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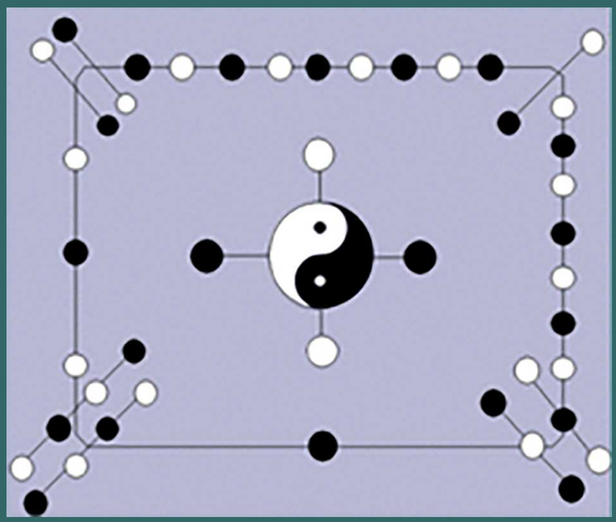

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Aims and Scope: The *mathematical combinatorics* is a subject that applying combinatorial notion to all mathematics and all sciences for understanding the reality of things in the universe, motivated by *CC Conjecture* of Dr.Linfan MAO on mathematical sciences. The **International J.Mathematical Combinatorics** (*ISSN 1937-1055*) is a fully refereed international journal, sponsored by the *MADIS of Chinese Academy of Sciences* and published in USA quarterly, which publishes original research papers and survey articles in all aspects of mathematical combinatorics, Smarandache multi-spaces, Smarandache geometries, non-Euclidean geometry, topology and their applications to other sciences. Topics in detail to be covered are:

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Famous Words:

If you want to know everything all at once, you will know nothing.

By Ivan Pavlov, a former Soviet physiologist, psychologist.

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Graphs, Networks and Natural Reality

– from Intuitive Abstracting to Theory

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Abstract: In the view of modern science, a matter is nothing else but a complex network $\overrightarrow{G}$, i.e., the reality of matter is characterized by complex network. However, there are no such a mathematical theory on complex network unless local and statistical results. *Could we establish such a mathematics on complex network*? The answer is affirmative, i.e., *mathematical combinatorics* or mathematics over topological graphs. Then, *what is a graph*? *How does it appears in the universe*? *And what is its role for understanding of the reality of matters*? The main purpose of this paper is to survey the progressing process and explains the notion from graphs to complex network and then, abstracts mathematical elements for understanding reality of matters. For example, L.Euler's solving on the problem of Königsberg seven bridges resulted in graph theory and embedding graphs in compact n-manifold, particularly, compact 2-manifold or surface with combinatorial maps and then, complex networks with reality of matters. We introduce 2 kinds of mathematical elements respectively on living body or non-living body for self-adaptive systems in the universe, i.e., continuity flow and harmonic flow $\overrightarrow{G}^L$ which are essentially elements in Banach space over graphs with operator actions on ends of edges in graph $\overrightarrow{G}$. We explain how to establish mathematics on the 2 kinds of elements, i.e., vectors underling a combinatorial structure $\overrightarrow{G}$ by generalize a few well-known theorems on Banach or Hilbert space and contribute mathematics on complex networks. All of these imply that graphs expand the mathematical field, establish the foundation on holding on the nature and networks are closer more to the real but without a systematic theory. However, its generalization enables one to establish mathematics over graphs, i.e., mathematical combinatorics on reality of matters in the universe.

Key Words: Graph, 2-cell embedding of graph, combinatorial map, complex network, reality, mathematical element, Smarandache multispace, mathematical combinatorics.

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§1. Introduction

What is the role of mathematics to natural reality? Certainly, as the science of quantity, mathematics is the main tool for humans understanding matters, both for the macro and the micro in the universe. Generally, it builds a model and characterizes the behavior of a matter

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for holding on reality and then, establishes a theory, such as those shown in Fig.1.

