2 = \{uv \in E(T_1(G)) : d_{T_1(G)}(u) = d_{T_1(G)}(v) = 2r\}$.

The cardinalities of edge sets E_1, E_2 are nr, $\frac{nr}{2}$, respectively. Then, by deploying these cardinalities for the definition of misbalance type indices the required results are obtained. □

Theorem 4.2 *Let $K_{r,s}$ be a complete bipartite graph with $1 \leq r \leq s$ vertices. Then*

$$\alpha_m(T_1(K_{r,s})) = \begin{cases} rs\left[\left|\frac{1-\sqrt{r}}{\sqrt{2r}}\right| + \left|\frac{1-\sqrt{s}}{\sqrt{2s}}\right|\right] & when\ m = -\frac{1}{2}; \\ rs\left[\left|\sqrt{2}\,[\sqrt{r}-1]\right| + \left|\sqrt{2}\,[\sqrt{s}-1]\right|\right] & when\ m = \frac{1}{2}; \\ rs\left[\left|\frac{1-r}{2r}\right| + \left|\frac{1-s}{2s}\right|\right] & when\ m = -1 \\ rs\left[|2[r-1]| + |2[s-1]|\right] & when\ m = 1, \end{cases}$$

$$MHD(T_1(K_{r,s})) = rs\left[\left|2^{-2r} - 2^{-2}\right| + \left|2^{-2s} - 2^{-2}\right|\right].$$

Proof If $K_{r,s}$ is a complete bipartite graph with $(r+s)$ - vertices and rs - edges, the cardinality for vertex and edge set is $r+s+rs$ and $3rs$ respectively.

The three partitions of the edge set $E(T_1(K_{r,s}))$ as follows:

$$E_1 = \{uv \in E(T_1(K_{r,s})) : d_{T_1(K_{r,s})}(u) = 2r, d_{T_1(K_{r,s})}(v) = 2\},$$
$$E_2 = \{uv \in E(T_1(K_{r,s})) : d_{T_1(K_{r,s})}(u) = 2s, d_{T_1(K_{r,s})}(v) = 2\},$$
$$E_3 = \{uv \in E(T_1(K_{r,s})) : d_{T_1(K_{r,s})}(u) = 2r, d_{T_1(K_{r,s})}(v) = 2s\}.$$

The cardinalities of edge sets E_1, E_2 and E_3 are rs. Then by deploying these cardinalities for the definition of misbalance type indices obtained the required results. □

By above result with $r = s$, the complete regular bipartite graph $K_{r,r}$ with $r > 2$.

§5. Edge-Semitotal Graph

In this section, misbalance type degree based adriatic indices of edge-semitotal graph of regular and complete bipartite graph are studied.

An edge-semitotal graph $T_2(G)$ with vertex set $V(G) \cup E(G)$ and edge set $E(S(G)) \cup E(L(G))$ is the graph obtained from G by inserting a new vertex into each edge of G and by joining with edges those pairs of these new vertices which lie on adjacent edges of G. Generally, a Smarandachely edge-semitotal graph $T^{S2}_{E_1}(G)$ on edge set $E_1 \subset E(G)$ is such a graph with vertex set $V(G) \cup E_1(G)$ and edge set $E_1(S(G)) \cup E\left(E_1 \cap L(G)\right)$. Clearly, $T^{S2}_{E_1}(G) = T_2(G)$ if $E_1 = E(G)$.

Theorem 5.1 *Let G be a r- regular graph with $n \geq 2$ vertices. Then*

$$\alpha_m(T_2(G)) = \begin{cases} nr\left|\frac{\sqrt{2}-1}{\sqrt{2r}}\right| & when\ m = \frac{-1}{2}; \\ nr\left|\sqrt{r}\left[1-\sqrt{2}\right]\right| & when\ m = \frac{1}{2}; \\ \frac{n}{2} & when\ m = -1; \\ r^2 & when\ m = 1, \end{cases}$$

$$MHD(T_2(G)) = r\left|2^{-r} - 2^{-2r}\right|.$$

Proof Let G be a r- regular graph with $n \geq 2$ vertices. By algebraic method, the cardinality for vertex and edge set is $\frac{nr}{2} + n$ and $\frac{nr}{2}(r+1)$ respectively.

The two partitions of the edge set $E(T_2(H))$ as follows:

$$E_1 = \{uv \in E(T_2(G)) : d_{T_2(G)}(u) = r, d_{T_2(G)}(v) = 2r\},$$
$$E_2 = \{uv \in E(T_2(G)) : d_{T_2(G)}(u) = d_{T_2(G)}(v) = 2r\}.$$

Then, the cardinalities of edge sets E_1 and E_2 are rn and $\dfrac{rn}{2}(r-1)$ respectively. By deploying these cardinalities for the definition of misbalance type indices, obtained the required results. □

Theorem 5.2 *Let $K_{r,s}$ be a complete bipartite graph with $1 \leq r \leq s$ vertices. Then*

$$\alpha_m(T_2(K_{r,s})) = \begin{cases} rs\left[\left|\frac{\sqrt{r+s}-\sqrt{r}}{\sqrt{r}(\sqrt{r+s})}\right| + \left|\frac{\sqrt{r+s}-\sqrt{s}}{\sqrt{s}(\sqrt{r+s})}\right|\right] & when\ m = \frac{-1}{2}; \\ rs\left[\left|\sqrt{r}-\sqrt{r+s}\right| + \left|\sqrt{s}-\sqrt{r+s}\right|\right] & when\ m = \frac{1}{2}; \\ rs\left[\left|\frac{s}{r(r+s)}\right| + \left|\frac{r}{s(r+s)}\right|\right] & when\ m = -1; \\ rs[s+r] \quad when\ m = 1, \end{cases}$$

$$MHD(T_2(K_{r,s})) = rs\left[\left|2^{-r} - 2^{-(r+s)}\right| + \left|2^{-s} - 2^{-(r+s)}\right|\right].$$

Proof Let $K_{r,s}$ be complete bipartite graph with $(r+s)$ vertices. By algebraic method, the cardinality for vertex and edge set is $r+s+rs$ and $sr[1+\frac{1}{2}(r+s)]$ respectively.

The three partitions of the edge set $E(T_2(K_{r,s}))$ as follows:

$$E_1 = \{uv \in E(T_2(K_{r,s})) : d_{T_2(K_{r,s})}(u) = r, d_{T_2(K_{r,s})}(v) = r+s\},$$
$$E_2 = \{uv \in E(T_2(K_{r,s})) : d_{T_2(K_{r,s})}(u) = s, d_{T_2(K_{r,s})}(v) = r+s\},$$
$$E_3 = \{uv \in E(T_2(K_{r,s})) : d_{T_2(K_{r,s})}(u) = d_{T_2(K_{r,s})}(v) = r+s\}.$$

The cardinalities of edge sets E_1 and E_2 are rs and the cardinality for edge set E_3 is $\dfrac{1}{2}rs\ [r+s-2]$. Then by utilizing these cardinalities for the definition of misbalance type indices, obtained the required results. □

By above result with $r = s$, the complete regular bipartite graph $K_{r,r}$ with $r > 1$.

§6. Total Graph

In this section, the misbalance type degree based adriatic indices of total graph of regular and complete bipartite graph are reckoned.

The total graph of a graph G is denoted by $T(G)$ with vertex set $V(G) \cup E(G)$ and any two vertices of $T(G)$ are adjacent if and only if they are either incident or adjacent in G. For more details, refer to [1].

Corollary 6.1 *Let G be a r- regular graph with $n \geq 2$ vertices. Then $\alpha_m(T(G)) = 0, \forall m \in \{-\frac{1}{2}, \frac{1}{2}, -1, 1\}$ iff the equality holds for $MHD(T(G))$.*

Theorem 6.2 *Let $K_{r,s}$ be a complete bipartite graph with $1 \leq r \leq s$ vertices. Then*

$$\alpha_m(T(K_{r,s}) = \begin{cases} rs\left[\left|\frac{\sqrt{2r}-\sqrt{2s}}{2\sqrt{rs}}\right| + \left|\frac{\sqrt{r+s}-\sqrt{2s}}{\sqrt{2s(r+s)}}\right| + \left|\frac{\sqrt{r+s}-\sqrt{2r}}{\sqrt{2r(r+s)}}\right|\right] & when\ m = -\frac{1}{2}; \\ rs\left[\left|\sqrt{2}\left[\sqrt{s}-\sqrt{r}\right]\right| + \left|\sqrt{2s}-\sqrt{r+s}\right| + \left|\sqrt{2r}-\sqrt{r+s}\right|\right] & when\ m = \frac{1}{2}; \\ rs\left[\left|\frac{r-2}{sr}\right| + \left|\frac{r-s}{2s(r+s)}\right| + \left|\frac{s-r}{2r(r+s)}\right|\right] & when\ m = -1; \\ rs[|2(s-r)| + |s-r| + |r-s|] \quad when\ m = 1, \end{cases}$$

$$MHD(T(K_{r,s}) \quad = \quad rs\left[\left|2^{-2}-2^{-2r}\right|+\left|2^{-2s}-2^{-(r+s)}\right|+\left|2^{-2r}-2^{-(r+s)}\right|\right].$$

Proof Let $K_{r,s}$ be complete bipartite graph with $(r+s)$ vertices. By algebraic method, the cardinality for vertex and edge set is $r+s+rs$ and $\frac{1}{2}rs(r+s-2)+3rs$ respectively. The four partitions of the edge set $E(T(K_{r,s}))$ as follows:

$E_1 = \{uv \in E(T(K_{r,s})) : d_{T(K_{r,s})}(u) = 2s, d_G(v) = 2r\}$,
$E_2 = \{uv \in E(T(K_{r,s})) : d_{T(K_{r,s})}(u) = 2s, d_{T(K_{r,s})}(v) = r+s\}$,
$E_3 = \{uv \in E(T(K_{r,s})) : d_{T(K_{r,s})}(u) = 2r, d_{T(K_{r,s})}(v) = r+s\}$,
$E_4 = \{uv \in E(T(K_{r,s})) : d_{T(K_{r,s})}(u) = d_{T(K_{r,s})}(v) = r+s\}$.

Then, the cardinalities of edge sets E_1, E_2 and E_3 are rs and the cardinality of edge set E_4 is $\frac{1}{2}rs\ (r+s-2)$. Then by deploying these cardinalities for the definition of misbalance type indices, obtained the required results. □

By above result with $r = s$, the complete regular bipartite graph $K_{r,r}$ with $r > 2$.

§7. Jump Graph

In this section, the misbalance type degree based adriatic indices of jump graph of regular and complete bipartite graph are studied.

The jump graph $J(G)$ of a graph G defined on $E(G)$ and in which two vertices are adjacent if and only if they are not adjacent in G.

Corollary 7.1 *Let G be a r- regular graph with $n \geq 2$ vertices. Then $\alpha_m(J(G)) = 0, \forall m \in \{-\frac{1}{2}, \frac{1}{2}, -1, 1\}$ iff the equality holds for $MHD(J(G))$.*

Corollary 7.2 *Let $K_{r,s}$ be a complete bipartite graph with $1 \leq r \leq s$ vertices. Then $\alpha_m(J(K_{r,s})) = 0,\ \forall m \in \{-\frac{1}{2}, \frac{1}{2}, -1, 1\}$ iff the equality holds for $MHD(J(K_{r,s}))$.*

§8. Para-Line Graph

In this section, the misbalance type degree based adriatic indices of para-line graph of regular and complete bipartite graph are reckoned.

Given a graph G, insert two vertices to each edge xy of G. Those two vertices will be denoted by (x,y), (y,x) where (x,y) $(resp.(y,x))$ is the one incident to $x(resp.$y). The vertex set and the edge set as follows:

$V(P(G)) = (x,y) \in V(G) \times V(G); xy \in E(G)$,
$E(P(G)) = (((x,w),(x,z)); (x,w),(x,z) \in V(P(G)), w \neq z) \cup ((x,y),(y,x); xy \in E(G))$.

The resultant graph is called a para-line graph and denoted by $P(G)$.

Corollary 8.1 *Let G be a r- regular graph with $n \geq 2$ vertices. Then $\alpha_m(P(G)) = 0, \forall m \in \{-\frac{1}{2}, \frac{1}{2}, -1, 1\}$ iff the equality holds for $MHD(P(G))$.*

Theorem 8.2 *Let $K_{r,s}$ be a complete bipartite graph with $1 \leq r \leq s$ vertices. Then,*

$$\alpha_m(P(K_{r,s})) = \begin{cases} rs\left|\frac{\sqrt{}-\sqrt{r}}{\sqrt{rs}}\right| & when\ m = -\frac{1}{2}; \\ |\sqrt{r} - \sqrt{s}| & when\ m = \frac{1}{2}; \\ |s - r| & when\ m = -1 \\ rs|r - s| & when\ m = 1, \end{cases}$$

$$MHD(P(K_{r,s})) = rs\left|2^{-r} - 2^{-s}\right|$$

Proof Let $K_{r,s}$ be complete bipartite graph with $(r+s)$ vertices. By algebraic method, the cardinality for vertex and edge set is $2rs$ and $\frac{rs(r+s)}{2}$ respectively. The three partitions of the edge set $E(P(K_{r,s}))$ as follows:

$E_1 = \{uv \in E(P(K_{r,s})) : d_{P(K_{r,s})}(u) = r, d_G(v) = s\}$,
$E_2 = \{uv \in E(P(K_{r,s})) : d_{P(K_{r,s})}(u) = s, d_{P(K_{r,s})}(v) = r\}$,
$E_3 = \{uv \in E(P(K_{r,s})) : d_{P(K_{r,s})}(u) = d_{P(K_{r,s})}(v) = s\}$.

Then the cardinalities of edge sets E_1, E_2 and E_3 are rs, $rs(\frac{r-1}{2})$ and $rs(\frac{s-1}{2})$, respectively. By deploying these cardinalities for the definition of misbalance type indices, the required results are obtained. □

§9. Conclusion

In this paper we established misbalance degree based adriatic indices of regular and complete bipartite graph using some operators such as line, subdivision, semi total (vertex and edge) graph, total, jump and para-line graphs. In future we will pay attention to some new classes of operations on graphs and study their adriatic indices which will be practically helpful to identify underlying topologies.

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Reciprocal Transmission Hosoya Polynomial of Graphs

Harishchandra S. Ramane and Saroja Y. Talwar

(Department of Mathematics, Karnatak University, Dharwad - 580003, India)

E-mail: hsramane@yahoo.com, sarojaytalwar@gmail.com

Abstract: In this paper we define reciprocal transmission Hosoya polynomial of graphs and obtain general formula for some graphs. Also calculate reciprocal transmission Hosoya polynomial of cluster graphs and of reciprocal transmission distance balanced graphs.

Key Words: Distance, reciprocal transmission of a vertex, reciprocal transmission distance balanced graphs.

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§1. Introduction

The concept of counting polynomial was first introduced in chemistry by Polya [5] in 1936. However the subject received attention from chemists for several decades even though the spectra of the characteristic polynomial of graphs were studied extensively by numerical means in order to obtain the molecular orbitals of unsaturated hydrocarbons.

The Hosoya polynomial of a graph was introduced in the Hosoya's seminal paper [4] in 1988 and received a lot of attention afterwards. The polynomial was later independently introduced and considered by Sagan, Yeh and Zhang [7] under the name Wiener polynomial of a graph. Both names are still used for the polynomial but the term Hosoya polynomial is nowadays used by majority of researchers. The main advantage of the Hosoya polynomial is that it contains a wealth of information about distance based graph invarients. For instance, knowing the Hosoya polynomial of graph, it is straightforward to determine the Wiener index of a graph as the first derivative of the polynomial at variable $x = 1$. Cash [1] noticed that the hyper-Wiener index can be obtained from the Hosoya polynomial in a similar simple manner. Also, Estrada et al. [2] studied several chemical applications of the Hosoya polynomial.

Let G be a connected graph on n vertices with vertex set $V(G)$ and edge set $E(G)$. If $d(G,k)$ is the number of unordered pairs of its vertices that are at distance k, then the Hosoya polynomial is defined as

$$H(G,x) = \sum_{k\geq 0} d(G,k)x^k. \tag{1}$$

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The reciprocal transmission (status) of a vertex u of a graph G is defined as [6]

$$rs(u) = \sum_{v \in V(G), u \neq v} \frac{1}{d(u,v)}.$$

The first reciprocal transmission (status) connectivity index of a graph G is defined as [**?**]

$$RS_1(G) = \sum_{uv \in E(G)} \left[rs(u) + rs(v) \right].$$

The reciprocal transmission Hosoya polynomial of a graph G is defined as

$$H_{rs}(G,x) = \sum_{uv \in E(G)} x^{rs(u)+rs(v)}. \tag{2}$$

where $rs(u)$ is the reciprocal transmission of a vertex u.

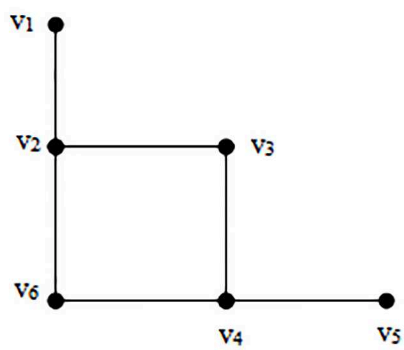

Figure 1

For a graph given in Figure 1, $rs(v_1) = 2.58$, $rs(v_2) = 3.83$, $rs(v_3) = 3.5$, $rs(v_4) = 3.83$, $rs(v_5) = 2.58$ and $rs(v_6) = 3.5$. Therefore

$$H_{rs}(G,x) = 2x^{6.41} + 4x^{7.33}.$$

§2. Reciprocal Transmission Hosoya Polynomial of Some Class of Graphs

Proposition 2.1 *Let G be a connected graph with n vertices and m edges. Let $diam(G) \leq 2$ and $d(u)$ be the degree of a vertex u in G. Then*

$$H_{rs}(G,x) = x^{n-1} \sum_{uv \in E(G)} x^{\frac{1}{2}(d(u)+d(v))}. \tag{3}$$

Proof If $diam(G) \leq 2$, then $d(u)$ number of vertices are at distance 1 from the vertex u and the remaining $n-1-d(u)$ vertices are at distance 2. Hence $rs(u) = d(u) + \frac{1}{2}\left(n-1-d(u)\right)$.

Therefore,

$$rs(u)+rs(v)=(n-1)+\frac{1}{2}(d(u)+d(v)).$$

Hence, from Eq.(2) we get

$$\begin{aligned} H_{rs}(G,x) &= \sum_{uv\in E(G)} x^{rs(u)+rs(v)} \\ &= \sum_{uv\in E(G)} x^{\left((n-1)+\frac{1}{2}(d(u)+d(v))\right)} = x^{(n-1)} \sum_{uv\in E(G)} x^{\frac{1}{2}(d(u)+d(v))}. \end{aligned} \qquad \square$$

Proposition 2.2 *Let G be a connected graph on n vertices and m edges. Then the first reciprocal transmission connectivity index $RS_1(G)=\frac{d}{dx}H_{rs}(G,x)|_{x=1}$.*

Corollary 2.3 *Let G be a connected r-regular graph on n vertices and m edges. Let $diam(G)\le 2$. Then*

$$H_{rs}(G,x)=mx^{r+n-1}. \tag{4}$$

Proof Since degree of each vertex is r, then by Proposition 2.1 we have,

$$H_{rs}(G,x)=x^{n-1}\sum_{uv\in E(G)} x^r = mx^{r+n-1}. \qquad \square$$

Corollary 2.4 *For a complete bipartite graph $K_{p,q}$ on $n=p+q$ vertices,*

$$H_{rs}(K_{p,q},x)=pqx^{\frac{3}{2}(p+q)-1}. \tag{5}$$

Proof The graph $K_{p,q}$ has $n=p+q$ vertices and $m=pq$ edges. Also $diam(K_{p,q})\le 2$. The vertex set $V(K_{p,q})$ can be partitioned into two sets V_1 and V_2 such that for every edge uv of $K_{p,q}$, the vertex $u\in V_1$ and $v\in V_2$, where $|V_1|=p$ and $|V_2|=q$. Therefore $d(u)=q$ and $d(v)=p$. Therefore, by Proposition 2.1 we have

$$\begin{aligned} H_{rs}(K_{p,q},x) &= x^{n-1}\sum_{uv\in E(K_{p,q})} x^{\frac{1}{2}(d(u)+d(v))} \\ &= x^{p+q-1}\sum_{uv\in E(K_{p,q})} x^{\frac{1}{2}(p+q)} = pqx^{\frac{3}{2}(p+q)-1}. \end{aligned} \qquad \square$$

Proposition 2.5 *For a cycle C_n on $n\ge 3$ vertices,*

$$H_{rs}(C_n,x)=\begin{cases} nx^{4\left(\frac{1}{n}+\sum_{i=1}^{\frac{n-2}{2}}\frac{1}{i}\right)}, & \textit{if } n \textit{ is even} \\ nx^{4\sum_{i=1}^{\frac{n-1}{2}}\frac{1}{i}}, & \textit{if } n \textit{ is odd.} \end{cases} \tag{6}$$

Proof If n is even number, then for every vertex u of C_n,

$$rs(u) = \frac{2}{n} + 2\sum_{i=1}^{\frac{n-2}{2}} \frac{1}{i}.$$

Therefore, from Eq.(2) we have

$$\begin{aligned} H_{rs}(C_n, x) &= \sum_{uv \in E(C_n)} x^{rs(u)+rs(v)} \\ &= \sum_{uv \in E(C_n)} x^{4\left(\frac{1}{n}+\sum_{i=1}^{\frac{n-2}{2}} \frac{1}{i}\right)} = nx^{4\left(\frac{1}{n}+\sum_{i=1}^{\frac{n-2}{2}} \frac{1}{i}\right)}. \end{aligned}$$

If n is odd number, then for every vertex u of C_n,

$$rs(u) = 2\sum_{i=1}^{\frac{n-1}{2}} \frac{1}{i}.$$

Therefore from Eq.(2) we have

$$\begin{aligned} H_{rs}(C_n, x) &= \sum_{uv \in E(C_n)} x^{rs(u)+rs(v)} \\ &= \sum_{uv \in E(C_n)} x^{4\sum_{i=1}^{\frac{n-1}{2}} \frac{1}{i}} = nx^{4\sum_{i=1}^{\frac{n-1}{2}} \frac{1}{i}}. \end{aligned}$$ □

Proposition 2.6 *For a wheel W_{n+1}, $n \geq 3$,*

$$H_{rs}(W_{n+1}, x) = n\left[x^{\frac{3}{2}(n+1)} + x^{n+3}\right]. \tag{7}$$

Proof A wheel graph W_{n+1} has $n+1$ vertices and $2n$ edges. Also $diam(W_{n+1}) \leq 2$. The edge set $E(W_{n+1})$ can be partitioned into two sets E_1, E_2, such that $E_1 = \{uv \mid d(u) = n \text{ and } d(v) = 3\}$ and $E_2 = \{uv \mid d(u) = 3 \text{ and } d(v) = 3\}$. It is easy to check that $|E_1| = n$ and $|E_2| = n$ and $diam(W_{n+1}) \leq 2$. Therefore from Proposition 2.1 we get

$$\begin{aligned} H_{rs}(W_{n+1}, x) &= x^{n+1-1} \sum_{uv \in E(W_{n+1})} x^{\frac{1}{2}(d(u)+d(v))} \\ &= x^n \left[\sum_{uv \in E_1(W_{n+1})} x^{\frac{1}{2}(d(u)+d(v))} + \sum_{uv \in E_2(W_{n+1})} x^{\frac{1}{2}(d(u)+d(v))}\right] \\ &= x^n \left[nx^{\frac{1}{2}(n+3)} + nx^{\frac{1}{2}(3+3)}\right] \\ &= x^n n \left[x^{\frac{1}{2}(n+3)} + x^3\right] \\ &= n\left[x^{\frac{3}{2}(n+1)} + x^{n+3}\right]. \end{aligned}$$ □

Proposition 2.7 *For a friendship graph F_n, $n \geq 2$,*

$$H_{rs}(F_n, x) = n\left[2x^{3n+1} + x^{2(n+1)}\right]. \tag{8}$$

Proof The edge set $E(F_n)$ can be partitioned into two sets E_1 and E_2, such that $E_1 = \{uv \mid d(u) = 2n \quad \text{and} \quad d(v) = 2\}$ and $E_2 = \{uv \mid d(u) = 2 \text{ and } d(v) = 2\}$. It is easy to check that $|E_1| = 2n$ and $|E_2| = n$ and $diam(F_n) = 2$. Therefore by Proposition 2.1, we have

$$\begin{aligned} H_{rs}(F_n, x) &= x^{2n+1-1} \sum_{uv \in E(F_n)} x^{\frac{1}{2}(d(u)+d(v))} \\ &= x^{2n}\left[\sum_{uv \in E_1(F_n)} x^{\frac{1}{2}(d(u)+d(v))} + \sum_{uv \in E_2(F_n)} x^{\frac{1}{2}(d(u)+d(v))}\right] \\ &= x^{2n}\left[\sum_{uv \in E_1(F_n)} x^{\frac{1}{2}(2n+2)} + \sum_{uv \in E_2(F_n)} x^{\frac{1}{2}(2+2)}\right] \\ &= x^{2n}\left[2nx^{n+1} + nx^2\right] \\ &= n\left[2x^{3n+1} + x^{2(n+1)}\right]. \end{aligned}$$

□

§3. Reciprocal Transmission Hosoya Polynomial of Cluster Graphs

Graphs with large number of edges are referred as cluster graphs [3].

Definition 3.1([3]) *Let e_i, $i = 1, 2, \cdots, k$, $1 \leq k \leq n-2$, be the distinct edges of a complete graph K_n, $n \geq 3$, all being incident to a single vertex. The graph $Ka_n(k)$ is obtained by deleting $e_i, i = 1, 2, \cdots, k$ from K_n. In addition $Ka_n(0) \cong K_n$.*

Definition 3.2([3]) *Let $f_i, i = 1, 2, \cdots, k, 1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ be independent edges of the complete graph $K_n, n \geq 3$. The graph $Kb_n(k)$ is obtained by deleting f_i, $i = 1, 2, \cdots, k$ from K_n. In addition $Kb_n(0) \cong K_n$.*

Definition 3.3([3]) *Let V_k be a k-element subset of the vertex set of the complete graph K_n, $2 \leq k \leq n-1, n \geq 3$. The graph $Kc_n(k)$ is obtained by deleting from K_n all the edges connecting pairs of vertices from V_k. In addition $Kc_n(0) \cong Kc_n(1) \cong K_n$.*

Definition 3.4([3]) *Let $3 \leq k \leq n, n \geq 3$. The graph $Kd_n(k)$ is obtained by deleting from the complete graph K_n, the edges belonging to a k-membered cycle.*

Proposition 3.5 *For $n \geq 3$ and $1 \leq k \leq n-2$,*

$$\begin{aligned} H_{rs}(Ka_n(k), x) &= x^{n-1}\left[(n-k-1)x^{\frac{1}{2}(2n-k-2)} + \frac{k(k-1)}{2}x^{n-2}\right. \\ &\quad \left. + (n-k-1)kx^{\frac{1}{2}(2n-3)} + \frac{(n-k-1)(n-k-2)}{2}x^{n-1}\right]. \end{aligned}$$

Proof The graph $Ka_n(k)$ has n vertices, $\left(\frac{n(n-1)}{2} - k\right)$ edges. The edge set $E(Ka_n(k))$ can be partitioned into four sets E_1, E_2, E_3 and E_4, where $E_1 = \{uv \mid d(u) = n-1-k \text{ and } d(v) = n-1\}$, $E_2 = \{uv \mid d(u) = n-2 \text{ and } d(v) = n-2\}$, $E_3 = \{uv \mid d(u) = n-2 \text{ and } d(v) = n-1\}$ and $E_4 = \{uv \mid d(u) = n-1 \text{ and } d(v) = n-1\}$. It is easy to check that $|E_1| = n-k-1$, $|E_2| = \frac{k(k-1)}{2}$, $|E_3| = (n-k-1)k$ and $|E_4| = \frac{(n-k-1)(n-k-2)}{2}$. Also $diam(Ka_n(k)) \leq 2$. Therefore, from Proposition 2.1 we have

$$\begin{aligned}
H_{rs}(Ka_n(k), x) &= x^{n-1} \sum_{uv \in E(Ka_n(k))} x^{\frac{1}{2}(d(u)+d(v))} \\
&= x^{n-1} \Bigg[\sum_{uv \in E_1(Ka_n(k))} x^{\frac{1}{2}(d(u)+d(v))} + \sum_{uv \in E_2(Ka_n(k))} x^{\frac{1}{2}(d(u)+d(v))} \\
&\quad + \sum_{uv \in E_3(Ka_n(k))} x^{\frac{1}{2}(d(u)+d(v))} + \sum_{uv \in E_4(Ka_n(k))} x^{\frac{1}{2}(d(u)+d(v))} \Bigg] \\
&= x^{n-1} \Bigg[\sum_{uv \in E_1(Ka_n(k))} x^{\frac{1}{2}(2n-k-2)} + \sum_{uv \in E_2(Ka_n(k))} x^{n-2} \\
&\quad + \sum_{uv \in E_3(Ka_n(k))} x^{\frac{1}{2}(2n-3)} + \sum_{uv \in E_4(Ka_n(k))} x^{n-1} \Bigg] \\
&= x^{n-1} \Bigg[(n-k-1)x^{\frac{1}{2}(2n-k-2)} + \frac{k(k-1)}{2} x^{n-2} \\
&\quad + (n-k-1)k x^{\frac{1}{2}(2n-3)} + \frac{(n-k-1)(n-k-2)}{2} x^{n-1} \Bigg].
\end{aligned}$$

$\square$

Proposition 3.6 *For $n \geq 3$ and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$,*

$$\begin{aligned}
H_{rs}(Kb_n(k), x) &= x^{n-1} \Bigg[2k(n-2k)x^{\frac{1}{2}(2n-3)} + \frac{(n-2k)(n-2k-1)}{2} x^{n-1} \\
&\quad + \left(\frac{2k(2k-1)}{2} - k \right) x^{n-2} \Bigg].
\end{aligned}$$

Proof The graph $Kb_n(k)$ has n vertices and $\left(\frac{n(n-1)}{2} - k\right)$ edges and $diam(Kb_n(k)) = 2$. The edge set $E(Kb_n(k))$ can be partitioned into three sets E_1, E_2 and E_3, where $E_1 = \{uv \mid d(u) = n-2 \text{ and } d(v) = n-1\}$, $E_2 = \{uv \mid d(u) = n-1 \text{ and } d(v) = n-1\}$ and $E_3 = \{uv \mid d(u) = n-2 \text{ and } d(v) = n-2\}$. It is easy to check that $|E_1| = 2k(n-2k)$, $|E_2| = (n-2k)(n-2k-1)/2$ and $|E_3| = (2k(2k-1)/2) - k$.

Therefore, from Proposition 2.1 we have

$$\begin{aligned}
H_{rs}(Kb_n(k),x) &= x^{n-1}\sum_{uv\in E(Kb_n(k))} x^{\frac{1}{2}(d(u)+d(v))}\\
&= x^{n-1}\left[\sum_{uv\in E_1(Kb_n(k))} x^{\frac{1}{2}(d(u)+d(v))}+\sum_{uv\in E_2(Kb_n(k))} x^{\frac{1}{2}(d(u)+d(v))}\right.\\
&\quad\left.+\sum_{uv\in E_3(Kb_n(k))} x^{\frac{1}{2}(d(u)+d(v))}\right]\\
&= x^{n-1}\left[\sum_{uv\in E_1(Kb_n(k))} x^{\frac{1}{2}(2n-3)}+\sum_{uv\in E_2(Kb_n(k))} x^{n-1}\right.\\
&\quad\left.+\sum_{uv\in E_3(Kb_n(k))} x^{(n-2)}\right]\\
&= x^{n-1}\left[2k(n-2k)x^{\frac{1}{2}(2n-3)}+\frac{(n-2k)(n-2k-1)}{2}x^{n-1}\right.\\
&\quad\left.+\left(\frac{2k(2k-1)}{2}-k\right)x^{n-2}\right].
\end{aligned}$$

□

Proposition 3.7 *For $n\geq 3$ and $2\leq k\leq n-1$,*

$$H_{rs}(Kc_n(k),x) = x^{n-1}\left[(n-k)kx^{\frac{1}{2}(2n-k-1)}+\frac{(n-k)(n-k-1)}{2}x^{n-1}\right].$$

Proof The graph $Kc_n(k)$ has n vertices and $\frac{1}{2}(n-k)(n+k-1)$ edges. Also $diam(Kc_n(k)) = 2$. The edge set $E(Kc_n(k))$ ca be partitioned into two sets E_1 and E_2, where $E_1=\{uv \mid d(u) = n-k \text{ and } d(v)=n-1\}$ and $E_2=\{uv \mid d(u)=n-1 \text{ and } d(v)=n-1\}$. It is easy to check that $|E_1|=(n-k)k$ and $|E_2|=(n-k)(n-k-1)/2$. Therefore, from Proposition 2.1 we have

$$\begin{aligned}
H_{rs}(Kc_n(k),x) &= x^{n-1}\sum_{uv\in E(Kc_n(k))} x^{\frac{1}{2}(d(u)+d(v))}\\
&= x^{n-1}\left[\sum_{uv\in E_1(Kc_n(k))} x^{\frac{1}{2}(d(u)+d(v))}+\sum_{uv\in E_2(Kc_n(k))} x^{\frac{1}{2}(d(u)+d(v))}\right]\\
&= x^{n-1}\left[\sum_{uv\in E_1(Kc_n(k))} x^{\frac{2}{2}(n-k-+n-1)}+\sum_{uv\in E_2(Kc_n(k))} x^{\frac{1}{2}(2(n-1))}\right]\\
&= x^{n-1}\left[(n-k)kx^{\frac{1}{2}(2n-k-1)}+\frac{(n-k)(n-k-1)}{2}x^{n-1}\right].
\end{aligned}$$

□

Proposition 3.8 *For $3 \le k \le n$ and $n \ge 5$,*

$$\begin{aligned} H_{rs}(Kd_n(k),x) &= x^{n-1}\left[\left((k(k-1)/2)-k\right)x^{n-3}+(n-k)kx^{n-2}\right. \\ &\quad \left.+\left((n-k)(n-k-1)/2\right)x^{n-1}\right]. \end{aligned}$$

Proof The graph $Kd_n(k)$ has n vertices and $n(n-1)/2-k$ edges. Also $diam(Kd_n(k)) = 2$. The edge set $E(Kd_n(k))$ can be partitioned into three sets E_1, E_2 and E_3, where $E_1 = \{uv \mid d(u) = n-3 \text{ and } d(v) = n-3\}$, $E_2 = \{uv \mid d(u) = n-3 \text{ and } d(v) = n-1\}$ and $E_3 = \{uv \mid d(u) = n-1 \text{ and } d(v) = n-1\}$. It is easy to check that $|E_1| = (k(k-1)/2) - k$, $|E_2| = (n-k)k$ and $|E_3| = (n-k)(n-k-1)/2$. Therefore, from Proposition 2.1 we have,

$$\begin{aligned} H_{rs}(Kd_n(k),x) &= x^{n-1}\sum_{uv\in E(Kd_n(k))} x^{\frac{1}{2}(d(u)+d(v))} \\ &= x^{n-1}\left[\sum_{uv\in E_1(Kd_n(k))} x^{\frac{1}{2}(d(u)+d(v))}+\sum_{uv\in E_2(Kd_n(k))} x^{\frac{1}{2}(d(u)+d(v))}\right. \\ &\quad \left.+\sum_{uv\in E_3(Kd_n(k))} x^{\frac{1}{2}(d(u)+d(v))}\right] \\ &= x^{n-1}\left[\sum_{uv\in E_1(Kd_n(k))} x^{\frac{1}{2}(2(n-3))}+\sum_{uv\in E_2(Kd_n(k))} x^{\frac{1}{2}(2n-4)}\right. \\ &\quad \left.+\sum_{uv\in E_3(Kd_n(k))} x^{\frac{1}{2}(2(n-1))}\right] \\ &= x^{n-1}\left[\left((k(k-1)/2)-k\right)x^{n-3}+(n-k)kx^{n-2}\right. \\ &\quad \left.+\left((n-k)(n-k-1)/2\right)x^{n-1}\right]. \end{aligned}$$

□

§4. Reciprocal Transmission Hosoya Polynomial of Some Reciprocal Transmission Distance Balanced Graphs

A bijection $\propto$ on $V(G)$ is called automorphism of G if it preserves $E(G)$. In other words, $\propto$ is an automorphism if for each $u, v \in V(G)$, $e = uv \in E(G)$ if and only if

$$\propto(e) = \propto(u)\propto(v) \in E(G).$$

Let $Aut(G) = \{\propto \mid \propto : V(G) \to V(G) \text{ is a bijection, which preserves the adjacency}\}$.

It is known that $Aut(G)$ forms a group under the composition of mappings. A graph G is called vertex-transitive if for every two vertices u and v of G, there exists an automorphism $\propto$ of G such that $\propto(u) = \propto(v)$.

Theorem 4.1([6]) *Let G be a connected graph on n vertices with the automorphism group $Aut(G)$ and the vertex set $V(G)$. Let $V_1, V_2, \cdots, V_t$ be all orbits of the action $Aut(G)$ on $V(G)$. Suppose that for each $1 \leq i \leq t$, k_i are the reciprocal transmission of vertices in the orbit V_i, respectively. Then*

$$H(G) = \frac{1}{2}\sum_{i=1}^{t} |V_i| k_i.$$

Specially if G is vertex-transitive (i.e., $t = 1$), then

$$H(G) = \frac{1}{2} nk,$$

where k is the reciprocal transmission of each vertex of G.

Analogous to Theorem 4.1 and as a consequence of Proposition 2.1, we have the following.

Lemma 4.2 *Let G be a connected k-reciprocal transmission regular graph with m edges and $diam(G) \leq 2$. Then*

$$H_{rs}(G, x) = mx^{n+k-1}.$$

Proof For any k-reciprocal transmission distance balanced graph, $rs(u) = k$ for every vertex $u \in V(G)$. Therefore, from Eq.(2) we have,

$$\begin{aligned} H_{rs}(G, x) &= x^{n-1} \sum_{uv \in E(G)} x^{\frac{1}{2}(rs(u)+rs(v))} \\ &= x^{n-1} \sum_{uv \in E(G)} x^{\frac{1}{2}(2k)} = x^{n-1} m x^k = mx^{k+n-1}. \end{aligned}$$ □

Theorem 4.3 *Let G be a connected graph on n vertices with automorphism group $Aut(G)$ and the vertex set $V(G)$. Let $V_1, V_2, \cdots, V_t$ be all orbits of the action $Aut(G)$ on $V(G)$. Suppose that for each $1 \leq i \leq t$, d_i and k_i are the vertex degree and the reciprocal transmission of vertices in the orbit V_i, respectively. Then*

$$H_{rs}(G, x) = \frac{nd}{2} x^{n+k-1},$$

where d and k are the degree and the reciprocal transmission of each vertex of G respectively.

Proof Applying Theorem 4.1 and Lemma 4.2, we get the result. □

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Generalized Common Neighbor Polynomial of Graphs

Shikhi M.

(Department of Mathematics, University of Calicut, Malappuram, Kerala 673 635, India)

Anil Kumar V.

(Department of Mathematics, University of Calicut, Malappuram, Kerala 673 635, India)

E-mail: shikhianil@gmail.com, anil@uoc.ac.in

Abstract: Let $G(V, E)$ be a simple graph of order n with vertex set V and edge set E. Let (u, v) denotes an unordered vertex pair of distinct vertices of G. The i-common neighbor set of G is defined as $N(G, i) := \{(u, v) : u, v \in V, u \neq v$ and $|N(u) \cap N(v)| = i\}$, for $0 \leq i \leq n-2$. The polynomial

$$N[G; x] = \sum_{i=0}^{(n-2)} |N(G, i)|x^i$$

is defined as the common neighbor polynomial of G. In this paper we introduce the generalized i-common neighbor set and the generalized common neighbor polynomial of a graph.

Key Words: Generalized i-common neighbor set, generalized common neighbor polynomial.

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§1. Introduction

A group of individuals who belong to various social, economical and occupational status, may show consensus in some areas. The similarity strength of such groups can be measured by the number of areas in which they are mutually interested. Visualizing the situation graphically, the individuals and different areas of interest may represented by nodes of the bipartite sets A and B of a bipartite graph and a node in A is connected to a node in B if the corresponding individual have the particular area of interest. Then the similarity strength of a group of r individuals is given by the number of common neighbors shared by the corresponding nodes. These concepts motivated the authors to define generalized i-common neighbor sets and common neighbor polynomial of graphs.

Let $G(V, E)$ be a simple graph of order n with vertex set V and edge set E. Let (u, v) denotes an unordered vertex pair of distinct vertices of G. The i-common neighbor set of G is defined by the present authors as $N(G, i) := \{(u, v) : u, v \in V, u \neq v$ and $|N(u) \cap N(v)| = i\}$, for $0 \leq i \leq n-2$. The polynomial $N[G; x] = \sum_{i=0}^{(n-2)} |N(G, i)|x^i$ is defined as the common

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neighbor polynomial of G [2].