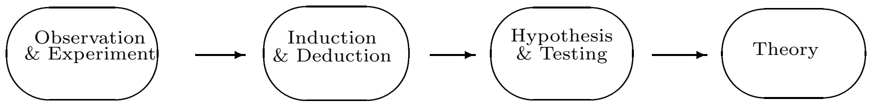


Fig.1

This scientific method on matters in the universe is completely reflected in the solving process of L.Euler on the problem of Königsberg seven bridges. Geographically, the city of Königsberg is located on both sides of Pregel River, including two large islands which were connected to each other and the mainland by seven bridges, such as those shown in Fig.2. The residents of Königsberg usually wished to pass through each bridge once without repeat, initialing at point of the mainland or islands.

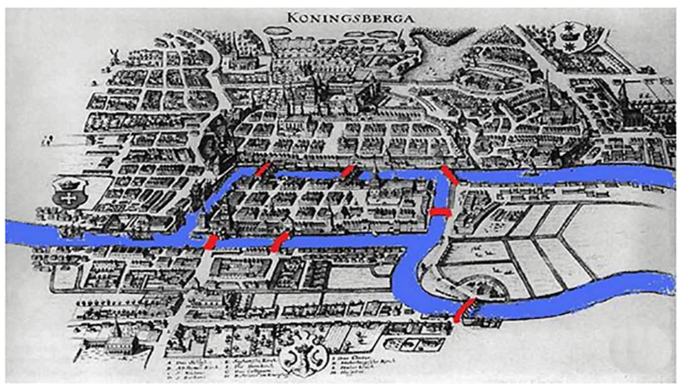
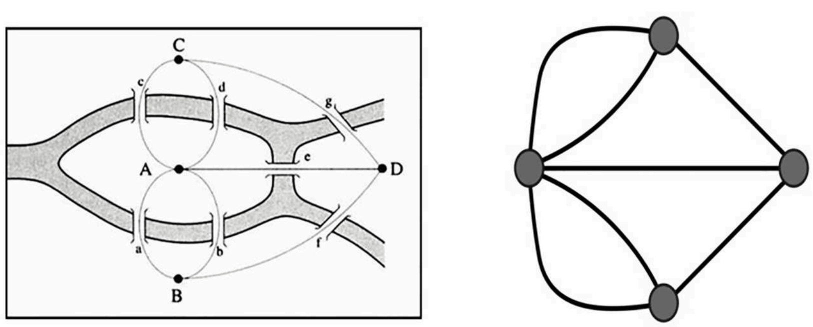


Fig.2

However, no one traveled in such a way once. Then, *a resident should how to travel for such a walk*? L.Euler solved this problem, and answered it had no solution in 1736. *How did he do it*? Let A,B,C,D be the two sides and islands. Then, he abstracted this problem on (a) equivalent to finding a traveling passing through each lines on (b) without repeating.



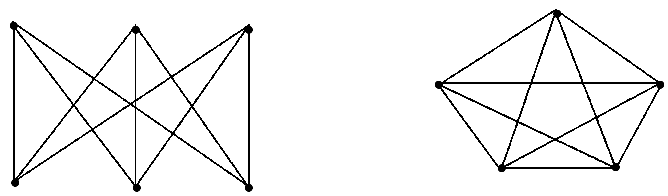
(a) (b)

Fig.3

Clearly, such a traveling must be with the same in and out times at each point A,B,C or D. But, (b) is not fitted with such conditions. So, there are no such a traveling in the problem on

Königsberg seven bridges.

Euler's solving method on the problem of Königsberg seven bridges finally resulted graph theory into beings today. A *graph* G is an ordered 3-tuple $(V, E; I)$, where V, E are finite sets, $V \neq \emptyset$ and $I : E \to V \times V$. Call V the *vertex set* and E the *edge set* of G, denoted by $V(G)$ and $E(G)$, respectively. For example, two graphs $K(3,3)$ and K_5 are shown in Fig.4.