A family Δ of finite subsets of a set V is a simplicial complex [6] if it satisfy the condition that whenever $\sigma \in \Delta$ and $\tau \subset \sigma$ then, $\tau \in \Delta$. If $\sigma \in \Delta$ is of cardinality $k+1$, then σ is called a k-simplex and every $\tau \subset \sigma$ is a face of the simplex. The dimension of a simplex is one less than its cardinality. If a simplex is not a proper subset of any other simplexes in the complex, then it is a facet of the complex.

In this paper we introduce the generalized i-common neighbor set and the generalized common neighbor polynomial of graphs. Also we derive the generalized common neighbor polynomial of some well known graph classes. Moreover, we define the simplicial complex of a graph G and introduce the cluster of a vertex $v \in G$ as a simplicial complex of G. These concepts are used to deduce some interesting properties of generalized i-common neighbor sets.

§2. Main Results

In this section we first introduce the definition of *generalized i-common-neighbor set* and then define the *generalized common neighbor polynomial* of a graph. Moreover, we discuss some properties of generalized $i-$ common neighbor sets and also derive the generalized common neighbor polynomial of some well known graph classes. Also we express generalized common neighbor sets using the theory of simplicial complexes in order to deduce some interesting properties of the sets.

2.1 Generalized Common Neighbor Sets and Common Neighbor Polynomial of Graphs

Definition 2.1 *Let $G(V,E)$ be a graph of order n. Let $\mathscr{L}_r$ denotes the set of all unordered r-tuples of distinct elements of V. For $0 \le i \le n-r$, the generalized i-common neighbor set of G is defined as follows:*

$$N_r(G,i) = \{(u_1, u_2, \cdots, u_r) \in \mathscr{L}_r : |\cap_{k=1}^{r} N(u_k)| = i\}.$$

Definition 2.2 *Let G be a graph of order n. For $0 < r \le n$ the generalized common neighbor polynomial, $N_r[G;x]$, of G is defined as*

$$N_r[G;x] = \sum_{i=0}^{(n-r)} |N_r(G,i)| x^i.$$

Throughout this paper, r denotes an integer such that $1 \le r \le n$. We observe the following simple properties of $N_r[G;x]$:

(i) $N_2[G;x] = N[G;x]$, the common neighbor polynomial of the graph G;

(ii) $N_r[G;x]$ is a polynomial of degree at most $(n-r)$;

(iii) Isomorphic graphs have same generalized common neighbor polynomials;

(iv) $N_r(G, i) = \phi$ for $n - r + 1 \leq i \leq n$;
(v) $N_r[G; 1] = \sum_{i=1}^{n-r} |N_r(G, i)| = \binom{n}{r}$;
(vi) $N_r[G; 0]$ gives the number of elemnts in $\mathscr{L}_r$ having no common neighbors.

Theorem 2.3 *For any graph G, we have $|N_r(G, 0)| \leq |N_s(G, 0)|$ if $r \leq s \leq n$.*

Proof It is enough to show that corresponding to each r-tuple of vertices in $N_r(G, 0)$, there are one or more s-tuples of vertices in $N_s(G, 0)$. Let $(u_1, u_2, \cdots, u_r) \in N_r(G, 0)$. Let $u_{r+1}, u_{r+2}, \cdots, u_s$ be any $s - r$ vertices in $V - \{u_1, u_2, \cdots, u_r\}$. Then the s-tuple of vertices $(u_1, u_2, \cdots, u_r, u_{r+1}, \cdots, u_s)$ have no common neighbors in G since the first r vertices have no common neighbors in G. Then $(u_1, u_2, \cdots, u_r, u_{r+1}, \cdots, u_s) \in N_s(G, 0)$. This completes the proof. □

Theorem 2.4 *For any graph G, if $(u_1, u_2, \cdots, u_r) \in N_r(G, i)$, then $(u_1, u_2, \cdots, u_r, u_{r+1}, \cdots, u_s) \notin N_s(G, j)$, where $r < s$ and $0 < i < j$.*

Proof Let $(u_1, u_2, \cdots, u_r) \in N_r(G, i)$. Let $u_{r+1}, u_{r+2}, \cdots, u_s$ be any $s - r$ vertices in $V - \{u_1, u_2, \cdots, u_r\}$ such that $(u_1, u_2, \cdots, u_r, u_{r+1}, \cdots, u_s) \in N_s(G, j)$ where $r < s$ and $0 < i < j$. Then the vertices $u_1, u_2, \cdots, u_r, \cdots, u_s$ have j common neighbors in G where $j > i$. In particular, the vertices $u_1, u_2, \cdots, u_r$ have at least j common neighbors in G, a contradiction since $j > i$ and $(u_1, u_2, \cdots, u_r) \in N_r(G, i)$. □

Theorem 2.5 *For a complete graph K_n $(n \geq r)$,we have*

$$N_r[K_n; x] = \binom{n}{r} x^{n-r}.$$

Proof The proof follows from the fact that any r-tuple of vertices of K_n have $(n - r)$ common neighbors and there are $\binom{n}{r}$ such r-tuples of vertices. □

Theorem 2.6 *For a path graph P_n, we have $N_r[P_n; x] = \binom{n}{r}$, $r \geq 3$.*

Proof The result follows from the fact that no r-tuple of vertices in P_n where $r \geq 3$ have common neighbors in P_n. □

Theorem 2.7 *For a cycle graph C_n, we have $N_r[C_n; x] = \binom{n}{r}$, $r \geq 3$.*

Proof The result follows from the fact that no r-tuple of vertices in C_n where $r \geq 3$ have common neighbors in C_n. □

Theorem 2.8 *For a complete bipartite graph $K_{m,n}$, we have*

$$N_r[K_{m,n}; x] = \binom{m}{r} x^n + \binom{n}{r} x^m + \sum_{j=1}^{r-1} \binom{m}{j} \binom{n}{r-j}.$$

Proof Let M, N be the bipartite sets of vertices of $K_{m,n}$ where $|M| = m$ and $|N| = n$. We consider the following 3 cases according to the selection of vertices in the $r-$tuple $(u_1, u_2, \cdots, u_r)$.

Case 1. $u_k \in M$ for $1 \le k \le r$.

In this case, each of the $r-$tuple of vertices $(u_1, u_2, \cdots, u_r)$ have n common neighbors contributing the term $\binom{m}{r}x^n$ in the generalized common neighbor polynomial.

Case 2. $u_k \in N$ for $1 \le k \le r$.

In this case, each of the $r-$tuple of vertices $(u_1, u_2, \cdots, u_r)$ have m common neighbors contributing the term $\binom{n}{r}x^m$ in the generalized common neighbor polynomial.

Case 3. After a sufficient rearrangement of terms, let $u_k \in M$ for $1 \le k \le j$ and $u_k \in N$ for $j+1 \le k \le r$.

For each j where $1 \le j \le r-1$, the $r-$tuple of vertices $(u_1, u_2, \cdots, u_r)$ has no common neighbor in $K_{m,n}$ and there are $\binom{m}{j}\binom{n}{r-j}$ such $r-$ tuples.

This completes the proof. □

Corollary 2.9 *For a star graph $K_{1,n}$, we have $N_r[K_{1,n}; x] = \binom{n}{r}x + \binom{n}{r-1}$ for $r \ge 2$.*

A bistar graph $B_{n,n}$is the union of two star graphs $K_{1,n}$ with centres u and v together with a new edge uv.

Theorem 2.10 *For a bistar graph $B_(n,n)$ we have*

$$N_r[B_{n,n}; x] = 2\binom{n+1}{r}x + 2\binom{n}{r-1} + \sum_{m=1}^{r-1}\binom{n}{m}\binom{n}{r-m} + \delta_{r2},$$

where $\delta_{rj} = \begin{cases} 1 & if\ r = j, \\ 0 & if\ r \ne j. \end{cases}$

Proof Let $S = \{s_1, s_2, \cdots, s_n\}$ and $T = \{t_1, t_2, \cdots, t_n\}$ be the pendent vertices of the star graphs with center vertices u and v respectively, which together with the edge uv constitute the bistar graph $B_{n,n}$. Let $(u_1, u_2, \cdots, u_r)$ be any r-tuple of vertices of $B_{n,n}$. We consider different cases according to the selection of vertices in the r-tuple where $r > 2$.

Case 1. $u_i \in S$ or $u_i \in T$ for all $i \in \{1, 2, \cdots, r\}$.

All the r-tuple of vertices under this case have exactly one common neighbor u or v according as $u_i \in S$ or $u_i \in T$. Hence this case contribute the term $2\binom{n}{r}x$ to the generalized common neighbor polynomial of $B_{n,n}$.

Case 2. For $i \in \{1, 2, \cdots, r\}$, $u_i = v$ for exactly one i and all other $u_i \in S$.

The r-tuple of vertices under this case have exactly one common neighbor u and there are $\binom{n}{r-1}$ such r-tuples thereby contributing the term $\binom{n}{r-1}x$ to $N_r[B_{n,n}; x]$.

Case 3. For $i \in \{1, 2, \cdots, r\}$, $u_i = u$ for exactly one i and all other $u_i \in T$.

By a similar argument as in Case 2, the r-tuples in this case also contributes the term $\binom{n}{r-1}x$ to $N_r[B_{n,n}; x]$.

Case 4. For $i \in \{1, 2, \cdots, r\}$, $u_i = u$ or $u_i = v$ for exactly one i where all other $u_i \in S$ or $\in T$ respectively.

All the r-tuple of vertices under this case have no common neighbors and there are $2\binom{n}{r-1}$ such r-tuples.

Case 5. After an appropriate rearrangement of terms of the r-tuple, let $u_1, u_2, \cdots, u_m \in S$ and $u_{m+1}, u_{m+2}, \cdots, u_r \in T$ where $1 \leq m \leq r - 1$.

All the r-tuple of vertices under this case have no common neighbors and this case contribute the term $\sum_{m=1}^{r-1} \binom{n}{m}\binom{n}{r-m}$ to $N_r[B_{n,n}; x]$.

It follows that

$$\begin{aligned} N_r[B_{n,n}; x] &= 2\binom{n}{r}x + 2\binom{n}{r-1}x + 2\binom{n}{r-1} + \sum_{m=1}^{r-1}\binom{n}{m}\binom{n}{r-m} \\ &= 2\binom{n+1}{r}x + 2\binom{n}{r-1} + \sum_{m=1}^{r-1}\binom{n}{m}\binom{n}{r-m} \end{aligned}$$

This completes the proof with a sufficient remark that when $r = 2$, there is a pair of vertices (u, v) having no common neighbors. $\square$

Theorem 2.11 *Every graph G contains $|N_r(G, i)|$ number of complete bipartite subgraphs $K_{i,r}$ where $1 \leq i \leq n - r$.*

Proof Note that corresponding to each $r-$tuples of vertices $(u_1, u_2, \cdots, u_r) \in N_r(G, i)$, the vertices $u_1, u_2, \cdots, u_r$ together with their i common neighbors constitute a complete bipartite subgraph $K_{i,r}$. Hence the result follows. $\square$

Theorem 2.12 *The generalized common neighbor polynomial of a graph G is non constant if and only if there exists a star $K_{1,r}$ in G where $1 \leq r \leq n$.*

Proof Let $N_r[G; x]$ be a non constant polynomial of degree $m \geq 1$. Then there exists an r-tuple of vertices $(u_1, u_2, \cdots, u_r)$ in G which has at least one common neighbor, say w in G. Then w together with the vertices $u_1, u_2, \cdots, u_r$ produces a star $K_{1,r}$ in G.

Conversely let there exists a star $K_{1,r}$ in G where $1 \leq r \leq n$. Let $u_1, u_2, \cdots, u_r$ be the pendent vertices of $K_{1,r}$. Then the center of the star graph $K_{1,r}$ is a common neighbor of the r-tuple $(u_1, u_2, \ldots, u_r)$. The result follows from the fact that $N_r(G, i) \neq \phi$ for some $i \geq 1$. $\square$

Corollary 2.13 *If a graph G doesn't contain any star graph $K_{1,r}$ as a subgraph where $1 \leq r \leq n$, then the generalized common neighbor polynomial $N_r[G; x] = \binom{n}{r}$.*

Theorem 2.14 *The generalized common neighbor polynomial $N_r[G; x]$ of a graph G is of degree*

$k \geq 1$ if and only if k is the largest integer such that G has a complete bipartite subgraph $K_{r,k}$.

Proof Assume that $N_r[G;x]$ of a graph G is of degree $k \geq 1$. Then, $|N_r(G,k)| \neq \phi$. Take $(u_1, u_2, \cdots, u_r) \in N_r(G,k)$. Then the vertices $u_1, u_2, \cdots, u_r$ together with their k common neighbors constitute a complete bipartite subgraph $K_{r,k}$ of G. Moreover, if G contains $K_{r,j}$ as a subgraph,where $j \geq k+1$, then G contains an r-tuple of vertices having j common neighbors where $j \geq k+1$ which is a contradiction since $N_r[G;x]$ is of degree k. This proves the necessary part of the theorem.

Conversely, we assume that k is the largest integer such that G has a complete bipartite subgraph $K_{r,k}$. If possible, let $N_r[G;x]$ is of degree $j \geq k+1$. Then G contains an r-tuple of vertices having at least $k+1$ common neighbors. These r vertices together with their $k+1$ common neighbors constitute a complete bipartite subgraph $K_{r,k+1}$ of G which is a contradiction to the assumption. □

Definition 2.15 *Two graphs G and H are said to be CNP_r equivalent if $N_r[G;x] = N_r[H;x]$. The set of all graphs which are CNP_r equivalent to G is denoted by $[G]_{\mathcal{N}_r}$.*

Theorem 2.16 *For any graph G, $\overline{G} \in [G]_{\mathcal{N}_r}$ if and only if there are $|N_r(G,i)|$ number of $r-$tuple of vertices in G which dominate $n-i$ vertices of G.*

Proof First, suppose that $\overline{G} \in [G]_{\mathcal{N}_r}$. Then $|N_r(G,i)| = \left|N_r(\overline{G},i)\right|$ for $0 \leq i \leq n-r$. Let $(u_1, u_2, \cdots, u_r) \in N_r(\overline{G},i)$. Since the vertices $u_1, u_2, \cdots, u_r$ have only i common neighbors in $\overline{G}$, all the remaining $n-i$ vertices in G are adjacent to at least one of the vertices in $\{u_1, u_2, \cdots, u_r\}$. Then $\{u_1, u_2, \cdots, u_r\}$ dominate exactly $n-i$ vertices of G. Since $|N_r(G,i)| = |N_r(\overline{G},i)|$, it follows that G has $|N_r(G,i)|$ number of $r-$tuple of vertices which dominate $n-i$ vertices of G.

Conversely assume that there are $|N_r(G,i)|$ number of $r-$tuple of vertices in G which dominate $n-i$ vertices of G. From the proof of first part of the theorem, the $r-$tuples of vertices in G which dominate exactly $n-i$ vertices of G are those which belongs to $N_r(\overline{G},i)$. It follows that $|N_r(G,i)| = |N_r(\overline{G},i)|$ and hence $N_r[G;x] = N_r[\overline{G};x]$. This completes the proof.□

Corollary 2.17 *Let G be a graph of order n. If $\overline{G} \in [G]_{\mathcal{N}_r}$, then $|N_r(G,0)|$ gives the number of dominating sets in G of order r.*

Lemma 2.18 *Let G be a connected graph with $n > 3$ vertices. If all the pairs of edges of G have a common end vertex, then G is a star graph.*

Proof Since $n > 3$ and G is connected, the number of edges m should be greater than or equal to 3. We will prove the result by using method of induction on the number of edges m of G. Clearly the result is true for $m = 3$. Let the result be true for all graphs G with less than m edges. And let G be a graph with m edges such that all the pairs of edges have a common end vertex. By deleting any edge e from G, we have a graph with $m-1$ edges. Clearly all the pairs of edges of $G-e$ are incident to a common vertex. Hence by induction assumption, $G-e$ is a star. Let v be the center vertex of the star so that the edges of $G-e$ be represented by $e_i = vv_i$ where $i = 1, 2, \cdots, m-1$. Since the edges e and e_1 of G are incident to a common vertex, either $e = vw$ or $e = v_1w$. In the first case G is a star and the proof is complete. And in the second case, there are two possibilities according as w belongs to $\{v_1, v_2, \cdots, v_{m-1}\}$ or not. If w belongs to the set, let $w = v_i$ where $i \in \{1, 2, \cdots, m-1\}$. Then the edges v_1w and and vv_{i+1} have no common end vertex. which ruled out the possibility. If w doesn't belong to the set, then the edges v_1w and vv_3 have no common end vertex. Hence by the induction assumption, the second case is ruled out. Hence the result follows. □

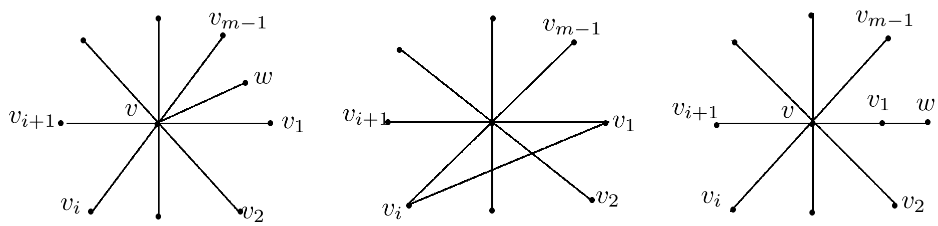

Figure 1 Figure showing different cases of Lemma 2.18

The line graph $L(G)$ of a graph G is the graph with vertex set the set of all edges of G and two vertices of $L(G)$ are adjacent if the corresponding edges of G are incident to a common vertex.

Theorem 2.19 *Let G be a connected graph of order $n > 3$. The number k of cliques of size $r > 1$ in the line graph of G is given by $k = \sum\limits_{i=1}^{n-r} i|N_r(G, i)|$.*

Proof Let S be the collection of all r-tuples of vertices $(u_1, u_2, \cdots, u_r)$ of G which have at least one common neighbor in G. Also let the r-tuple $(u_1, u_2, \cdots, u_r)$ repeat as many times in S as its number of common neighbors. Then, $|S| = \sum_{i=1}^{n-2} i|N_r(G, i)|$. Let P be the collection of all cliques of size r in the line graph $L(G)$ of G. Let the vertices of $L(G)$ be denoted by uv where u, v are adjacent vertices of G. Define $\phi : S \to P$ as follows.

Let $u = (u_1, u_2, \cdots, u_r) \in S$ which repeats i-times in S. Let these i members be represented by $u_k = (u_1, u_2, \cdots, u_r)^{(k)}$ where $k = 1, 2, \cdots, i$. Then each $(u_1, u_2, \cdots, u_r)^{(k)}$ can be assigned to exactly one common neighbor w_k of $(u_1, u_2, \cdots, u_r)$ in G. It follows that all the pairs of vertices u_lw_k and u_mw_k where $l, m \in \{1, 2, \cdots, r\}$ and $l \neq m$ are adjacent vertices of $L(G)$ which forms a clique C_{uk} of size r in $L(G)$.

Now define $\phi : S \to P$ as $\phi((u_1, u_2, \cdots, u_r)^{(k)} = C_{uk}$. Clearly ϕ is one-one. We claim that $\phi : S \to P$ is onto. Let C be a clique of size r in the line graph $L(G)$ of G. Since any pair of vertices of C are adjacent in $L(G)$, all the pairs of edges in G which constitute the vertex set of C, have a common end vertex in G. Hence by Lemma 2.18, those edges form a star in G whose pendent vertices forms an r-tuple $(u_1, u_2, \cdots, u_r) \in S$ such that $\phi(u_1, u_2, \cdots, u_r) = C$. Thus ϕ is onto.

It follows that ϕ is a bijection from S to P and $|S| = |P|$. This completes the proof. □

Corollary 2.20 *Let G be a graph of order n. Then the number of edges of the line graph $L(G)$ of G equals $\sum\limits_{i=1}^{n-2} i|N(G, i)|$.*

Proof The result follows from the fact that the 2-cliques of any graph are the edges of the graph. □

Theorem 2.21(Schwartz 1969 and Ghirlanda 1963) *A graph is isomorphic to its line graph if and only if it is regular of degree two.*

Corollary 2.22 *If a graph G is regular of degree two, then the number of edges of G equals* $\sum\limits_{i=1}^{n-2} i|N(G,i)|$.

2.2 Simplicial Complexes of Graphs and Common Neighbor Sets

In this section, we first define the simplicial complex of a graph G and introduce the cluster of a vertex $v \in G$ as a simplicial complex of G. Then we incorporate the concept of generalized i-common neighbor set of a graph with the cluster of vertices in it, to deduce some interesting properties of generalized i-common neighbor sets.

Definition 2.23 *Let $G(V,E)$ be a graph and let Δ be a collection of subsets of V. The elements of Δ are called simplexes. Let τ be an element in Δ. Then the subsets of τ are called its faces. We say that Δ is a simplicial complex of G if for every τ in Δ, all its faces are in Δ.*

Let G be a simple finite graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. For each vertex v_i, the cluster of v_i is defined as

$$\mathrm{clr}(v_i) =: \{W \subset V : v_i \in \cap_{v \in W} N(v)\}.$$

Then each $\mathrm{clr}(v_i)$ where $i \in \{1, 2, \cdots, n\}$ is a simplicial complex of G. We may consider $clr(v_i)$ as a simplicial complex of G generated by the vertex v_i. Note that each simplex W of $clr(v_i)$ spans a subgraph of G which is a star graph with center vertex v_i. So these simplexes are called the stars of v_i denoted by $str(v_i)$. The facets of $clr(v_i)$ are the maximal stars in $clr(v_i)$.

Lemma 2.24 *Let v be a vertex of the graph G having degree d. Then the cluster of v contains $\binom{d}{r}$ number of $(r-1)$-simplexes.*

Proof Let S be the set of all neighbors of the vertex v such that $|S| = d$. Any subset S_1 of S with cardinality $r \leq d$ will act as a r-tuple of vertices with v as a common neighbor. There are exactly $\binom{d}{r}$ distinct subsets of S with cardinality r and these subsets are exactly the $(r-1)$-simplexes of the cluster of v. Hence the result follows. □

Theorem 2.25 *Let $G(V,E)$ be a simple graph and let $v \in V$. Let f_i, $i = 1, 2, \cdots, m$ be the facets of the simplicial complex $clr(v)$. If the facet f_i is of cardinality d_i, then $clr(v)$ contains* $\sum\limits_{i=1}^{m}\sum\limits_{r=1}^{d_i}\binom{d_i}{r}$ *distinct simplexes.*

Proof According to the definition of a simplicial complex, all the subsets of its facets must also be simplexes of the complex. If the facet f_i of $clr(v)$ is of cardinality d_i, there are $\binom{d_i}{r}$

simplexes of dimension r in $clr(v)$. Thus corresponding to each facet f_i, there are $\sum_{r=1}^{d_i}\binom{d_i}{r}$ distinct simplexes in $clr(v)$. As there are m facets, the result follows. □

Theorem 2.26 *If G is a graph having degree sequence $(d_1, d_2, \cdots, d_n)$, then we have the following:*

$$\sum_{i=1}^{n-r} i|N_r(G,i)| = \sum_{i=1}^{n}\binom{d_i}{r}.$$

Proof Let $clr(v_i)$, $i = 1, 2, \cdots, n$ be the simplicial complexes generated by the vertices $v_1, v_2, \cdots, v_n$ of the graph G. We will show that the expression on both sides of the equation equates the total number of $(r-1)$-simplexes of $clr(v_i)$ where $i = 1, 2, \cdots, n$.

By Lemma 2.24, the number of $(r-1)$-simplexes in $clr(v_i)$ is given by $\binom{d_i}{r}$ where d_i is the degree of the vertex v_i which generates $clr(v_i)$. Hence if all the simplicial complexes $clr(v_i)$, $i \in \{1, 2, \cdots, n\}$ are taken into account, there are altogether $\sum_{i=1}^{n}\binom{d_i}{r}$ number of $(r-1)$-simplexes.

Now, for a fixed $i \in \{1, 2, \cdots, n\}$, the $(r-1)$-simplexes of $clr(v_i)$ are exactly r-tuples of vertices with v_i as a common neighbor. Hence the total number of $(r-1)$-simplexes of $clr(v_i)$, $i = 1, 2, \cdots, n$ equals the number of r-tuples of vertices with at least one common neighbor where the r-tuple with i common neighbors has to be counted i times. From the definition of generalized i-common neighbor set of G, the number of such r-tuple of vertices is given by $\sum_{i=1}^{n-r} i|N_r(G,i)|$. This completes the proof. □

Theorem 2.27 *The generalized i-common neighbor set $N_r(G,i)$ is the set of all $(r-1)$-simplexes which belongs to the intersection of exactly i of the clusters of vertices of G.*

Proof Let W be a $(r-1)$-simplex which belongs to a simplicial complex $clr(v_j)$, for some $j \in \{1, 2, \cdots, n\}$. From the definition of $clr(v_j)$, it is clear that the members of W constitute a r-tuple of vertices of G having v_j as a common neighbor. Now fix an integer i such that $1 \leq i \leq n-2$. W belongs to exactly i of the $clr(v_j)$, if and only if the corresponding r-tuple of vertices has exactly i common neighbors. It follows that $W \in N_r(G,i)$. □

Remark 2.28 *We observe the following properties of the simplicial complexes $clr(v_i)$ generated by the vertices v_i of a simple graph G.*

For $i, j, k \in \{1, 2, \cdots, n\}$,

(1) *If a simplicial complex $clr(v_i)$ is generated by a vertex v_i, then, $\{v_i\} \notin clr(v_i)$;*

(2) *$clr(v_i)$ contains all possible unions of the 0-simplexes containing in it;*

(3) *If $\{v_i\} \in clr(v_j)$, then $\{v_j\} \in clr(v_i)$.*

The first statement follows from the fact that a vertex cannot be adjacent to itself as we are considering only simple graphs. The second and third statements directly follows from the definition of $clr(v_i)$.

The following theorem shows that these are the sufficient conditions for a collection of simplicial complexes $\{clr(v_i)\}, i \in \{1, 2, \cdots, n\}$ on a set of cardinality n to be generated by a set of vertices $V = \{v_1, v_2, \cdots, v_n\}$ of a simple graph G.

Theorem 2.29 *Let $V = \{v_1, v_2, \cdots, v_n\}$ be any set of n elements. If $clr(v_i)$, $i \in \{1, 2, \cdots, n\}$ are simplicial complexes on the set V satisfying the conditions (1),(2) and (3) stated in above remark, then there exists a simple graph G with vertex set V where $clr(v_i)$ is the simplicial complex generated by the vertex v_i of G.*

Proof Given a set of elements $V = \{v_1, v_2, \cdots, v_n\}$ and a collection of simplicial complexes $\{clr(v_i)\}, i \in \{1, 2, \cdots, n\}$ on the set V, construct a graph with vertex set V and edge set E where an edge $v_iv_j \in E$ if and only if $\{v_j\} \in clr(v_i)$.

By condition (1), $\{v_i\} \notin clr(v_i)$ which implies that G has no loops. Also by condition (3), if $\{v_i\} \in clr(v_j)$, then $\{v_j\} \in clr(v_i)$ which implies that the adjacency of vertices of the graph is well defined in the sense that whenever v_i adjacent to v_j, v_j is adjacent to v_i also.

Now we will prove that $\{clr(v_i)\}$ are the simplicial complexes generated by the vertices $\{v_i\}$ of the graph G. Let V_1 be a subset of V which belongs to $clr(v_i)$. Then $V_1 = \{v_{j_1}, v_{j_2}, \cdots, v_{j_m}\}$ where each of the vertices in the set are adjacent to a vertex $v_i \in V$ in G. Then $v_iv_{j_k} \in E$ and $\{v_{j_k}\} \in clr(v_i)$ for all $k \in \{1, 2, \cdots, m\}$. Hence by condition(3), all the subsets of V_1 are in $clr(v_i)$. It follows that $clr(v_i)$ is a simplicial complex on V. And by definition of edge set of G, it is generated by v_i. This completes the proof. $\square$

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On Status Coindex Distance Sum and Status Connectivity Coindices of Graphs

Kishori P. Narayankar, Dickson Selvan and Afework T. Kahsay

(Department of Mathematics, Mangalore University, Mangalagangothri, Mangalore-574199, India)

E-mail: kishori_pn@yahoo.co.in, dickson.selvan@gmail.com, kafew17@gmail.com

Abstract: The status of a vertex u in a connected graph G, denoted by $\sigma(u)$ is defined as the sum of the distance between u and all other vertices of a graph G. Let G be a connected graph of order $n \geq 3$ and size m. The first and second status coindices distance sum of graph G, denoted by $S_1^d(G)$ and $S_2^d(G)$, are defined as

$$\begin{aligned} S_1^d(G) &= \sum_{uv \notin E(G)} [\sigma(u) + \sigma(v)]d(u,v), \\ S_2^d(G) &= \sum_{uv \notin E(G)} [\sigma(u)\sigma(v)]d(u,v) \end{aligned}$$

respectively. In this paper the first and second status coindex distance sum of some graphs are obtained. Status connectivity coindices of some standard graphs are computed. The bounds of the first and second status coindex distance sum and status connectivity coindices are established.

Key Words: Distance, status of a vertex, status coindex distance sum, status connectivity coindices.

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§1. Introduction

Let G be a connected graph with n vertices and m edges. Let $V(G)$ and $E(G)$ be its vertex and edge sets, respectively. The edge joining the vertices u and v is denoted by uv. The complement $\overline{G}$ of the graph G is the graph with vertex set $V(G)$ in which two vertices are adjacent if and only if they are not adjacent in G. The *degree* of a vertex u in a graph G is the number of edges joining to u and is denoted by $d(u)$ or d_u. The *distance* between the vertex u and v is the length of the shortest path joining u and v and is denoted by $d_G(u,v)$ [6]. For well known graph and terminology, we refer the books [6], [17].

The *status* of a vertex $u \in V(G)$, denoted by $\sigma_G(u)$ is defined as [8],

$$\sigma_G(u) = \sum_{v \in V(G)} d(u,v).$$

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The Wiener index $W(G)$ of a connected graph G is defined as [12],

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v) = \frac{1}{2}\sum_{v\in V(G)} \sigma_G(u).$$

The first and second Zagreb indices of a graph G are defined as [13]

$$M_1(G) = \sum_{u\in V(G)} d_u^2 \quad \text{and} \quad M_2(G) = \sum_{uv\in E(G)} d(u)d(v).$$

Results on the Zagreb indices can be found in [5, 18, 15, 22, 24, 20, 21].

The first and second Zagreb coindices of a graph G are defined as [15]

$$\overline{M}_1(G) = \sum_{uv\notin E(G)} d(u)+d(v) \quad \text{and} \quad \overline{M}_2 = \sum_{uv\notin E(G)} d(u)d(v).$$

$M_1(G)$ can be written also as [25], [26]

$$M_1(G) = \sum_{uv\in E(G)} [d_u + d_v].$$

More results on Zagreb coindices can be found in [1], [2].

Furtula and Gutman [3] introduced the forgotten topological index of a graph G, also called as F-index, which is defined as

$$F(G) = \sum_{u\in V(G)} (d(u))^3.$$

The first status connectivity index, $S_1(G)$ and second status connectivity index, $S_2(G)$ of a connected graph is defined as [9]

$$\begin{aligned} S_1(G) &= \sum_{uv\in E(G)} [\sigma_G(u) + \sigma_G(v)], \\ S_2(G) &= \sum_{uv\in E(G)} [\sigma_G(u)\sigma_G(v)]. \end{aligned}$$

The first and second status connectivity coindex of a graph G are defined by [10]

$$\begin{aligned} \overline{S}_1(G) &= \sum_{uv\notin E(G)} [\sigma_G(u) + \sigma_G(v)], \\ \overline{S}_2(G) &= \sum_{uv\notin E(G)} [\sigma_G(u)\sigma_G(v)]. \end{aligned}$$

Definition 1.1([19]) *Let G be a connected graph of order $n \geq 3$. The first and second status coindex distance sum of G are defined as*

$$S_1^d(G) = \sum_{uv\notin E(G)} (\sigma(u)+\sigma(v))d(u,v)$$

and

$$S_2^d(G) = \sum_{uv \notin E(G)} \sigma(u)\sigma(v)d(u,v)$$

respectively.

§2. Status Coindex Distance Sum

In this section, we obtain status coindices distance sum of connected graphs in terms of Wiener index and also status coindices distance sum of complements of graphs.

Proposition 2.1 *Let G be a connected graph on n vertices with $diam(G) = 2$. Then,*

$$S_1^d(G) = 4(n-1)W(G) - 2S_1(G)$$

and

$$S_2^d(G) = 4(W(G))^2 - \sum_{u \in V(G)} (\sigma_G(u))^2 - 2S_2(G).$$

Proof By definition, we know that

$$\begin{aligned}
S_1^d(G) &= \sum_{uv \notin E(G)} [\sigma_G(u) + \sigma_G(v)]d(u,v) = \sum_{uv \notin E(G)} [\sigma_G(u) + \sigma_G(v)]2 \\
&= \left[\sum_{\{u,v\} \subseteq V(G)} [\sigma_G(u) + \sigma_G(v)] - \sum_{uv \in E(G)} [\sigma_G(u) + \sigma_G(v)] \right] 2 \\
&= [(n-1) \sum_{u \in V(G)} \sigma_G(u) - S_1(G)]2 \\
&= [2(n-1)W(G) - S_1(G)]2 = 4(n-1)W(G) - 2S_1(G)
\end{aligned}$$

Also,

$$\begin{aligned}
S_2^d(G) &= \sum_{uv \notin E(G)} [\sigma_G(u)\sigma_G(v)]d(u,v) = \sum_{uv \notin E(G)} [\sigma_G(u)\sigma_G(v)]2 \\
&= \left[\sum_{\{u,v\} \subseteq V(G)} [\sigma_G(u)\sigma_G(v)] - \sum_{uv \in E(G)} [\sigma_G(u)\sigma_G(v)] \right] 2 \\
&= \left[\frac{1}{2} \left(\left(\sum_{u \in V(G)} \sigma_G(u) \right)^2 - \sum_{u \in V(G)} \sigma_G(u)^2 \right) - S_2(G) \right] 2 \\
&= [2(W(G))^2 - \frac{1}{2} \sum_{u \in V(G)} (\sigma_G(u))^2 - S_2(G)]2.
\end{aligned}$$

$$S_2^d(G) = 4(W(G))^2 - \sum_{u \in V(G)} (\sigma_G(u))^2 - 2S_2(G). \qquad \square$$

Proposition 2.2 *Let G be a graph of order n and size m. Let $\overline{G}$, the complement of G, be connected. Then*

$$S_1^d(\overline{G}) \geq 4m(n-1) + 2M_1(G) \tag{2.1}$$

and

$$S_2^d(\overline{G}) \geq 2m(n-1)^2 + 2(n-1)M_1(G) + 2M_2(G) \tag{2.2}$$

with equality holds if and only if diam($\overline{G}$)=2.

Proof For any vertex u in $\overline{G}$ there are $n-1-d_G(u)$ vertices which are at distance 1 and the remaining $d_G(u)$ vertices are at distance at least 2. Therefore,

$$\sigma_{\overline{G}}(u) \geq [n-1-d_G(u)] + 2d_G(u) = n-1+d_G(u).$$

Therefore,

$$\begin{aligned}
S_1^d(\overline{G}) &= \sum_{uv \notin E(\overline{G})} [\sigma_{\overline{G}}(u) + \sigma_{\overline{G}}(v)] d_{\overline{G}}(u,v) \\
&\geq \sum_{uv \notin E(\overline{G})} [n-1+d_G(u)+n-1+d_G(v)] d_{\overline{G}}(u,v) \\
&= \sum_{uv \notin E(\overline{G})} [2n-2+d_G(u)+d_G(v)] d_{\overline{G}}(u,v) \\
&= 2m(2n-2) + \sum_{uv \notin E(\overline{G})} [d_G(u)+d_G(v)] d_{\overline{G}}(u,v) \\
&= 4m(n-1) + \sum_{uv \in E(G)} [d_G(u)+d_G(v)]2 \\
&= 4m(n-1) + 2M_1(G).
\end{aligned}$$

And

$$\begin{aligned}
S_2^d(\overline{G}) &= \sum_{uv \notin E(\overline{G})} [\sigma_{\overline{G}}(u)\sigma_{\overline{G}}(v)] d_{\overline{G}}(u,v) \\
&\geq \sum_{uv \notin E(\overline{G})} [n-1+d_G(u)][n-1+d_G(v)] d_{\overline{G}}(u,v) \\
&= \sum_{uv \notin E(\overline{G})} \left[(n-1)^2 + (n-1)[d_G(u)+d_G(v)] + [d_G(u)d_G(v)]\right] d_{\overline{G}}(u,v) \\
&= 2m(n-1)^2 + \sum_{uv \in E(G)} (n-1)[d_G(u)+d_G(v)] d_{\overline{G}}(u,v)
\end{aligned}$$

$$+ \sum_{uv \in E(G)} [d_G(u) d_G(v)] d_{\overline{G}}(u, v)$$
$$= 2m(n-1)^2 + 2(n-1)M_1(G) + 2M_2(G). \qquad \square$$

Corollary 2.3 *Let G be a graph with n vertices, m edges and diam ≥ 2 and let $\overline{G}$, the complement of G, be connected. Then,*

$$S_1^d(\overline{G}) \geq 2\left[4m(n-1) - \overline{M}_1(\overline{G})\right]$$

and

$$S_2^d(\overline{G}) \geq 2[4m(n-1)^2 - 2(n-1)\overline{M}_1(\overline{G}) + \overline{M}_2(\overline{G})],$$

with equality holds if and only if diam$(\overline{G}) = 2$.

Proof By definition, we have [16]

$$\overline{M}_1(\overline{G}) = 2m(n-1) - M_1(G) \tag{2.3}$$

and

$$\overline{M}_2(\overline{G}) = m(n-1)^2 - (n-1)M_1(G) + M_2(G). \tag{2.4}$$

Substituting (2.3) in (2.1) and (2.4) in (2.2) we get the required result. $\square$

§3. Bounds for Status Coindex Distance Sum

Theorem 3.1 *Let G be a connected graph with n vertices, m edges and diam$(G) = D \geq 2$. Then,*

$$4(n-1)W(G) - 2S_1(G) \leq S_1^d(G) \leq 2D(n-1)W(G) - DS_1(G)$$

and

$$4(W(G))^2 - \sum_{u \in V(G)} (\sigma_G(u))^2 - 2S_2(G)$$
$$\leq S_2^d(G) \leq 2D(W(G))^2 - \frac{D}{2} \sum_{u \in V(G)} [(\sigma_G(u))^2 - DS_2(G)]$$

with equality holds if and only if $D = 2$.

Proof Let us first prove the lower bound. When $uv \notin E(G)$, the minimum distance between

u and v is 2. Therefore

$$\begin{aligned}
S_1^d(G) &= \sum_{uv \notin E(G)} [\sigma_G(u) + \sigma_G(v)] d(u,v) \\
&\geq \sum_{uv \notin E(G)} [\sigma_G(u) + \sigma_G(v)] 2 \\
&= \left[\sum_{\{u,v\} \subseteq V(G)} [\sigma_G(u) + \sigma_G(v)] - \sum_{uv \in E(G)} [\sigma_G(u) + \sigma_G(v)] \right] 2 \\
&= \left[(n-1) \left(\sum_{u \in V(G)} \sigma_G(u) \right) - S_1(G) \right] 2 \\
&= [2(n-1)W(G) - S_1(G)]2.
\end{aligned}$$

i.e.,

$$S_1^d(G) \geq 4(n-1)W(G) - 2S_1(G).$$

And

$$\begin{aligned}
S_2^d(G) &= \sum_{uv \notin E(G)} [\sigma_G(u)\sigma_G(v)] d(u,v) \\
&\geq \sum_{uv \notin E(G)} [\sigma_G(u)\sigma_G(v)] 2 \\
&= \left[\sum_{\{u,v\} \subseteq V(G)} [\sigma_G(u)\sigma_G(v)] - \sum_{uv \in E(G)} [\sigma_G(u)\sigma_G(v)] \right] 2 \\
&= \left[\frac{1}{2} \left[\left(\sum_{u \in V(G)} \sigma_G(u) \right)^2 - \sum_{u \in V(G)} \sigma_G(u)^2 \right] - S_2(G) \right] 2 \\
&= [2(W(G))^2 - \frac{1}{2} \sum_{u \in V(G)} (\sigma_G(u)^2) - S_2(G)]2.
\end{aligned}$$

i.e.,

$$S_2^d(G) \geq 4(W(G))^2 - \sum_{u \in V(G)} (\sigma_G(u))^2 - 2S_2(G).$$

Now let us prove the upper bound. When $uv \notin E(G)$, the maximum distance between u and v be $D(diameter)$. Then

$$\begin{aligned}
S_1^d(G) &= \sum_{uv \notin E(G)} [\sigma_G(u) + \sigma_G(v)]d(u,v) \\
&\leq \sum_{uv \notin E(G)} [\sigma_G(u) + \sigma_G(v)]D \\
&= \left[\sum_{\{u,v\} \subseteq V(G)} [\sigma_G(u) + \sigma_G(v)] - \sum_{uv \in E(G)} [\sigma_G(u) + \sigma_G(v)] \right] D \\
&= \left[(n-1) \left(\sum_{u \in V(G)} \sigma_G(u) \right) - S_1(G) \right] D,
\end{aligned}$$

i.e.,

$$S_1^d(G) \leq 2D(n-1)W(G) - DS_1(G).$$

And similarly

$$S_2^d(G) = \sum_{uv \notin E(G)} [\sigma_G(u)\sigma_G(v)]d(u,v).$$

We get that

$$S_2^d(G) \leq 2D(W(G))^2 - \frac{D}{2} \sum_{u \in V(G)} [(\sigma_G(u))^2 - DS_2(G)].$$

Thus the result follows and in both upper and lower bounds of $S_1^d(G)$ and $S_2^d(G)$, the equality holds for $D = 2$. $\square$

§4. First Status Coindex Distance Sum of Line Graphs

Theorem 4.1([14]) *Let G be a graph with n-vertices and m-edges. Then,*

$$M_1(\overline{G}) = M_1(G) + n(n-1)^2 - 4m(n-1).$$

Proposition 4.2([14]) *Let L be the line graph of the graph G. Then*

$$M_1(L(G)) = F - 4M_1 + 2M_2 + 4m$$

where, M_1, M_2, F are the first Zagreb index, second Zagreb index, and forgotten topological index of the parent graph G respectively.