$K(3,3)$ K_5

Fig.4

Usually, if $(u, v) = (v, u)$ for $\forall u, v \in V(G)$, then G is called a graph. Otherwise, it is called a directed graph with an orientation $u \to v$ on each edge (u, v), denoted by $\overrightarrow{G}$.

Let $G_1 = (V_1, E_1, I_1)$, $G_2 = (V_2, E_2, I_2)$ be 2 graphs. If there exists a $1-1$ mapping $\phi : V_1 \to V_2$ and $\phi : E_1 \to E_2$ such that $\phi I_1(e) = I_2\phi(e)$ for $\forall e \in E_1$ with the convention that $\phi(u, v) = (\phi(u), \phi(v))$, then we say that G_1 is *isomorphic* to G_2, denoted by $G_1 \cong G_2$ and ϕ an *isomorphism* between G_1 and G_2. Clearly, all automorphisms $\phi : V(G) \to V(G)$ of graph G form a group under the composition operation, and denoted by Aut$G$ the automorphism group of graph G. A few automorphism groups of well-known graphs are listed in Table 1.

G	Aut$G$	order
P_n	Z_2	2
C_n	D_n	$2n$
K_n	S_n	$n!$
$K_{m,n}(m \neq n)$	$S_m \times S_n$	$m!n!$
$K_{n,n}$	$S_2[S_n]$	$2n!^2$

Table 1

Certainly, an edge $e = uv \in E(G)$ can be divided into two semi-arcs e_u, e_v such as those shown in Fig.5.

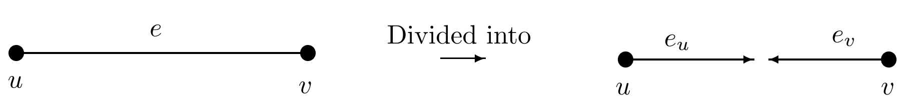


Fig.5

Similarly, two semi-arcs e_u, f_v are called v-incident or e-incident if $u = v$ or $e = f$. Denote all semi-arcs of a graph G by $X_{\frac{1}{2}}(G)$. A $1-1$ mapping ξ on $X_{\frac{1}{2}}(G)$ such that $\forall e_u, f_v \in X_{\frac{1}{2}}(G)$, $\xi(e_u)$ and $\xi(f_v)$ are $v-$incident or $e-$incident if e_u and f_v are $v-$incident or $e-$incident, is called a semi-arc automorphism of the graph G. Clearly, all semi-arc automorphisms of a graph also form a group, denoted by $\mathrm{Aut}_{\frac{1}{2}}G$.

Certainly, *graph theory* studies properties of graphs. A property is nothing else but a family of graph, i.e., $\mathscr{P} = \{G_1, G_2, \cdots, G_n, \cdots\}$ but closed under isomorphisms ϕ of graphs, i.e., $G^\phi \in \mathscr{P}$ if $G \in \mathscr{P}$. For example, hamiltonian graphs, Euler graphs and also interesting parameters, such as those of connectivity, independent number, covering number, girth, level number, $\cdots$ of a graph.

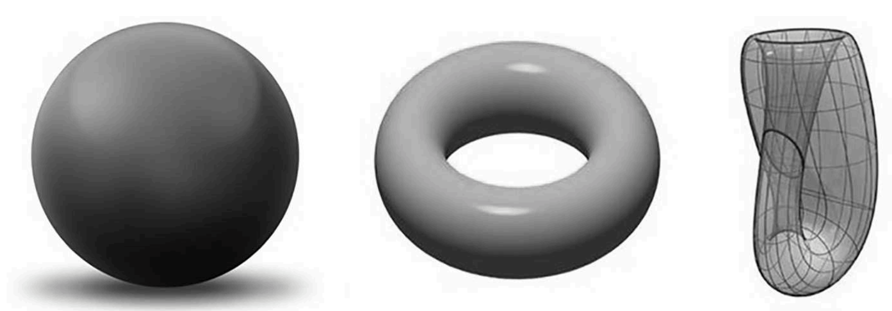
The main purpose of this paper is to survey the progressing process and explains the notion from graphs to complex network and then, abstracts mathematical elements for understanding reality of matters. For example, L.Euler's solving on the problem of Königsberg seven bridges resulted in graph theory and embedding graphs in compact n-manifold, particularly, compact 2-manifold or surface with combinatorial maps and then, complex networks with reality of matters. We introduce 2 kinds of mathematical elements respectively on living or non-living body in the universe, i.e., continuity and harmonic flows $\overrightarrow{G}^L$ which are essentially elements in Banach space over graphs with operator actions on ends of edges in graph $\overrightarrow{G}$. We explain how to establish mathematics on the 2 kinds of elements, i.e., vectors underling a combinatorial structure $\overrightarrow{G}$ by generalize a few well-known theorems on Banach or Hilbert space and contribute a mathematics on complex networks.