Theorem 4.3([19]) *Let G be a connected graph with n vertices, m edges and $diam(G) = D \geq 2$. Then,*

$$4(n-1)(n(n-1)-2m) - D\overline{M_1}(G) \leq S_1^d(G) \leq (n-1)D^2(n(n-1)-2m) - 2(D-1)\overline{M_1}(G)$$

with equality holds for $Diam(G) = 2$.

F :

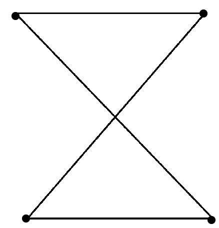

Figure 1

Theorem 4.4 *Let $L(G)$ be the line graph of the graph G with n-vertices, m-edges and $diam(G) = 2$. Then*

$$S_1^d(L(G)) = 4m[(m-1)^2 - 3(m-1) + 2] - M_1(G)[6(m-1) + 8] + 4M_2(G) + 2F.$$

Proof From the definition of line graphs [4], the number of vertices of $L(G)$ is $n_1 = m$ and the number of edges of $L(G)$ is [7] $m_1 = \frac{1}{2}\sum_{i=1}^{n} d_i^2 - m$. Since from [11], if $diam(G) \leq 2$ and G does not contain F (Figure 1.) as an induced subgraph of G and also G is not a star Graph S_n, then $diam(L(G)) = 2$. From [19],

$$S_1^d(G) = 4(n-1)[n(n-1) - 3m] + 2M_1(G)$$

Therefore, the status coindex distance sum of line graphs can be written as,

$$\begin{aligned} S_1^d(L(G)) &= 4(n_1-1)[n_1(n_1-1) - 3m_1] + 2M_1(L(G)) \\ &= 4(m-1)[m(m-1) - 3(\frac{1}{2}\sum_{i=1}^{n} d_i^2 - m)] + 2M_1(L(G)) \end{aligned}$$

From Proposition 4.2 and definition of Zagreb index [13]

$$M_1(G) = \sum_{i=1}^{n} d_i^2.$$

Hence,

$$S_1^d(L(G)) = 4m[(m-1)^2 - 3(m-1) + 2] - M_1(G)[6(m-1) + 8] + 4M_2(G) + 2F. \qquad \square$$

The following corollary directly follows from the Theorem 4.4.

Corollary 4.5 *Let G be a connected regular graph of degree r on n-vertices and m-edges and let $diam(G) = 2$. Then,*

$$S_1^d(L(G)) = 2m[4r^2 - 2r(3m-7) + 2(m-1)^2 - 6(m-1) + 4].$$

Proposition 4.6 *The first status coindex distance sum of line graph of complete bipartite graph $K_{p,q}$*

$$\begin{aligned} S_1^d(L(K_{p,q})) &= 4pq[(pq-1)^2 - 3(pq-1) + 2] \\ &\quad -pq(p+q)[6(pq-1) + 8] + 4(pq)^2 + 2pq(p^2+q^2). \end{aligned}$$

Proof The graph $K_{p,q}$ has $n = p + q$ vertices and $m = pq$ edges. Also $diam(K_{p,q}) \leq 2$. The vertex set $V(K_{p,q})$ can be partitioned into two sets V_1 and V_2 such that for every edge uv of $K_{p,q}$, the vertex $u \in V_1$ and $v \in V_2$, where $|V_1| = p$ and $|V_2| = q$. Therefore $d(u) = q$ and $d(v) = p$ and hence,

$$M_1(K_{p,q}) = pq(p+q),\ M_2(K_{p,q}) = (pq)^2,\ F = pq(p^2+q^2).$$

Therefore by the Theorem 4.4 the result holds. □

Theorem 4.7 *Let G be a graph whose line graph $L(G)$ has $diam(L(G)) > 3$, then*

$$S_1^d(\overline{L(G)}) = 4(m-1)[m(m-1) - M_1(G) + 2m] - D[(M_1(G) - 2m)(n-1) - M_1(\overline{L(G)}).$$

Proof Let G be any graph with n-vertices and m-edges whose line graph $L(G)$ has $diam(L(G)) > 3$. Let $\overline{L(G)}$ be the complement of line graph.

We know from Theorem 4.3 that

$$4(n-1)(n(n-1) - 2m) - D\overline{M_1}(G) \leq S_1^d(G)$$

i.e.,

$$4(n-1)(n(n-1) - 2m) - D\left[2m(n-1) - M_1(G)\right] \leq S_1^d(G)$$

with equality holds for graphs of $diam = 2$. Since there exist a fact that for any graph G, if $diam(G) > 3$ then $diam(\overline{G}) \leq 2$ [27]. Since G is connected graph and $diam(L(G)) > 3$, then $\overline{L(G)}$ is connected and has diameter $D = 2$, then by Theorem 4.3,

$$\begin{aligned} S_1^d(\overline{L(G)}) &= 4(n_1-1)\left[n_1(n_1-1) - 2m_1\right] - D\left[2m_1(n_1-1) - M_1(\overline{L(G)}\right] \\ S_1^d(\overline{L(G)}) &= 4(m-1)[m(m-1) - M_1(G) + 2m] \\ &\quad -D[(M_1(G) - 2m)(n-1) - M_1(\overline{L(G)}). \end{aligned}$$ □

§5. Status Connectivity Coindices of Some Standard Graphs

A simple calculation enables us getting status connectivity coindice on a few standard graphs following.

Proposition 5.1 *For a complete bipartite graph $K_{s.t}$*

$$\overline{S}_1(K_{s,t}) = (2s+t-2)s(s-1) - (2t+s-2)t(t-1)$$

and

$$\overline{S}_2(K_{s,t}) = \frac{s(s-1)}{2}(2s+t-2)^2 + \frac{t(t-1)}{2}(2t+s-2)^2.$$

Proposition 5.2 *For a cycle C_n on $n \geq 4$ vertices*

$$\overline{S}_1(C_n) = \begin{cases} \frac{n^2}{4}[n(n-1)-2m], & if\ n\ is\ even; \\ \frac{n^2-1}{4}[n(n-1)-2m], & if\ n\ is\ odd. \end{cases}$$

and

$$\overline{S}_2(C_n) = \begin{cases} \frac{n^4}{32}(n(n-1)-2m), & if\ n\ is\ even; \\ \frac{(n^2-1)^2}{32}[n(n-1)-2m], & if\ n\ is\ odd. \end{cases}$$

Proposition 5.3 *For a wheel $W_{n+1}, n \geq 4$*

$$\overline{S}_1(W_{n+1}) = 3n(n-3) + 2n(n-3)^2$$

and

$$\overline{S}_2(W_{n+1}) = \frac{9n(n-3)}{2} + 6n(n-3)^2 + 2n(n-3)^3.$$

Proposition 5.4 *For a helm $H_n, n \geq 3$*

$$\overline{S}_1(H_n) = 2[12n^3 - 27n^2 + 18n]$$

and

$$\begin{aligned}\overline{S}_2(H_n) \quad &= \quad (21n^3 - 24n^2) + \frac{(n^2-n)}{2}(7n-8)^2 \\ &\quad + \frac{n^2-3n}{2}(5n-7)^2 + (7n^3 - 15n^2 + 8n)(5n-7).\end{aligned}$$

Proposition 5.5 *For a friendship graph F_n, $n \geq 2$*

$$\overline{S}_1(F_n) = 8n(2n-1)(n-1) \quad and \quad \overline{S}_2(F_n) = (4n-2)^2(2n^2-2n).$$

§6. Bounds for Status Connectivity Coindices

Theorem 6.1 *Let G be a connected graph with n vertices, m edges and $diam(G) = D \geq 2$. Then,*

$$2(n-1)(n(n-1)-2m) - \overline{M}_1(G) \leq \overline{S}_1(G) \leq D(n-1)(n(n-1)-2m) - (D-1)\overline{M}_1(G)$$

and

$$\begin{aligned}
&2(n-1)^2(n(n-1)-2m) - 2(n-1)\overline{M}_1(G) + \overline{M}_2(G) \\
&\leq \overline{S}_2(G) \leq D^2(n-1)^2\left(\frac{n(n-1)}{2} - m\right) - D(D-1)(n-1)\overline{M}_1(G) + (D-1)^2\overline{M}_2(G)
\end{aligned}$$

with equality holds if and only if $D = 2$.

Proof Let us first prove the lower bound. For any vertex u of G there are $d(u)$ which are at a distance 1 from the vertex u and the remaining $(n-1-d(u))$ vertices are at distance at least 2. Therefore

$$\sigma(u) \geq d(u) + 2(n-1-d(u)) = 2(n-1) - d(u) = 2(n-1) - d(u).$$

Therefore,

$$\begin{aligned}
\overline{S}_1(G) &= \sum_{uv \notin E(G)} [\sigma(u) + \sigma(v)] \geq \sum_{uv \notin E(G)} [4n - 4 - (d(u) + d(v))] \\
&= \sum_{uv \notin E(G)} 4(n-1) - \sum_{uv \notin E(G)} d(u) + d(v) \\
&= 4(n-1)\left(\frac{n(n-1)}{2} - m\right) - \overline{M}_1(G) \\
&= 2(n-1)(n(n-1)-2m) - \overline{M}_1(G)
\end{aligned}$$

and

$$\begin{aligned}
\overline{S}_2(G) &= \sum_{uv \notin E(G)} [\sigma(u)\sigma(v)] \\
&\geq \sum_{uv \notin E(G)} (2n-2-d(u))(2n-2-d(u)) \\
&= \sum_{uv \notin E(G)} [4(n-1)^2 - 2(n-1)(d(u)+d(v)) + d(u)d(v)] \\
&= \sum_{uv \notin E(G)} 4(n-1)^2 - (2n-2) \sum_{uv \notin E(G)} (d(u)+d(v)) + \sum_{uv \notin E(G)} d(u)d(v) \\
&= 4(n-1)^2\left(\frac{n(n-1)}{2} - m\right) - (2n-2)\overline{M}_1(G) + \overline{M}_2(G) \\
&= 2(n-1)^2(n(n-1)-2m) - 2(n-1)\overline{M}_1(G) + \overline{M}_2(G).
\end{aligned}$$

Now we prove the upper bound. For any vertex u of G there are $d(u)$ which are at a distance 1 from the vertex u and the remaining $(n-1-d(u))$ vertices are at distance at most D. Hence,

$$\sigma(u) \leq d(u) + D(n-1-d(u)) = D(n-1) - (D-1)d(u).$$

Therefore

$$\begin{aligned}
\overline{S}_1(G) &= \sum_{uv \notin E(G)} [\sigma(u) + \sigma(v)] \\
&\leq \sum_{uv \notin E(G)} [2D(n-1) - (D-1)(d(u)+d(v))] \\
&= 2D(n-1)\left(\frac{n(n-1)}{2} - m\right) - (D-1)\overline{M}_1(G) \\
&= D(n-1)(n(n-1)-2m) - (D-1)\overline{M}_1(G)
\end{aligned}$$

and

$$\begin{aligned}
\overline{S}_2(G) &= \sum_{uv \notin E(G)} [\sigma(u)\sigma(v)] \\
&\leq \sum_{uv \notin E(G)} [D(n-1) - (D-1)d(u)][D(n-1) - (D-1)d(v)] \\
&= \sum_{uv \notin E(G)} [D^2(n-1)^2 - D(D-1)(n-1)(d(u)+d(v)) + (D-1)^2 d(u)d(v)] \\
&= D^2(n-1)^2\left(\frac{n(n-1)}{2} - m\right) - D(D-1)(n-1)\overline{M}_1(G) + (D-1)^2\overline{M}_2(G).
\end{aligned}$$

In both upper and lower bounds of $\overline{S}_1(G)$ and $\overline{S}_2(G)$, the equality holds for $D = 2$. $\square$

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Some Results on 4-Total Prime Cordial Graphs

R.Ponraj

(Department of Mathematics, Sri Paramakalyani College, Alwarkurichi–627 412, India)

J.Maruthamani and R.Kala

(Department of Mathematics, Manonmaniam Sundarnar University, Tirunelveli-627012, Tamilnadu, India)

E-mail: ponrajmaths@gmail.com, mmani2011@gmail.com, karthipyi91@yahoo.co.in

Abstract: Let G be a (p,q) graph. Let $f : V(G) \to \{1,2,\cdots,k\}$ be a map where $k \in \mathbb{N}$ and $k > 1$. For each edge uv, assign the label $\gcd(f(u),f(v))$. f is called k-total prime cordial labeling of G if $|t_f(i) - t_f(j)| \leq 1$, $i,j \in \{1,2,\cdots,k\}$ where $t_f(x)$ denotes the total number of vertices and the edges labeled with x. A graph with a k-total prime cordial labeling is called k-total prime cordial graph. In this paper we investigate the 4-total prime cordial labeling of certain graphs like shadow graph, P_n^2, $T_n \odot K_2$ and subdivision of $T_n \odot K_1$.

Key Words: k-Total prime cordial labeling, Smarandachely k-total prime cordial labeling, corona, P_n^2, shadow graph.

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§1. Introduction

Graphs considered here are finite, simple and undirected. Ponraj et al. [4], have been introduced the concept of k-total prime cordial labeling and investigate the k-total prime cordial labeling of certain graphs. Also in [4, 5, 6, 7, 8, 9, 10, 12], the 4-total prime cordial labeling behavior of path, cycle, star, bistar, some complete graphs, comb, double comb, triangular snake, double triangular snake, ladder, friendship graph, flower graph, gear graph, Jelly fish, book, irregular triangular snake, prism, helm, dumbbell graph, sunflower graph, corona of irregular triangular snake, dragon, Möbius ladder, corona of some graphs and subdivision of some graphs. 3-total prime cordial labeling behavior of some graphs have been investigated [11]. In this paper we investigate the 4-total prime cordial labeling of certain graphs like shadow graph, P_n^2, $T_n \odot K_2$ and subdivision of $T_n \odot K_1$.

§2. Preliminary Results

Definition 2.1 *Let G_1, G_2 respectively be (p_1,q_1), (p_2,q_2) graphs. A corona of G_1 with G_2 is the graph $G_1 \odot G_2$ obtained by taking one copy of G_1 , p_1 copies of G_2 and joining the i^{th}*

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vertex of G_1 *by an edge to every vertex in the* i^{th} *copy of* G_2 *where* $1 \leq i \leq p_1$.

Definition 2.2 *A shadow graph* $D_2(G)$ *of a connected graph* G *is constructed by taking two copies of* G, G' *and* G'' *and joining each vertex* u' *in* G' *to the neighbors of the corresponding vertex* u'' *in* G''.

Definition 2.3 *If* $e = uv$ *is an edge of* G *then* e *is said to be subdivided when it is replaced by the edges* uw *and* wv. *The graph obtained by subdividing each edge of a graph* G *is called the subdivision graph of* G *and is denoted by* $S(G)$.

Definition 2.4 *For a simple connected graph* G *the square of graph* G *is denoted by* G^2 *and defined as the graph with the same vertex set as of* G *and two vertices are adjacent in* G^2 *if they are at a distance* 1 *or* 2 *apart in* G.

Theorem 2.5([4]) *A cycle* C_n *is 4-total prime cordial iff* $n \notin \{4, 6, 8\}$.

Remark 2.6 *A* 2*-total prime cordial graph is* 2*-total product cordial graph.*

§3. k-Total Prime Cordial Labeling

Definition 3.1 *Let* G *be a* (p, q) *graph. Let* $f : V(G) \to \{1, 2, \cdots, k\}$ *be a function where* $k \in \mathbb{N}$ *and* $k > 1$. *For each edge* uv, *assign the label* $\gcd(f(u),\ f(v))$. f *is called* k*-total prime cordial labeling of* G *if* $|t_f(i) - t_f(j)| \leq 1$, $i, j \in \{1, 2, \cdots, k\}$ *where* $t_f(x)$ *denotes the total number of vertices and the edges labeled with* x. *Conversely, a non-*k*-total prime cordial labeling of* G *is called a Smarandachely* k*-total prime cordial labeling* f, *i.e.,* $|t_f(i) - t_f(j)| \geq 2$ *for an integer pair* $\{i, j\}$, *where* $i, j \in \{1, 2, \cdots, k\}$.

A graph with a k*-total prime cordial labeling is called* k*-total prime cordial graph.*

Theorem 3.2 *If* $n \equiv 1 \pmod 4$, *then* P_n^2 *is* 4*-total prime cordial.*

Proof Let $u_1 u_2 \cdots u_n$ be the path. Let u_i is adjacent to u_{i+2}, $(1 \leq i \leq n-2)$. Clearly $\left|V(P_n^2)\right| + \left|E(P_n^2)\right| = 3n - 3$.

Let $n = 4r + 1$, $r \in \mathbb{N}$. Assign the label 4 to the vertices $u_1, u_2, \cdots, u_{r+1}$ and assign the label 2 to the vertices $u_{r+2}, u_{r+3}, \cdots, u_{2r+1}$. Next we assign the label 3 to the vertices $u_{2r+2}, u_{2r+3}, \cdots, u_{3r+2}$. Finally we assign the label 1 to the vertices $u_{3r+3}, u_{3r+4}, \cdots, u_{4r}$. It is easy to verify that $t_f(1) = t_f(2) = t_f(3) = t_f(4) = 3r$. □

Theorem 3.3 *The shadow graph of* P_n, $D_2(P_n)$ *is* 4*-total prime cordial iff* $n \notin \{2, 4\}$.

Proof Let $u_1 u_2 \cdots u_n$ and $v_1 v_2 \cdots v_n$ be the two copies of the path P_n. Let u_i is adjacent to v_{i+1} and v_i is adjacent to u_{i+1}, $(1 \leq i \leq n-1)$. Clearly $|V(D_2(P_n))| + |E(D_2(P_n))| = 6n - 4$.

Case 1. $n \equiv 0 \pmod 4$.

Let $n = 4r$, $r > 1$ and $r \in \mathbb{N}$. Assign the label 4 to the vertices $u_1, u_2, \cdots, u_r$ and assign the label 2 to the vertices $u_{r+1}, u_{r+2}, \cdots, u_{2r}$. Next we assign the label 3 to the vertices

$u_{2r+1}, u_{2r+2}, \cdots, u_{3r}$ then we assign the label 1 to the vertices $u_{3r+1}, u_{3r+2}, \cdots, u_{4r-2}$. Finally we assign the labels 4, 2 to the vertices u_{4r-1} and u_{4r} respectively. Now we move to the vertices v_i $(1 \leq i \leq n)$. Assign the label 4 to the vertices $v_1, v_2, \cdots, v_r$ and assign the label 2 to the vertices $v_{r+1}, v_{r+2}, \cdots, v_{2r-1}$. Next we assign the label 3 to the vertices $v_{2r}, v_{2r+1}, \cdots, v_{3r}$. Next we assign the label 1 to the vertices $v_{3r+1}, v_{3r+2}, \cdots, v_{4r-1}$. Finally we assign the label 4 to the vertex v_{4r}. Here $t_f(1) = t_f(2) = t_f(3) = t_f(4) = 6r - 1$.

Case 2. $n \equiv 1 \pmod 4$.

Let $n = 4r + 1$, $r > 1$ and $r \in \mathbb{N}$. As in Case 1, assign the label to the vertices u_i, v_i $(1 \leq i \leq 4r - 1)$. Finally we assign the labels 4, 3 respectively to the vertices u_{4r} and v_{4r}. Clearly $t_f(1) = t_f(4) = 6r + 1$ and $t_f(2) = t_f(3) = 6r$.

Case 3. $n \equiv 2 \pmod 4$.