For terminologies and notations not mentioned here, we follow references [1],[2] and [4] for graphs, [3] for complex network, [6] for automorphisms of graph, [24] for algebraic topology, [25] for elementary particles and [6],[26] for Smarandache systems and multispaces.

§2. **Embedding Graphs on Surfaces**

2.1 Surface

A surface is a 2-dimensional compact manifold without boundary. For example, a few surfaces are shown in Fig.6.



Sphere Torus Klein bottle

Fig.6

Clearly, the intuition imagination is difficult for determining surface of higher genus. However, T.Radó showed the following result, which is the fundamental of combinatorial topology, ro topological graphs on surfaces.

Theorem 2.1(T.Radó 1925,[24]) *For any compact surface S, there exist a triangulation $\{T_i, i \geq 1\}$ on S.*

T.Radó's result on triangulation of surface enables one to present a surface by listing every triangle with each side a label and a direction, i.e., the polygon representation. Then, the surface is assembled by identifying the two sides with the same label and direction. This way results in a polygon representation on a surface finally. For examples, the polygon representations on surfaces in Fig.6 are shown in Fig.7.

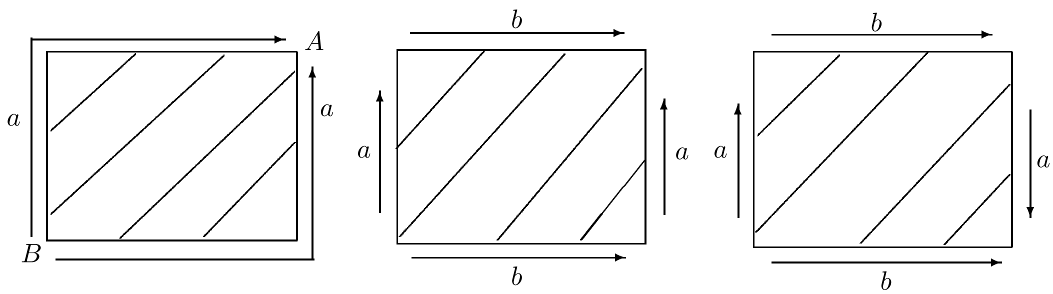


Fig.7

We know the classification theorem of surfaces following.

Theorem 2.2([24]) *Any connected compact surface S is either homeomorphic to a sphere, or to a connected sum of tori, or to a connected sum of projective planes, i.e., its surface presentation $\mathcal{S}$ is elementary equivalent to one of the standard surface presentations following:*

(1) *The sphere* $S^2 = \langle a|aa^{-1}\rangle$;

(2) *The connected sum of p tori*

$$\underbrace{T^2\#T^2\#\cdots\#T^2}_{p} = \left\langle\ a_i, b_i, 1 \leq i \leq p \mid \prod_{i=1}^{p} a_i b_i a_i^{-1} b_i^{-1} \right\rangle;$$

(3) *The connected sum of q projective planes*

$$\underbrace{P^2\#P^2\cdots\#P^2}_{q} = \left\langle\ a_i, 1 \leq i \leq q \mid \prod_{i=1}^{q} a_i \right\rangle.$$

A combinatorial proof on Theorem 2.2 can be found in [6]. By definition, the *Euler characteristic* of $\mathcal{S}$ is

$$\chi(\mathcal{S}) = |V(\mathcal{S})| - |E(\mathcal{S})| + |F(\mathcal{S})|,$$

where $V(\mathcal{S}), E(\mathcal{S})$ and $F(\mathcal{S})$ are respective the set of vertex set, edge set and face set of the polygon representation of surface $\mathcal{S}$. Then, we know the next result.

Theorem 2.3([24]) *Let S be a connected compact surface with a presentation $\mathcal{S}$. Then*

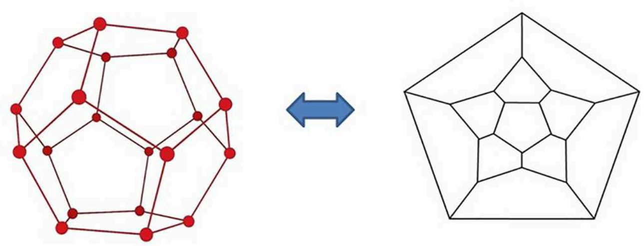
$$\chi(S)=\begin{cases} 2, & \text{if } \mathcal{S}\sim_{El} S^2,\\ 2-2p, & \text{if } \mathcal{S}\sim_{El} \underbrace{T^2\#T^2\#\cdots\#T^2}_{p},\\ 2-q, & \text{if } \mathcal{S}\sim_{El} \underbrace{P^2\#P^2\#\cdots\#P^2}_{q}.\end{cases}$$

Theorem 2.3 enables one to define the genus of orientable or non-orientable surface S by numbers p and q, respectively, and the genus of sphere is defined to be 0.

2.2 Embedding Graph

A graph G is said to be embeddable into a topological space $\mathscr{T}$ if there is a $1-1$ continuous mapping $\phi: G\to\mathscr{T}$ with $\phi(p)\neq\phi(q)$ if p,q are different points on graph G. Particularly, if $\mathscr{T}=\mathbb{R}^2$ is a Euclidean plane, we say that G is a planar graph.

A most interesting case on the embedding problem of graphs is the case of surface, which is essentially to search the polyhedral structures on surfaces. Clearly, many results on embedding graphs is on these surfaces with small genus. For example, embedding results on $p=0$ the sphere, $p=1$ the torus, $\cdots$ of orientable surfaces, or on $q=1$ the projective plane, $q=2$ the Kein bottle, $\cdots$ of non-orientable surfaces. The most simple case is embedding graphs on sphere which is equivalent to a planar graph, such as the dodecahedron shown in Fig.8.



Dodecahedron Planar graph

Fig.8

We have known a few criterions on planar graphs following.

Theorem 2.4(Euler,1758, [2]) *Let G be a planar graph with p vertices, q edges and r faces. Then, $p-q+r=2$.*

Theorem 2.5(Kuratowski,1930, [1]) *A graph is planar if and only if it contains no subgraphs homeomorphic with K_5 or $K(3,3)$.*

A 2-cell embedding of G on surface S is defined to be a continuous 1-1 mapping $\tau: G\to S$ such that each component in $S\setminus\tau(G)$ homeomorphic to an open 2-disk $\{\ (x,y)\mid x^2+y^2<1\}$. Certainly, the image $\tau(G)$ is contained in the 1-skeleton of a triangulation on the surface S.

For example, the embedding of K_4 on the sphere and Klein bottle are shown in Fig.9 (a) and (b), respectively.