Let $n = 4r + 2$, $r > 1$ and $r \in \mathbb{N}$. Assign the label to the vertices u_i $(1 \leq i \leq 4r - 3)$, v_i $(1 \leq i \leq 4r - 4)$ by in Case 1. Finally we assign the labels 4, 3, 2, 4, 3, 2, 4 respectively to the vertices u_{4r-2}, u_{4r-1}, u_{4r}, v_{4r-3}, v_{4r-2}, v_{4r-1} and v_{4r}. It is easy to verify that $t_f(1) = t_f(2) = t_f(3) = t_f(4) = 6r + 2$.

Case 4. $n \equiv 3 \pmod 4$.

Let $n = 4r + 3$, $r > 1$ and $r \in \mathbb{N}$. In this case, assign the label to the vertices u_i, v_i $(1 \leq i \leq 4r - 1)$ by in Case 3. Finally we assign the labels 3, 4 to the vertices u_{4r} and v_{4r} respectively. Here $t_f(1) = t_f(4) = 6r + 4$ and $t_f(2) = t_f(3) = 6r + 3$.

Case 5. $n = 2$.

Theorem 2.5 gives $n = 2$ is not a 4-total prime cordial.

Case 6. $n = 4$.

Suppose f is a 4-total prime cordial labeling of $D_2(P_4)$. Then $t_f(1) = t_f(2) = t_f(3) = t_f(4) = 5$. Under the labeling f, we have $t_f(4) = 5$. For this, it is easy to verify that 4 must be labeled to 3 consecutive vertices of $D_2(P_4)$. That is, 4 must be labeled to all the three vertices of an induced subpath P_3 of $D_2(P_4)$. Similarly for $t_f(3) = 5$, 3 must be labeled to all the three vertices of another induced subpath P_3' of $D_2(P_4)$ which is disjoint from P_3. Now, we have only two vertices are remaining in $D_2(P_4)$. If the two vertices are labeled by 2, then $t_f(2) > 5$ or $t_f(2) < 5$, according as 2 is labels of adjacent vertices (or) 2 is labels of non-adjacent vertices, a contradiction.

Case 7. $n = 4, 5, 6, 7$.

A 4-total prime cordial labeling follows from Table 1.

n	4	5	6	7
u_1	4	2	4	4
u_2	2	4	4	4

u_3	3	2	2	2
u_4		4	3	3
u_5		3	3	3
u_6			2	1
u_7				4
v_1	4	4	4	4
v_2	3	4	4	4
v_3	4	3	2	2
v_4		3	3	2
v_5		3	3	3
v_6			2	3
v_7				3

Table 1

This completes the proof. □

Theorem 3.4 *The corona of T_n with K_2, $T_n \odot K_2$ is 4-total prime cordial for all $n \geq 2$.*

Proof Let $u_1u_2\cdots u_n$ be the path and v_i is adjacent to u_i, u_{i+1}. Let x_i, y_i be the vertices adjacent to v_i and x_i, y_i be adjacent. Let z_i, w_i be the vertices adjacent to u_i and z_i, w_i be adjacent. Clearly $|V(T_n \odot K_2)| + |E(T_n \odot K_2)| = 15n - 9$.

Case 1. $n \equiv 0 \pmod 4$.

Let $n = 4r$, $r > 1$ and $r \in \mathbb{N}$. Assign the label 4 to the vertices $u_1, u_2, \cdots, u_r$ and assign the label 2 to the vertices $u_{r+1}, u_{r+2}, \cdots, u_{2r}$. Next we assign the label 3 to the vertices u_{2r+1}, $u_{2r+2}, \cdots$, u_{3r} then we assign the label 1 to the vertices $u_{3r+1}, u_{3r+2}, \cdots, u_{4r-1}$. Finally, we assign the label 2 to the vertices u_{4r}. Next we consider the vertices v_i $(1 \leq i \leq n-1)$. Assign the label 4 to the vertices $v_1, v_2, \cdots, v_r$ and assign the label 2 to the vertices $v_{r+1}, v_{r+2}, \cdots, v_{2r-1}$. Next we assign the label 3 to the vertices $v_{2r}, v_{2r+1}, \cdots, v_{3r-1}$. Then we assign the label 1 to the vertices $v_{3r}, v_{3r+1}, \cdots, v_{4r-2}$. Finally, we assign the label 2 to the vertices v_{4r-1}. Now we move to the vertices x_i, y_i $(1 \leq i \leq n-1)$. Assign the label 4 to the vertices $x_1, x_2, \cdots, x_r$ and $y_1, y_2, \cdots, y_r$ and assign the label 2 to the vertices $x_{r+1}, x_{r+2}, \cdots, x_{2r-1}$ and $y_{r+1}, y_{r+2}, \cdots, y_{2r-1}$. Next we assign the label 3 to the vertices $x_{2r}, x_{2r+1}, \cdots, x_{3r-1}$ and $y_{2r}, y_{2r+1}, \cdots, y_{3r-1}$. Finally we assign the label 1 to the vertices $x_{3r}, x_{3r+1}, \cdots, x_{4r-1}$ and $y_{3r}, y_{3r+1}, \cdots, y_{4r-1}$. Next we consider the vertices z_i, w_i $(1 \leq i \leq n)$. Assign the label 4 to the vertices $z_1, z_2, \cdots, z_r$ and $w_1, w_2, \cdots, w_r$ and assign the label 2 to the vertices $z_{r+1}, z_{r+2}, \cdots, z_{2r}$ and $w_{r+1}, w_{r+2}, \cdots, w_{2r}$. Next we assign the label 3 to the vertices $z_{2r+1}, z_{2r+2}, \cdots, z_{3r}$ and $w_{2r+1}, w_{2r+2}, \cdots, w_{3r}$ then we assign the label 1 to the vertices $z_{3r+1}, z_{3r+2}, \cdots, z_{4r-1}$ and $w_{3r+1}, w_{3r+2}, \cdots, w_{4r-1}$. Finally we assign the labels 1, 2 respectively to the vertices z_{4r} and

w_{4r}. Clearly $t_f(1) = 15r - 3$ and $t_f(2) = t_f(3) = t_f(4) = 15r - 2$.

Case 2. $n \equiv 1 \pmod 4$.

Let $n = 4r + 1$, $r > 1$ and $r \in \mathbb{N}$. As in Case 1, assign the label to the vertices u_i $(1 \le i \le n-1)$, v_i $(1 \le i \le n-2)$, x_i $(1 \le i \le n-2)$, y_i $(1 \le i \le n-2)$, z_i $(1 \le i \le n-2)$ and w_i $(1 \le i \le n-1)$. Finally we assign the labels 4, 4, 2, 1, 2, 3, 3 respectively to the vertices u_{4r}, v_{4r-1}, x_{4r-1}, y_{4r-1}, z_{4r-1}, z_{4r} and w_{4r}. Here $t_f(1) = t_f(2) = 15r + 2$ and $t_f(3) = t_f(4) = 15r + 1$.

Case 3. $n \equiv 2 \pmod 4$.

Let $n = 4r + 2$, $r > 1$ and $r \in \mathbb{N}$. As in Case 2, assign the label to the vertices u_i $(1 \le i \le n-1)$, v_i $(1 \le i \le n-2)$, x_i $(1 \le i \le n-2)$, y_i $(1 \le i \le n-2)$, z_i $(1 \le i \le n-1)$ and w_i $(1 \le i \le n-1)$. Finally we assign the labels 2, 4, 4, 3, 3, 3 to the vertices u_{4r}, v_{4r-1}, x_{4r-1}, y_{4r-1}, z_{4r} and w_{4r} respectively. It is easy to verify that $t_f(1) = 15r + 6$ and $t_f(2) = t_f(3) = t_f(4) = 15r + 5$.

Case 4. $n \equiv 3 \pmod 4$.

Let $n = 4r + 3$, $r > 1$ and $r \in \mathbb{N}$. As in Case 3, assign the label to the vertices u_i $(1 \le i \le n-4)$, v_i $(1 \le i \le n-4)$, x_i $(1 \le i \le n-4)$, y_i $(1 \le i \le n-4)$, z_i $(1 \le i \le n-4)$ and w_i $(1 \le i \le n-4)$. Now we assign the labels 2, 4, 3, 3, 2, 4, 3 respectively to the vertices u_{4r-3}, u_{4r-2}, u_{4r-1}, u_{4r}, v_{4r-3}, v_{4r-2}, v_{4r-1}. Next we assign the labels 2, 4, 1, 4, 4, 3 to the vertices x_{4r-3}, x_{4r-2}, x_{4r-1}, y_{4r-3}, y_{4r-2} and y_{4r-1} respectively. Finally we assign the labels 2, 3, 1, 4, 2, 3, 4, 1 respectively to the vertices z_{4r-3}, z_{4r-2}, z_{4r-1}, z_{4r}, w_{4r-3}, w_{4r-2}, w_{4r-1} and w_{4r}. Here $t_f(1) = t_f(2) = t_f(3) = t_f(4) = 15r + 9$.

Case 5. $n = 2, 3, 4, 5, 6, 7$.

A 4-total prime cordial labeling follows from Table 2.

n	2	3	4	5	6	7
u_1	4	4	4	4	4	4
u_2	3	2	2	2	4	4
u_3		3	3	3	2	2
u_4			1	2	3	2
u_5				2	3	3
u_6					3	3
u_7						1
v_1	4	4	4	4	4	4
v_2		2	2	2	4	4
v_3			3	3	2	2
v_4				1	3	3

v_5					1	3
v_6						1
x_1	2	4	4	4	4	4
x_2		3	2	2	4	4
x_3			3	3	2	2
x_4				1	3	3
x_5					1	3
x_6						1
y_1	1	4	4	4	4	4
y_2		3	1	1	2	2
y_3			3	3	2	2
y_4				1	3	3
y_5					1	3
y_6						1
z_1	4	4	4	4	4	4
z_2	3	1	2	2	2	4
z_3		3	3	3	2	2
z_4			1	4	3	3
z_5				3	3	3
z_6					1	3
z_7						1
w_1	2	2	4	4	4	4
w_2	3	1	2	2	2	2
w_3		3	3	3	1	2
w_4			1	4	3	3
w_5				3	1	4
w_6					1	4
w_7						1

Table 2

This completes the proof. □

Theorem 3.5 *The subdivision of $T_n \odot K_1$, $S(T_n \odot K_1)$ is 4-total prime cordial for all $n \geq 2$.*

Proof Let P_n be the path $u_1u_2\cdots u_n$. Let $v_1, v_2, \cdots, v_n$ be the vertices such that v_i is adjacent to both u_i and u_{i+1} $(1 \leq i \leq n-1)$. Let w_i be the pendent vertices adjacent to v_i $(1 \leq i \leq n-1)$. Let p_i be the pendent vertices adjacent to u_i $(1 \leq i \leq n)$. Let s_i, x_i, y_i, z_i, q_i be the vertices which subdivide the edge u_iu_{i+1}, u_iv_i, v_iu_{i+1}, v_iw_i, u_ip_i respectively. It is easy to show that $|V(S(T_n \odot K_1))| + |E(S(T_n \odot K_1))| = 19n - 14$.

Case 1. $n \equiv 0 \pmod 4$.

Let $n = 4r$, $r > 1$ and $r \in \mathbb{N}$. Assign the label 4 to the vertices $u_1, u_2, \cdots, u_r$ and assign the label 2 to the vertices $u_{r+1}, u_{r+2}, \cdots, u_{2r}$. Next we assign the label 3 to the vertices u_{2r+1}, u_{2r+2},$\cdots$, u_{3r} then we assign the label 1 to the vertices $u_{3r+1}, u_{3r+2}, \cdots, u_{4r-1}$. Finally, we assign the label 4 to the vertices u_{4r}. Next we consider the vertices v_i $(1 \leq i \leq n-1)$. Assign the label 4 to the vertices $v_1, v_2, \cdots, v_r$ and assign the label 2 to the vertices $v_{r+1}, v_{r+2}, \cdots, v_{2r-1}$. Next we assign the label 3 to the vertices $v_{2r}, v_{2r+1}, \cdots, v_{3r-1}$. Then we assign the label 1 to the vertices $v_{3r}, v_{3r+1}, \cdots, v_{4r-2}$. Finally we assign the label 3 to the vertices v_{4r-1}. Now we move to the vertices s_i $(1 \leq i \leq n-1)$. Assign the label 4 to the vertices $s_1, s_2, \cdots, s_r$ and assign the label 2 to the vertices $s_{r+1}, s_{r+2}, \cdots, s_{2r}$. Next we assign the label 3 to the vertices s_{2r+1}, s_{2r+2},$\ldots$, s_{3r} then we assign the label 1 to the vertices $s_{3r+1}, s_{3r+2}, \cdots, s_{4r-1}$. Next we consider the vertices x_i, y_i $(1 \leq i \leq n-1)$. Assign the label 4 to the vertices $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \cdots, y_{r-1}$ and assign the label 2 to the vertices $x_{r+1}, x_{r+2}, \cdots, x_{2r}$ and $y_r, y_{r+1}, \cdots, y_{2r-1}$. Next we assign the label 3 to the vertices $x_{2r+1}, x_{2r+2}, \cdots, x_{3r-1}$ and $y_{2r}, y_{2r+1}, \cdots, y_{3r-1}$. Finally we assign the label 1 to the vertices $x_{3r}, x_{3r+1}, \ldots, x_{4r-1}$ and $y_{3r}, y_{3r+1}, \ldots, y_{4r-1}$. Now we move to the vertices z_i, w_i $(1 \leq i \leq n-1)$. Assign the label 4 to the vertices $z_1, z_2, \cdots, z_r$ and $w_1, w_2, \cdots, w_r$ and assign the label 2 to the vertices $z_{r+1}, z_{r+2}, \cdots, z_{2r-1}$ and $w_{r+1}, w_{r+2}, \cdots, w_{2r-1}$. Next we assign the label 3 to the vertices $z_{2r}, z_{2r+1}, \cdots, z_{3r-1}$ and $w_{2r}, w_{2r+1}, \cdots, w_{3r-1}$ then we assign the label 1 to the vertices $z_{3r}, z_{3r+1}, \ldots, z_{4r-1}$ and w_{3r}, w_{3r+1},$\ldots$,w_{4r-1}. Next we consider the vertices p_i, q_i $(1 \leq i \leq n)$. Assign the label 4 to the vertices $p_1, p_2, \ldots, p_r$ and $q_1, q_2, \cdots, q_r$ and assign the label 2 to the vertices $p_{r+1}, p_{r+2}, \cdots, p_{2r}$ and $q_{r+1}, q_{r+2}, \cdots, q_{2r}$. Next we assign the label 3 to the vertices $p_{2r+1}, p_{2r+2}, \cdots, p_{3r}$ and $q_{2r+1}, q_{2r+2}, \cdots, q_{3r}$ then we assign the label 1 to the vertices p_{3r+1} ,p_{3r+2},$\cdots$,p_{4r} and q_{3r+1}, q_{3r+2},$\cdots$,q_{4r-1}. Finally we assign the label 2 to the vertex q_{4r}. Clearly $t_f(1) = t_f(2) = 19r - 4$ and $t_f(3) = t_f(4) = 19r - 3$.

Case 2. $n \equiv 1 \pmod 4$.

Let $n = 4r + 1$, $r > 1$ and $r \in \mathbb{N}$. As in Case 1, assign the label to the vertices u_i $(1 \leq i \leq n-1)$, v_i $(1 \leq i \leq n-2)$, s_i $(1 \leq i \leq n-2)$, x_i $(1 \leq i \leq n-2)$, y_i $(1 \leq i \leq n-2)$, z_i $(1 \leq i \leq n-2)$, w_i $(1 \leq i \leq n-2)$, p_i $(1 \leq i \leq n-1)$, and q_i $(1 \leq i \leq n-1)$. Finally we assign the labels 1, 3, 4, 4, 1, 3, 3, 2, 2 respectively to the vertices u_{4r}, v_{4r-1}, s_{4r-1}, x_{4r-1}, y_{4r-1}, z_{4r-1}, w_{4r-1}, p_{4r} and q_{4r}. Here $t_f(1) = t_f(2) = t_f(4) = 19r + 1$ and $t_f(3) = 19r + 2$.

Case 3. $n \equiv 2 \pmod 4$.

Let $n = 4r + 2$, $r > 1$ and $r \in \mathbb{N}$. Assign the label to the vertices u_i $(1 \leq i \leq n-1)$, v_i $(1 \leq i \leq n-2)$, s_i $(1 \leq i \leq n-2)$, x_i $(1 \leq i \leq n-2)$, y_i $(1 \leq i \leq n-2)$, z_i $(1 \leq i \leq n-2)$, w_i $(1 \leq i \leq n-2)$, p_i $(1 \leq i \leq n-1)$, and q_i $(1 \leq i \leq n-1)$ by in Case 2. Finally we assign the labels 1, 4, 3, 4, 4, 2, 2, 3, 2 to the vertices u_{4r}, v_{4r-1}, s_{4r-1}, x_{4r-1}, y_{4r-1}, z_{4r-1}, w_{4r-1},

p_{4r} and q_{4r} respectively. It is easy to verify that $t_f(1) = t_f(2) = t_f(3) = t_f(4) = 19r + 6$.

Case 4. $n \equiv 3 \pmod 4$.

Let $n = 4r + 3$, $r > 1$ and $r \in \mathbb{N}$. As in Case 3, assign the label to the vertices u_i $(1 \le i \le n-1)$, v_i $(1 \le i \le n-2)$, s_i $(1 \le i \le n-2)$, x_i $(1 \le i \le n-2)$, y_i $(1 \le i \le n-2)$, z_i $(1 \le i \le n-2)$, w_i $(1 \le i \le n-2)$, p_i $(1 \le i \le n-1)$, and q_i $(1 \le i \le n-1)$. Finally we assign the labels 2, 4, 3, 3, 2, 4, 4, 2, 1 respectively to the vertices u_{4r}, v_{4r-1}, s_{4r-1}, x_{4r-1}, y_{4r-1}, z_{4r-1}, w_{4r-1}, p_{4r} and q_{4r}. Here $t_f(1) = t_f(2) = t_f(4) = 19r + 11$ and $t_f(3) = 19r + 10$.

Case 5. $n = 2, 3, 4, 5, 6, 7$.

A 4-total prime cordial labeling follows from Table 3.

n	2	3	4	5	6	7
u_1	4	4	4	4	4	4
u_2	3	2	2	4	4	4
u_3		3	3	2	2	2
u_4			1	3	3	2
u_5				1	3	3
u_6					1	1
u_7						1
v_1	4	4	4	4	4	4
v_2		3	2	2	2	4
v_3			3	3	2	2
v_4				1	3	3
v_5					1	3
v_6						1
s_1	2	4	4	4	4	4
s_2		3	3	2	2	4
s_3			1	3	2	2
s_4				3	3	3
s_5					1	3
s_6						1
x_1	4	4	4	4	4	4
x_2		2	2	2	2	4
x_3			3	3	2	2
x_4				1	3	3
x_5					1	3
x_6						1

y_1	3	2	2	4	4	4
y_2		3	3	3	2	2
y_3			1	3	1	2
y_4				1	3	3
y_5					1	3
y_6						1
z_1	2	4	4	4	4	4
z_2		1	2	2	2	2
z_3			3	3	1	2
z_4				1	3	3
z_5					1	3
z_6						1
w_1	2	4	4	4	4	4
w_2		1	3	2	2	2
w_3			2	3	4	2
w_4				1	3	3
w_5					3	3
w_6						1
p_1	1	2	4	4	4	4
p_2	4	1	2	2	4	4
p_3		3	3	2	2	2
p_4			1	3	3	3
p_5				1	3	3
p_6					1	1
p_7						1
q_1	3	2	4	4	4	4
q_2	3	1	2	2	4	4
q_3		3	3	2	2	2
q_4			1	3	3	2
q_5				1	3	3
q_6					1	1
q_7						1

Table 3

This completes the proof. □

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Super F-Centroidal Mean Graphs

S.Arockiaraj[1], A.Rajesh Kannan[2] and A.Durai Baskar[3]

1. Department of Mathematics, Government Arts & Science College, Sivakasi-626124, Tamil Nadu, India
2. Department of Mathematics, Mepco Schlenk Engineering College, Sivakasi- 626 005, Tamil Nadu, India
3. Department of Mathematics, C.S.I. Jayaraj Annapackiam College, Nallur - 627853, Tamil Nadu, India

E-mail: psarockiaraj@gmail.com, rajmaths@gmail.com, a.duraibaskar@gmail.com

Abstract: Let G be a graph and $f : V(G) \to \{1, 2, 3, \ldots, p + q\}$ be an injection. For each uv, the induced edge labeling f^* is defined as

$$f^*(uv) = \left\lfloor \frac{2\,[f(u)^2 + f(u)f(v) + f(v)^2]}{3\,[f(u) + f(v)]} \right\rfloor.$$

Then f is called a super F-centroidal mean labeling if $f(V(G)) \cup \{f^*(uv) : uv \in E(G)\} = \{1, 2, 3, \ldots, p+q\}$. A graph that admits a super F-centroidal mean labeling is called a super F-centroidal mean graph. In this paper, the super F-centroidal meanness of some standard graphs have been studied.