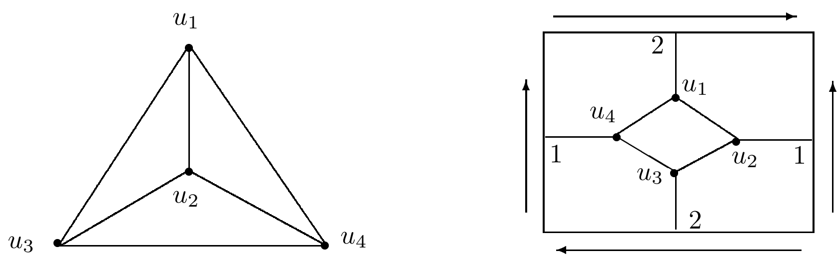


Fig.9

There is an algebraic representation for characterizing the 2-cell embedding of graphs. For $v \in V(G)$, denote by $N_G^e(v) = \{e_1, e_2, \cdots, e_{\rho(v)}\}$ all the edges incident with the vertex v. A permutation on $e_1, e_2, \cdots, e_{\rho(v)}$ is said to be a *pure rotation* and all pure rotations incident with v is denoted by $\varrho(v)$. Generally, a *pure rotation system* of the graph G is defined to be $\rho(G) = \{\varrho(v) | v \in V(G)\}$ which was observed and used by Dyck in 1888, Heffter in 1891 and then formalized by Edmonds in 1960. For example,

$$\rho(K_4) = \{(u_1u_4, u_1u_3, u_1u_2), (u_2u_1, u_2u_3, u_2u_4), (u_3u_1, u_3u_4, u_3u_2), (u_4u_1, u_4u_2, u_4u_3)\},$$
$$\rho(K_4) = \{(u_1u_2, u_1u_3, u_1u_4), (u_2u_1, u_2u_3, u_2u_4), (u_3u_2, u_3u_4, u_3u_1), (u_4u_1, u_4u_2, u_4u_3)\}$$

are respectively the pure rotation systems for embeddings of K_4 on the sphere and Klein bottle shown in Fig.9.

Theorem 2.6(Heffter 1891, Edmonds 1960, [4]) *Every embedding of a graph G on an orientable surface S induces a unique pure rotation system $\rho(G)$. Conversely, Every pure rotation system $\rho(G)$ of a graph G induces a unique embedding of G on an orientable surface S.*

Clearly, an embedding of graph G can be associated 0, 1 or 2-band respectively with vertices, edges and face on its surface. A band decomposition is called *locally orientable* if each 0-band is assigned an orientation, and a 1-band is called *orientation-preserving* if the direction induced on its ends by adjoining 0-bands are the same as those induced by one of the two possible orientations of the 1-band. Otherwise, *orientation-reversing*. An edge e in a graph G embedded on a surface S associated with a locally orientable band decomposition is said to be *type* 0 if its corresponding 1-band is orientation-preserving and otherwise, *type* 1. A *rotation system* $\rho^L(v)$ of $v \in V(G)$ to be a pair $(\mathcal{J}(v), \lambda)$, where $\mathcal{J}(v)$ is a pure rotation system and $\lambda : E(G) \to \mathbb{Z}_2$ is determined by $\lambda(e) = 0$ or $\lambda(e) = 1$ if e is *type* 0 or *type* 1 edge, respectively.

Theorem 2.7(Ringel 1950s, Stahl 1978,[4]) *Every rotation system on a graph G defines a unique locally orientable 2-cell embedding of $G \to S$. Conversely, every 2-cell embedding of a graph $G \to S$ defines a rotation system for G.*

A 2-cell embedding of a connected graph on surface is called map by W.T.Tutte. He characterized embeddings by purely algebra in 1973 ([5], [26]). By his definition, an embedding $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ is defined to be a basic permutation $\mathcal{P}$, i.e, for any $x \in \mathcal{X}_{\alpha,\beta}$, no integer k exists such that $\mathcal{P}^k x = \alpha x$, acting on $\mathcal{X}_{\alpha,\beta}$, the disjoint union of quadricells Kx of $x \in X$ (the base set), where $K = \{1, \alpha, \beta, \alpha\beta\}$ is the Klein group, satisfying the following two conditions:

(1) $\alpha\mathcal{P} = \mathcal{P}^{-1}\alpha$;

(2) the group $\Psi_J = \langle \alpha, \beta, \mathcal{P} \rangle$ is transitive on $\mathcal{X}_{\alpha,\beta}$.

Furthermore, if the group $\Psi_I = \langle \alpha\beta, \mathcal{P} \rangle$ is transitive on $\mathcal{X}_{\alpha,\beta}$, then M is non-orientable. Otherwise, orientable.

For example, the embedding $(\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ of graph K_4 on torus shown in Fig.10.

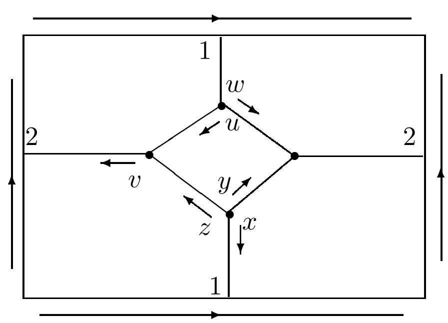


Fig.10

can be algebraic represented by

$$\begin{aligned}\mathcal{X}_{\alpha,\beta} &= \{x, y, z, u, v, w, \alpha x, \alpha y, \alpha z, \alpha u, \alpha v, \alpha w, \beta x, \beta y,\\ &\quad \beta z, \beta u, \beta v, \beta w, \alpha\beta x, \alpha\beta y, \alpha\beta z, \alpha\beta u, \alpha\beta v, \alpha\beta w\},\\ \mathcal{P} &= (x, y, z)(\alpha\beta x, u, w)(\alpha\beta z, \alpha\beta u, v)(\alpha\beta y, \alpha\beta v, \alpha\beta w)\\ &\quad \times(\alpha x, \alpha z, \alpha y)(\beta x, \alpha w, \alpha u)(\beta z, \alpha v, \beta u)(\beta y, \beta w, \beta v),\end{aligned}$$

with vertices

$$v_1 = \{(x, y, z), (\alpha x, \alpha z, \alpha y)\}, \qquad v_2 = \{(\alpha\beta x, u, w), (\beta x, \alpha w, \alpha u)\},$$
$$v_3 = \{(\alpha\beta z, \alpha\beta u, v), (\beta z, \alpha v, \beta u)\}, \quad v_4 = \{(\alpha\beta y, \alpha\beta v, \alpha\beta w), (\beta y, \beta w, \beta v)\},$$

edges

$$\{e, \alpha e, \beta e, \alpha\beta e\},\ e \in \{x, y, z, u, v, w\}$$

and faces

$$\begin{aligned}f_1 &= \{(x, u, v, \alpha\beta w, \alpha\beta x, y, \alpha\beta v, \alpha\beta z), (\beta x, \alpha z, \alpha v, \beta y, \alpha x, \alpha w, \beta v, \beta u)\},\\ f_2 &= \{(z, \alpha\beta u, w, \alpha\beta y), (\beta z, \alpha y, \beta w, \alpha u)\}.\end{aligned}$$

Two embeddings $M_1 = (\mathcal{X}^1_{\alpha,\beta}, \mathcal{P}_1)$ and $M_2 = (\mathcal{X}^2_{\alpha,\beta}, \mathcal{P}_2)$ are said to be *isomorphic* if there exists a bijection ξ

$$\xi : \mathcal{X}^1_{\alpha,\beta} \longrightarrow \mathcal{X}^2_{\alpha,\beta}$$

such that for $\forall x \in \mathcal{X}^1_{\alpha,\beta}$, $\xi\alpha(x) = \alpha\xi(x), \xi\beta(x) = \beta\xi(x), \xi\mathcal{P}_1(x) = \mathcal{P}_2\xi(x)$. Particularly, if $M_1 = M_2 = M$, an isomorphism between M_1 and M_2 is then called an *automorphism* of embedding M. Clearly, all automorphisms of a embedding M form a group, called the *automorphism group* of M, denoted by $\mathrm{Aut}M$.

There are two main problems on embedding of graphs on surfaces following.

Problem 2.1 *Let G be a graph and S a surface. Whether or not G can be embedded on S?*

This problem had been solved by Duke on orientable case in 1966, and Stahl on non-orientable case in 1978. They obtained the result following.