Key Words: F-centroidal mean graph, super F-centroidal mean labeling, Smarandachely super F-centroidal mean labeling, super F-centroidal mean graph.

AMS(2010): 05C78.

§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected and simple graph. Let $G(V, E)$ be a graph with p vertices and q edges. For notations and terminology, we follow [7]. For a detailed survey on graph labeling, we refer [6].

Path on n vertices is denoted by P_n and a cycle on n vertices is denoted by C_n. A star graph S_n is the complete bipartite graph $K_{1,n}$. The union $G_1 \cup G_2$ of any two graphs G_1 and G_2 with disjoint vertex sets, has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The middle graph $M(G)$ of a graph G is the graph whose vertex set is $\{v : v \in V(G)\} \cup \{e : e \in E(G)\}$ and the edge set is $\{e_1e_2 : e_1, e_2 \in E(G)$ and e_1 and e_2 are adjacent edges of $G\} \cup \{ve : v \in V(G), e \in E(G)$ and e is incident with $v\}$. The graph $G \circ S_m$ is obtained from G by attaching m pendant vertices to each vertex of G. A Twig $TW(P_n), n \geq 3$ is a graph obtained from a path by attaching exactly two pendant vertices to each internal vertices of the path P_n. A subdivision of a graph G, denoted by $S(G)$, is a graph obtained by subdividing edge of G by a vertex. An arbitrary subdivision of a graph G is a graph obtained from G by a sequence of elementary subdivisions forming edges into paths through new vertices of degree 2. Square of a graph G,

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denoted by G^2, has the vertex set as in G and two vertices are adjacent in G^2 if they are at a distance either 1 or 2 apart in G. The baloon of a graph G, $P_n(G)$ is the graph obtained from G by identifying an end vertex of P_n at a vertex of G. The graph $P_n(C_m)$ is called a dragon.

The concept of geometric mean labeling [1] and super geometric mean labeling [2] were introduced by Durai Baskar et al. and studied for some standard graphs. Arockiaraj et al. introduced the concept of F-root square labeling [3] and super F-root square labeling [4]. The concept of F-centroidal mean labeling [5] was introduced and developed its meanness for some standard graphs.

Arockiaraj et al. [5], defined the F-centroidal mean labeling as follows:

A function f is called an F-centroidal mean labeling of a graph $G(V, E)$ with p vertices and q edges if $f : V(G) \to \{1, 2, 3, \cdots, q+1\}$ is injective and the induced function $f^* : E(G) \to \{1, 2, 3, \cdots, q\}$ defined as

$$f^*(uv) = \left\lfloor \frac{2\ [f(u)^2 + f(u)f(v) + f(v)^2]}{3\ [f(u) + f(v)]} \right\rfloor$$

is bijective for all $uv \in E(G)$. A graph that admits an F-centroidal mean labeling is called an F-centroidal mean graph. Motivated by the works of so many authors in the area of graph labeling, we introduced a new type of labeling called a super F-centroidal mean labeling.

Let G be a graph and $f : V(G) \to \{1, 2, 3, \cdots, p+q\}$ be an injection. For each uv, the induced edge labeling f^* is defined as

$$f^*(uv) = \left\lfloor \frac{2\ [f(u)^2 + f(u)f(v) + f(v)^2]}{3\ [f(u) + f(v)]} \right\rfloor.$$

Then f is called a super F-centroidal mean labeling if $f(V(G)) \cup \{f^*(uv) : uv \in E(G)\} = \{1, 2, 3, \cdots, p+q\}$. A graph that admits a super F-centroidal mean labeling is called a super F-centroidal mean graph. Generally, let $C \subset \{1, 2, 3, \cdots, p+q\}$. If $f(V(G)) \bigcup \{f^*(uv) : uv \in E(G)\} = \{1, 2, 3, \cdots, p+q\} \setminus C$, such a f is called a Smarandachely super F-centroidal mean labeling on C. Clearly, if $C = \emptyset$, a Smarandachely super F-centroidal mean labeling on C is nothing else but the super F-centroidal mean labeling on G.

A super F-centroidal mean labeling of the graph C_4 is shown in Figure 1.

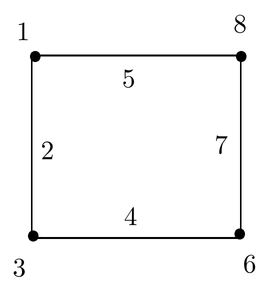

Figure 1 A super F-centroidal mean labeling of C_4

In this paper, we have studied the super F-centroidal meanness of some standard graphs.

§2. Main Results

Theorem 2.1 *A union of any number of paths is a super F-centroidal mean graph.*

Proof Let the graph G be the union of k paths. Let $\{v_j^{(i)} : 1 \leq j \leq p_i\}$ be the vertices of the i^{th} path P_{p_i} with $p_i \geq 2$ and $1 \leq i \leq k$. Define $f : V(G) \rightarrow \left\{1, 2, 3, \cdots, \sum\limits_{i=1}^{k} 2p_i - k\right\}$ as follows:

$$f(v_j^{(1)})) = 2j - 1, \text{ for } 1 \leq j \leq p_1 \text{ and}$$
$$f\left(v_j^{(i)}\right) = f(v_{p_{i-1}}^{(i-1)}) + 2j - 1, \text{ for } 2 \leq i \leq k \text{ and } 1 \leq j \leq p_i.$$

Then the induced edge labeling f^* is obtained as follows:

$$f^*(v_j^{(1)} v_{j+1}^{(1)}) = 2j, \text{ for } 1 \leq j \leq p_1 - 1 \text{ and}$$
$$f^*\left(v_j^{(i)} v_{j+1}^{(i)}\right) = f(v_{p_{i-1}}^{(i-1)}) + 2j, \text{ for } 2 \leq i \leq k \text{ and } 1 \leq j \leq p_1 - 1.$$

Hence, f is a super F-centroidal mean labeling of G. Thus the graph G is a super F-centroidal mean graph. □

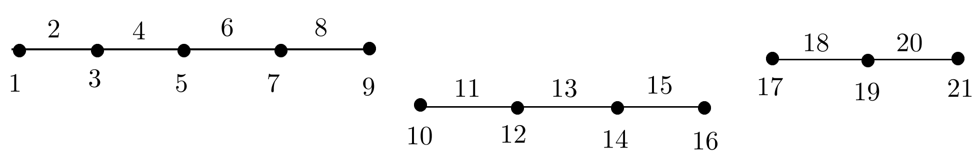

Figure 2 A super F-centroidal mean labeling of union of P_5, P_4 and P_3

Corollary 2.2 *Every path P_n is a super F-centroidal mean graph, for $n \geq 1$.*

Theorem 2.3 *The middle graph $M(P_n)$ of a path P_n is a super F-centroidal mean graph, for $n \geq 4$.*

Proof Let $V(P_n) = \{v_1, v_2, v_3, \cdots, v_n\}$ and $E(P_n) = \{e_i = v_i v_{i+1} : 1 \leq i \leq n-1\}$ be the vertex set and edge set of the path P_n. Then,

$$V(M(P_n)) = \{v_1, v_2, v_3, \ldots, v_n, e_1, e_2, e_3, \ldots, e_{n-1}\} \text{ and}$$
$$E(M(P_n)) = \{v_i e_i, e_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{e_i e_{i+1} : 1 \leq i \leq n-2\}.$$

Define $f : V(M(P_n)) \rightarrow \{1, 2, 3, \cdots, 5n-5\}$ as follows:

$$f(v_i) = 5i - 4, \text{ for } 1 \leq i \leq n-1,$$
$$f(v_n) = 5n - 5 \text{ and}$$
$$f(e_i) = 5i - 2, \text{ for } 1 \leq i \leq n-1.$$

Then, the induced edge labeling f^* is obtained as follows:

$$f^*(v_ie_i) = 5i-3, \text{ for } 1 \le i \le n-1$$
$$f^*(e_iv_{i+1}) = 5i-1, \text{ for } 1 \le i \le n-1$$
$$f^*(e_ie_{i+1}) = 5i, \text{ for } 1 \le i \le n-2.$$

Hence f is a super F-centroidal mean labeling of $M(P_n)$. Thus the middle graph $M(P_n)$ of a path P_n is a super F-centroidal mean graph, for $n \geq 4$. □

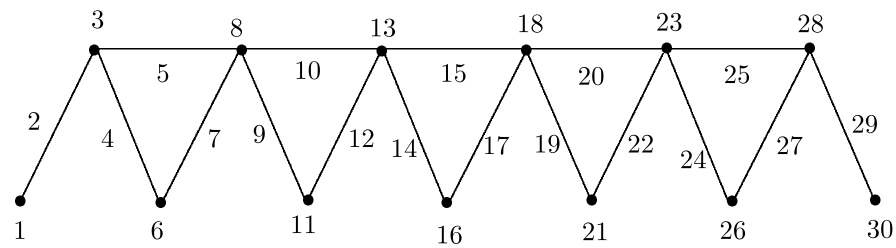

Figure 3 A super F-centroidal mean labeling of $M(P_7)$

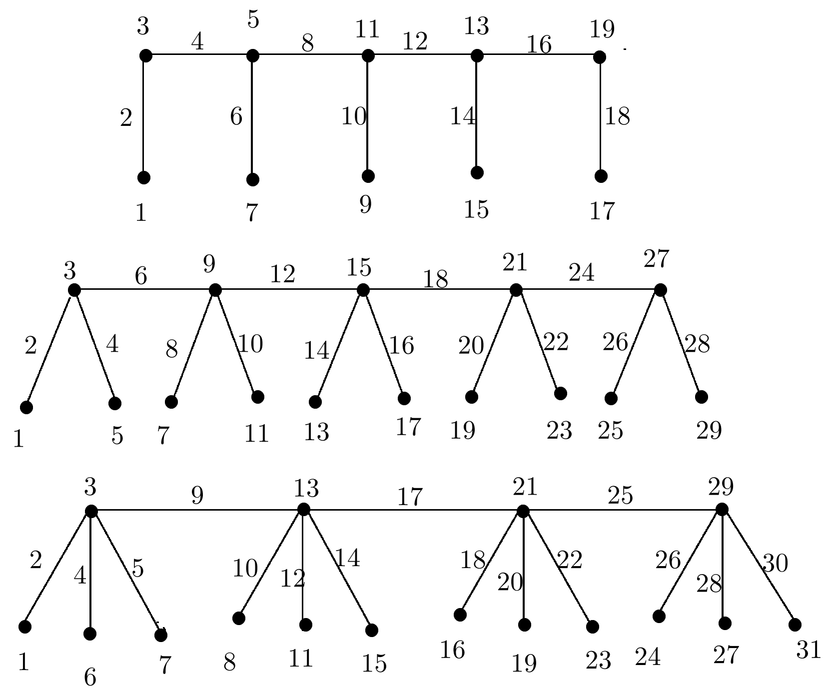

Figure 4. A super F-centroidal mean labeling of $P_5 \circ S_1$, $P_6 \circ S_2$ and $P_4 \circ S_3$

Theorem 2.4 *The graph $P_n \circ S_m$ is a super F-centroidal mean graph, for $n \geq 1$ and $m \leq 3$.*

Proof Let $u_1, u_2, \cdots, u_n$ be the vertices of the path P_n and $v_1^{(i)}, v_2^{(i)}, \ldots, v_m^{(i)}$ be the pendant vertices attached at each vertex u_i of the path P_n, for $1 \leq i \leq n$.

Case 1. $m = 1$.

Define $f : V(P_n \circ S_1) \to \{1, 2, 3, \cdots, 4n-1\}$ as follows:

$$f(u_i) = \begin{cases} 4i-1, & 1 \le i \le n \text{ and } i \text{ is odd} \\ 4i-3, & 2 \le i \le n \text{ and } i \text{ is even and} \end{cases}$$

$$f(v_1^{(i)}) = \begin{cases} 4i-3, & 1 \le i \le n \text{ and } i \text{ is odd} \\ 4i-1, & 2 \le i \le n \text{ and } i \text{ is even.} \end{cases}$$

Then, the induced edge labeling f^* is obtained as follows:

$$f^*(u_i u_{i+1}) = 4i, \text{ for } 1 \le i \le n-1 \text{ and}$$
$$f^*(v_1^{(i)} u_i) = 4i-2, \text{ for } 1 \le i \le n.$$

Case 2. $m = 2$.

Define $f : V(P_n \circ S_2) \to \{1, 2, 3, \cdots, 6n-1\}$ as follows:

$$f(u_i) = 6i-3, \text{ for } 1 \le i \le n,$$
$$f(v_1^{(i)}) = 6i-5, \text{ for } 1 \le i \le n \text{ and}$$
$$f(v_2^{(i)}) = 6i-1, \text{ for } 1 \le i \le n.$$

Then, the induced edge labeling f^* is obtained as follows:

$$f^*(u_i u_{i+1}) = 6i, \text{ for } 1 \le i \le n-1,$$
$$f^*(v_1^{(i)} u_i) = 6i-4, \text{ for } 1 \le i \le n \text{ and}$$

$$f^*(v_2^{(i)} u_i) = 6i-2, \text{ for } 1 \le i \le n.$$

Case 3. $m = 3$.

Define $f : V(P_n \circ S_3) \to \{1, 2, 3, \cdots, 8n-1\}$ as follows:

$$f(u_i) = \begin{cases} 3, & i = 1 \\ 8i-3, & 2 \le i \le n, \end{cases}$$

$$f(v_1^{(i)}) = \begin{cases} 1, & i = 1 \\ 8i-8, & 2 \le i \le n, \end{cases}$$

$$f(v_2^{(i)}) = \begin{cases} 6, & i = 1 \\ 8i-5, & 2 \le i \le n \text{ and} \end{cases}$$

$$f(v_3^{(i)}) = 8i-1, \text{ for } 1 \le i \le n.$$

Then, the induced edge labeling f^* is obtained as follows:

$$f^*(u_iu_{i+1}) = 8i+1, \text{ for } 1 \leq i \leq n-1,$$
$$f^*(v_1^{(i)}u_i) = 8i-6, \text{ for } 1 \leq i \leq n,$$
$$f^*(v_2^{(i)}u_i) = 8i-4, \text{ for } 1 \leq i \leq n \text{ and}$$
$$f^*(v_3^{(i)}u_i) = \begin{cases} 5, & i=1 \\ 8i-2, & 2 \leq i \leq n. \end{cases}$$

In each case, f is a super F-centroidal mean labeling of $P_n \circ S_m$. Thus the graph $P_n \circ S_m$ is a super F-centroidal mean graph, for $n \geq 1$ and $m \leq 3$. □

Theorem 2.5 *The twig graph $TW(P_n)$ of the path P_n is a super F-centroidal mean graph, only when $n \geq 4$.*

Proof Let $v_1, v_2, v_3, \cdots, v_n$ be the vertices of the path P_n and $u_1^{(i)}, u_2^{(i)}$ be the pendant vertices at each vertex v_i, for $2 \leq i \leq n-1$.

Assume that $n \geq 4$.

Define $f : V(TW(P_n)) \to \{1, 2, 3, \cdots, 6n-9\}$ as follows:

$$f(v_i) = \begin{cases} 2i-1, & 1 \leq i \leq 2 \\ 6i-7, & 3 \leq i \leq n-1 \\ 6i-9, & i=n, \end{cases}$$

$$f(u_1^{(i)}) = \begin{cases} 6, & i=2 \\ 6i-9, & 3 \leq i \leq n-1 \end{cases} \text{ and}$$

$$f(u_2^{(i)}) = \begin{cases} 8, & i=2 \\ 6i-5, & 3 \leq i \leq n-2 \\ 6i-4, & i=n-1. \end{cases}$$

Then, the induced edge labeling f^* is obtained as follows:

$$f^*(v_iv_{i+1}) = \begin{cases} 5i-3, & 1 \leq i \leq 2 \\ 6i-4, & 3 \leq i \leq n-2, \end{cases}$$
$$f^*(v_{n-1}v_n) = 6n-11,$$
$$f^*(v_iu_1^{(i)}) = 6i-8, \text{ for } 2 \leq i \leq n-1 \text{ and}$$
$$f^*(v_iu_2^{(i)}) = \begin{cases} 5, & i=2 \\ 6i-6, & 3 \leq i \leq n-1. \end{cases}$$

Hence f is a super F-centroidal mean labeling of $TW(P_n)$. Thus the twig graph $TW(P_n)$ is a super F-centroidal mean graph, for $n \geq 4$. □

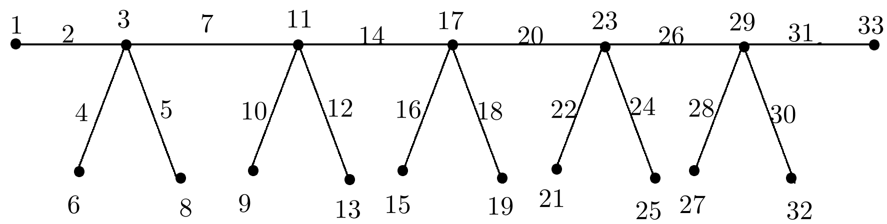

Figure 5. A super F-centroidal mean labeling of $TW(P_7)$

Theorem 2.6 *The graph $[P_n; S_1]$ is a super F-centroidal mean graph, for $n \geq 1$.*

Proof Let $u_1, u_2, u_3, \cdots, u_n$ be the vertices of the path P_n and $v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, \cdots, v_{m+1}^{(i)}$ be the vertices of the star graph S_m such that $v_1^{(i)}$ is the central vertex of the star graph $S_m, 1 \leq i \leq n$.

Assume that $m = 1$. Define $f : V([P_n; S_1]) \to \{1, 2, 3, \cdots, 6n - 1\}$ as follows:

$$f(u_i) = 6i - 1, \text{ for } 1 \leq i \leq n,$$
$$f(v_1^{(i)}) = 6i - 3, \text{ for } 1 \leq i \leq n \text{ and}$$
$$f(v_2^{(i)}) = \begin{cases} 1, & i = 1 \\ 6i - 6, & 2 \leq i \leq n. \end{cases}$$

Then, the induced edge labeling f^* is obtained as follows:

$$f^*(u_i u_{i+1}) = 6i + 2, \text{ for } 1 \leq i \leq n - 1,$$
$$f^*(u_i v_1^{(i)}) = 6i - 2, \text{ for } 1 \leq i \leq n \text{ and}$$
$$f^*(v_1^{(i)} v_2^{(i)}) = \begin{cases} 2, & i = 1 \\ 6i - 5, & 2 \leq i \leq n. \end{cases}$$

Hence f is a super F-centroidal mean labeling of $[P_n; S_1]$. Thus the graph $[P_n; S_1]$ is a super F-centroidal mean graph, for $n \geq 1$. □

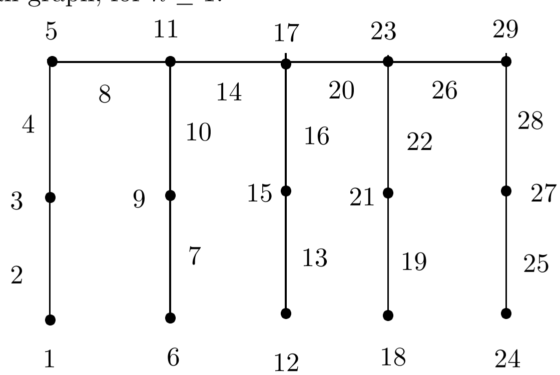

Figure 6 A super F-centroidal mean labeling of $[P_5; S_1]$

Theorem 2.7 *Arbitrary subdivision of $K_{1,3}$ is a super F-centroidal mean graph.*

Proof Let G be the graph of arbitrary subdivision of $K_{1,3}$. Let v_0, v_1, v_2 and v_3 be the vertices of $K_{1,3}$ in which v_0 is the central vertex and v_1, v_2 and v_3 are the pendent vertices of $K_{1,3}$. Let the edges v_0v_1, v_0v_2 and v_0v_3 of S_3 be subdivided by p_1, p_2 and p_3 number of vertices respectively.

Let $v_0, v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \cdots, v_{p_1+1}^{(1)} = v_1, v_0, v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \cdots, v_{p_2+1}^{(2)} = v_2$ and $v_0, v_1^{(3)}, v_2^{(3)}, v_3^{(3)}, \cdots, v_{p_3+1}^{(3)} (= v_3)$ be the vertices of G and $v_0 = v_0^{(i)}$ for $1 \leq i \leq 3$.

Let $e_j^{(i)} = v_{j-1}^{(i)} v_j^{(i)}$ for $1 \leq j \leq p_i + 1$ and $1 \leq i \leq 3$ be the edges with G and it has $p_1 + p_2 + p_3 + 4$ vertices and $p_1 + p_2 + p_3 + 3$ edges with $p_1 \leq p_2 \leq p_3$.

Case 1. $p_1 = p_2,\ p_1 \geq 1$ and $p_3 \geq 3$.

Define $f : V(G) \to \{1, 2, 3, \cdots, 2(p_1 + p_2 + p_3) + 7\}$ as follows:

$$f(v_0) = 2(p_1 + p_2) + 5,$$

$$f(v_j^{(1)}) = \begin{cases} 2(p_1 + p_2), & j = 1 \\ 2(p_1 + p_2) + 5 - 4j, & 2 \leq j \leq p_1 + 1, \end{cases}$$

$$f(v_j^{(2)}) = \begin{cases} 2(p_1 + p_2) + 7 - 4j, & 1 \leq j \leq 2 \\ 2(p_1 + p_2) + 6 - 4j, & 3 \leq j \leq p_2 + 1 \text{ and} \end{cases}$$

$$f^*(v_j^{(3)}) = 2(p_1 + p_2) + 5 + 2j \text{ for } 1 \leq j \leq p_3 + 1.$$

Then, the induced edge labeling f^* is obtained as follows:

$$f^*(v_0 v_1^{(i)}) = 2(p_1 + p_2) + 2i, \text{ for } 1 \leq i \leq 3,$$

$$f^*(v_j^{(1)} v_{j+1}^{(1)}) = \begin{cases} 2(p_1 + p_2) - 2, & j = 1 \\ 2(p_1 + p_2) + 3 - 4j, & 2 \leq j \leq p_1, \end{cases}$$

$$f^*(v_j^{(2)} v_{j+1}^{(2)}) = \begin{cases} 2(p_1 + p_2) + 1, & j = 1 \\ 2(p_1 + p_2) + 4 - 4j, & 2 \leq j \leq p_2 \ \ \textit{and} \end{cases}$$

$$f^*(v_j^{(3)} v_{j+1}^{(3)}) = 2(p_1 + p_2) + 6 + 2j, \text{ for } 1 \leq j \leq p_3.$$

Case 2. $p_1 < p_2$.

Define $f : V(G) \to \{1, 2, 3, \cdots, 2(p_1 + p_2 + p_3) + 7\}$ as follows:

$$f(v_0) = 2(p_1 + p_2) + 5,$$

$$f(v_j^{(1)}) = \begin{cases} 2(p_1+p_2)+3, & j=1 \\ 2(p_1+p_2)+8-4j, & 2 \le j \le p_1+1, \end{cases}$$

$$f(v_j^{(2)}) = \begin{cases} 2(p_1+p_2)+3-4j, & 1 \le j \le p_1 \\ 2p_2+3-2j, & p_1+1 \le j \le p_2+1 \text{ and} \end{cases}$$

$$f^*\left(v_j^{(3)}\right) = 2(p_1+p_2)+5+2j \text{ for } 1 \le j \le p_3+1.$$

Then, the induced edge labeling f^* is obtained as follows:

$$f^*(v_0 v_1^{(i)}) = 2(p_1+p_2)+2i, \text{ for } 1 \le i \le 3,$$

$$f^*(v_j^{(1)} v_{j+1}^{(1)}) = \begin{cases} 2(p_1+p_2)+5-4j, & j=1 \\ 2(p_1+p_2)+6-4j, & 2 \le j \le p_1, \end{cases}$$

$$f^*(v_j^{(2)} v_{j+1}^{(2)}) = \begin{cases} 2(p_1+p_2)+1-4j, & 1 \le j \le p_1-1 \\ 2p_2+2-2j, & p_1 \le j \le p_2 \quad \text{and} \end{cases}$$

$$f^*(v_j^{(3)} v_{j+1}^{(3)}) = 2(p_1+p_2)+6+2j, \text{ for } 1 \le j \le p_3.$$

In both cases, f is a super F-centroidal mean labeling of the arbitrary subdivision of S_3.

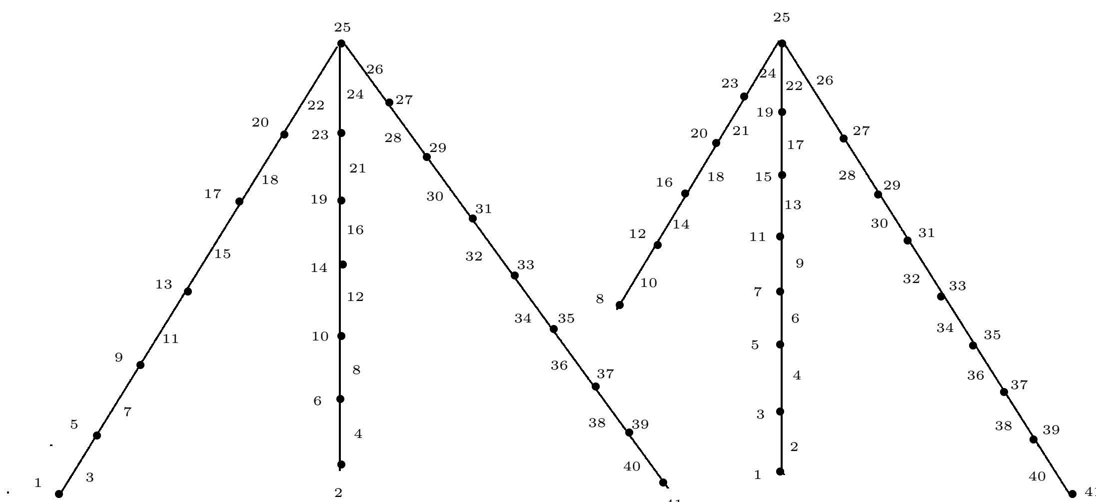

Figure 7. A super F-centroidal mean labeling of G with $p_1 = p_2 = 5, p_3 = 7$ and $p_1 = 4, p_2 = 6, p_3 = 7$

The graphs does not fall on the Case 1 are found to be a super F-centroidal mean graphs whose super F-centroidal mean labeling is shown in Figure 8. □

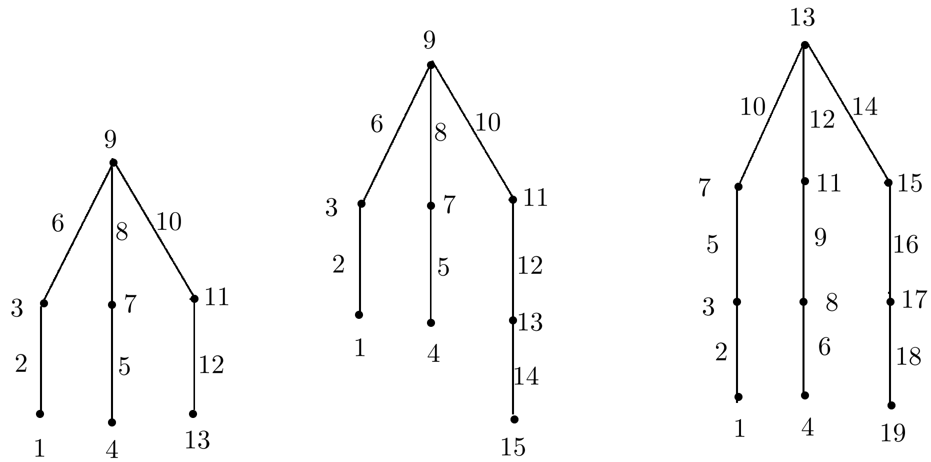

Figure 8 A super F-centroidal mean labeling of G with
$p_1 = p_2 = p_3 = 1$, $p_1 = p_2 = 1, p_3 = 2$ and $p_1 = p_2 = p_3 = 2$

Theorem 2.8 *Every cycle C_n is a super F-centroidal mean graph, for $n \geq 4$.*

Proof Let $u_1, u_2, \cdots, u_n$ be the vertices of the cycle C_n. Assume that $n \geq 5$.

A vertex labeling $f : V(C_n) \to \{1, 2, 3, \cdots, 2n\}$ is defined as

$$f(u_i) = \begin{cases} 1, & i = 1 \\ 4i - 4, & 2 \leq i \leq \lfloor \frac{n}{2} \rfloor + 1 \text{ and } n \text{ is odd} \\ 4i - 6, & i = \lfloor \frac{n}{2} \rfloor + 2 \text{ and } n \text{ is odd} \\ 4n - 4i + 5, & \lfloor \frac{n}{2} \rfloor + 3 \leq i \leq n \text{ and } n \text{ is odd} \\ 4i - 5, & 2 \leq i \leq \lfloor \frac{n}{2} \rfloor \text{ and } n \text{ is even} \\ 4i - 4, & i = \lfloor \frac{n}{2} \rfloor + 1 \text{ and } n \text{ is even} \\ 4n - 4i + 6, & \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n \text{ and } n \text{ is even.} \end{cases}$$

Then, the induced edge labeling f^* is obtained as follows:

$$f^*(u_i u_{i+1}) = \begin{cases} 2, & i = 1 \text{ and } n \text{ is odd} \\ 4i - 2, & 2 \leq i \leq \lfloor \frac{n}{2} \rfloor \text{ and } n \text{ is odd} \\ 4i - 3, & i = \lfloor \frac{n}{2} \rfloor + 1 \text{ and } n \text{ is odd} \\ 4n - 4i + 3, & \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n - 1 \text{ and } n \text{ is odd} \\ 3i - 1, & 1 \leq i \leq 2 \text{ and } n \text{ is even} \\ 4i - 3, & 3 \leq i \leq \lfloor \frac{n}{2} \rfloor \text{ and } n \text{ is even} \\ 4i - 5, & i = \lfloor \frac{n}{2} \rfloor + 1 \text{ and } n \text{ is even} \\ 4n - 4i + 4, & \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n - 1 \text{ and } n \text{ is even and} \end{cases}$$

$$f^*(u_n u_1) = \begin{cases} 3, & n \text{ is odd} \\ 4, & n \text{ is even.} \end{cases}$$

Hence f is a super F-centroidal mean labeling of C_n, for $n \geq 5$.
Thus the graph C_n is a super F-centroidal mean graph, for $n \geq 5$.

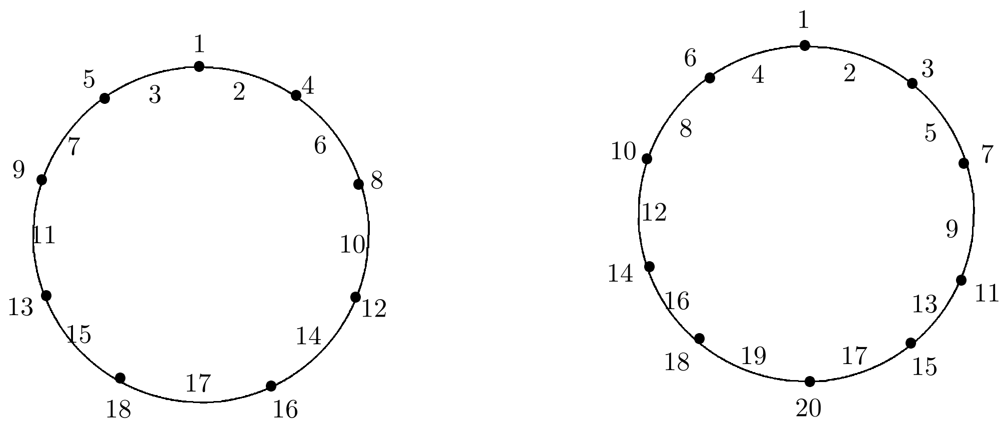

Figure 9 A super F-centroidal mean labeling of C_9 and C_{10}

For $n = 4$, a super F-centroidal mean labeling of C_4, is shown in Figure 1. But, the graph C_3 is not a super F-centroidal mean graph. □

Theorem 2.9 *$P_n \cup C_m$ is a super F-centroidal mean graph, for $n \geq 1$ and $m \geq 3$.*

Proof Let $u_1, u_2, \cdots, u_m$ and $v_1, v_2, \cdots, v_n$ be the vertices of the cycle C_m and the path P_n respectively.

Case 1. $m \geq 4$.

Define $f : V(P_n \cup C_m) \to \{1, 2, 3, \cdots, 2m + 2n - 1\}$ as follows:

$$f(u_i) = \begin{cases} 1, & i = 1 \\ 4i - 4, & 2 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor \\ 2m - 3, & i = \left\lfloor \frac{m}{2} \right\rfloor + 1 \text{ and } m \text{ is odd} \\ 2m, & i = \left\lfloor \frac{m}{2} \right\rfloor + 1 \text{ and } m \text{ is even} \\ 2m, & i = \left\lfloor \frac{m}{2} \right\rfloor + 2 \text{ and } m \text{ is odd} \\ 2m - 3, & i = \left\lfloor \frac{m}{2} \right\rfloor + 2 \text{ and } m \text{ is even} \\ 4m + 5 - 4i, & \left\lfloor \frac{m}{2} \right\rfloor + 3 \leq i \leq n \text{ and } n \text{ is even and} \end{cases}$$

$f(v_i) = 2m + 2i - 1$, for $1 \leq i \leq n$.

Then, the induced edge labeling f^* is obtained as follows:

$$f^*(u_iu_{i+1}) = \begin{cases} 4i-2, & 1 \le i \le \left\lfloor \frac{m}{2} \right\rfloor \\ 2m-1, & i = \left\lfloor \frac{m}{2} \right\rfloor + 1 \\ 2m-2, & i = \left\lfloor \frac{m}{2} \right\rfloor + 2 \text{ and } m \text{ is odd} \\ 2m-5, & i = \left\lfloor \frac{m}{2} \right\rfloor + 2 \text{ and } m \text{ is even} \\ 4m+3-4i, & \left\lfloor \frac{m}{2} \right\rfloor + 3 \le i \le m-1, \end{cases}$$

$$f^*(u_1u_m) = 3 \text{ and}$$
$$f^*(v_iv_{i+1}) = 2m + 2i, \text{ for } 1 \le i \le n-1.$$

Case 2. $m = 3$.

Define $f : V(P_n \cup C_3) \to \{1, 2, 3, \cdots, 2n+5\}$ as follows:

$$f(v_i) = 2i - 1, \text{ for } 1 \le i \le n,$$
$$f(u_1) = 2n,$$
$$f(u_2) = 2n + 3 \text{ and}$$
$$f(u_3) = 2n + 5.$$

Then, the induced edge labeling f^* is obtained as follows:

$$f^*(v_iv_{i+1}) = 2i, \text{ for } 1 \le i \le n-1,$$
$$f^*(u_1u_2) = 2n + 1,$$
$$f^*(u_2u_3) = 2n + 4 \text{ and}$$
$$f^*(u_1u_3) = 2n + 2.$$

Hence f is a super F-centroidal mean labeling of $P_n \cup C_m$. Thus the graph $P_n \cup C_m$ is a super F-centroidal mean graph for $n \ge 1$ and $m \ge 3$. □

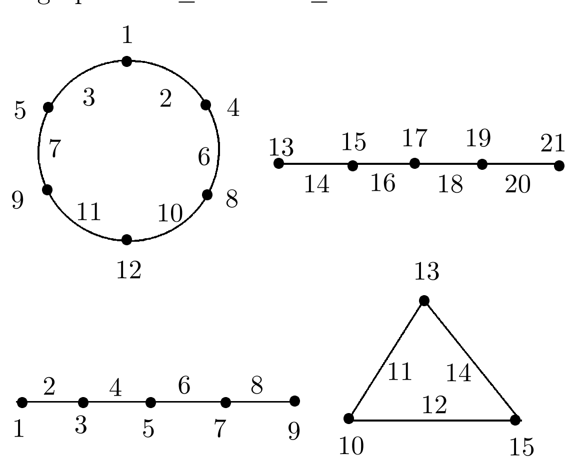

Figure 10 A super F-centroidal mean labeling of $P_5 \cup C_6$ and $P_4 \cup C_3$

Theorem 2.10 *P_n^2 is a super F-centroidal mean graph, for $n \geq 3$.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of the path P_n. Assume that $n \neq 5$. Define $f : V(P_n^2) \to \{1, 2, 3, \cdots, 3n-3\}$ as follows:

$$f(v_i) = \begin{cases} 3i-2, & 1 \leq i \leq n-2 \text{ and } i \text{ is odd} \\ 3i-3, & 1 \leq i \leq n-2 \text{ and } i \text{ is even,} \end{cases}$$

$$f(v_{n-1}) = 3n-5 \text{ and}$$

$$f(v_n) = 3n-3.$$

Then, the induced edge labeling f^* is obtained as follows:

$$f^*(v_iv_{i+1}) = 3i-1, \text{ for } 1 \leq i \leq n-1,$$

$$f^*(v_iv_{i+2}) = \begin{cases} 3i+1, & 1 \leq i \leq n-4 \text{ and } i \text{ is odd} \\ 3i, & 2 \leq i \leq n-4 \text{ and } i \text{ is even,} \end{cases}$$

$$f^*(v_{n-3}v_{n-1}) = \begin{cases} 3n-9, & n \text{ is odd} \\ 3n-8, & n \text{ is even and} \end{cases}$$

$$f^*(v_{n-2}v_n) = 3n-6.$$

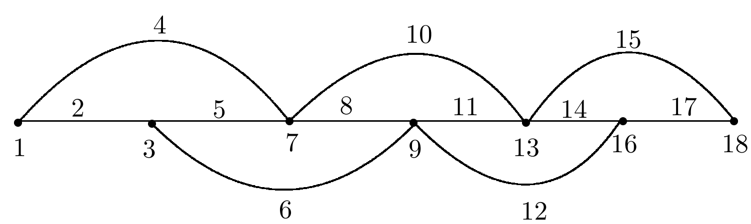

Figure 11 A super F-centroidal mean labeling of P_7^2

For $n = 5$, a super F-centroidal mean labeling of P_n^5 is shown the Figure 12.

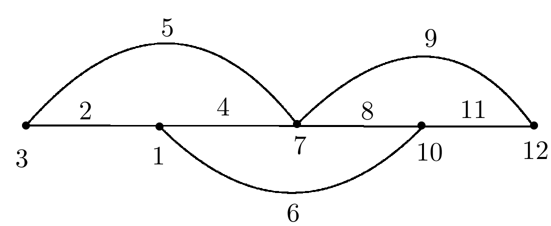

Figure 12 A super F-centroidal mean labeling of P_5^2

Hence f is a super F-centroidal mean labeling of P_n^2. Thus the graph P_n^2 is a super F-centroidal mean graph, for $n \geq 3$. □

Theorem 2.11 *If the graph G is a super F-centroidal mean graph, then $P_n(G)$ is also a super F-centroidal mean graph.*

Proof Let f be a super F-centroidal mean graph of G. Let $v_1, v_2, v_3, \cdots, v_p$ be the vertices and $e_1, e_2, e_3, \cdots, e_q$ be the edges of G so that the vertex having maximum vertex label is taken as v_p. Let $u_1, u_2, u_3, \cdots, u_n$ and $E_1, E_2, E_3, \cdots, E_{n-1}$ be the vertices and edges of P_n respectively and v_p is identified with u_1 in $P_n(G)$.

Define $g : V(P_n(G)) \to \{1, 2, 3, \cdots, p+q+2j-2\}$ as follows:

$$g(v_i) = f(v_i), \text{ for } 1 \leq i \leq p \text{ and}$$
$$g(u_j) = p+q+2j-2, \text{ for } 1 \leq j \leq n.$$

Then, the induced edge labeling g^* is obtained as follows:

$$g^*(e_i) = f(e_i), \text{ for } 1 \leq i \leq p \text{ and}$$
$$g^*(E_j) = p+q+2j-1, \text{ for } 1 \leq j \leq n-1.$$

Hence $P_n(G)$ is a super F-centroidal mean graph. Thus the graph G is a super F-centroidal mean graph then $P_n(G)$ is also a super F-centroidal mean graph. □

Corollary 2.12 *A dragon $P_n(C_m)$ is a super F-centroidal mean graph, for $m \geq 4$ and $n \geq 2$.*

§3. Conclusion

In this paper, the super F-centroidal meanness of some standard graphs have been studied. It is possible to investigate the super F-centroidal meanness for other graphs.

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Famous Words

Could mathematics characterizes the reality of all things T in the universe? The answer is negative at least for today's mathematics because all mathematical systems should be homogenous without contradictions in logic. We can not conclude that the mathematical reality equal to that of things both in theory and practice. (Extracted from the paper: Science's Dilemma - a Review on Science with Applications, *Progress in Physics*, Vol.15 (2019), 78-85.)

By Dr.Linfan MAO, a Chinese mathematician, philosophical critic.

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[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.
[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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September 2019

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