Theorem 2.8(Duke 1966, Stahl 1978,[4]) *Let G be a connected graph and let $GR(G), CR(G)$ be the respective genus range of G on orientable or non-orientable surfaces. Then, $GR(G)$ and $CR(G)$ both are unbroken interval of integers.*

Theorem 2.8 bring about to determine the minimum and maximum genus $\gamma(G)$, $\gamma_M(G)$ of graph G on surfaces. Among them, the most simple case is to determine the maximum genus $\gamma_M(G)$ on non-orientable case, which was obtained by Edmonds in 1965. It is the Betti number $\beta(G) = |E(G)| - |V(G)| + 1$. The maximum genus $\gamma_M(G)$ of G on orientable case is determined by Xuong with the deficiency $\xi(G)$, i.e., the minimum number of components in $G \setminus T$ for all spanning trees T in G in 1979. However, it is difficult for the minimum genus $\gamma(G)$, only a few results on typical graphs. For example, the genus of K_n and Km, n are listed following.

Theorem 2.9(Ringel and Youngs 1968, [4]) *The minimum genus of a complete graph is given by*

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil, \quad n \geq 3.$$

Theorem 2.10(Ringel 1965, [4]) *The minimum genus of a complete bipartite graph is given by*

$$\gamma(K(m,n)) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil, \quad m, n \geq 2.$$

Problem 2.2 *Let G be a graph and S a surface. How many non-isomorphic embeddings of G on S?*

This problem is difficult, only be partially solved until today. However, the following simple result enables one to enumerate rooted embeddings, where an embedding $(\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ is rooted on an element $r \in \mathcal{X}_{\alpha,\beta}$ if r is marked beforehand.

Theorem 2.11([5],[6]) *The auotmorphism group of a rooted embedding M, i.e.,* $\mathrm{Aut}M^r$ *is trivial.*

Theorem 2.12([5],[6]) $|\mathrm{Aut}M| \mid |\mathcal{X}_{\alpha,\beta}| = 4\varepsilon(M)$.

A root r in an embedding M is called an *i-root* if it is incident to a vertex of valency i. Two i-roots r_1, r_2 are *transitive* if there exists an automorphism $\tau \in$ AutM such that $\tau(r_1) = r_2$. Define the *enumerator* $v(D,x)$ and the *root polynomials* $r(M,x)$, $r(\mathcal{M}(D),x)$ as follows:

$$v(D,x) = \sum_{i\geq 1} i v_i x^i; \qquad r(M,x) = \sum_{i\geq 1} r(M,i)x^i,$$

where $r(M,i)$ denotes the number of non-transitive i-roots in M.

Theorem 2.12 enables us to get the following results by applying the enumerator and root polynomial of M.

Theorem 2.13(Mao and Liu, [21]) *The number $r^O(\Gamma)$ of non-isomorphic rooted maps on orientable surfaces underlying a simple graph Γ is*

$$r^O(\Gamma) = \frac{2\varepsilon(\Gamma) \prod\limits_{v\in V(\Gamma)} (\rho(v)-1)!}{|\mathrm{Aut}\Gamma|},$$

where $\varepsilon(\Gamma)$,$\rho(v)$ denote the size of Γ and the valency of the vertex v, respectively.

Theorem 2.14(Mao and Liu, [22]) *The number $r^N(\Gamma)$ of rooted maps on non-orientable surfaces underlying a graph Γ is*

$$r^N(\Gamma) = \frac{(2^{\beta(\Gamma)+1}-2)\varepsilon(\Gamma) \prod\limits_{v\in V(\Gamma)} (\rho(v)-1)!}{\left|Aut_{\frac{1}{2}}\Gamma\right|}.$$

For a few well-known graphs, Theorems 2.13 and 2.14 enables us to get Table 2.

G	$r^O(G)$	$r^N(G)$
P_n	$n-1$	0
C_n	1	1
K_n	$(n-2)!^{n-1}$	$(2^{\frac{(n-1)(n-2)}{2}}-1)(n-2)!^{n-1}$
$K_{m,n}(m\neq n)$	$2(m-1)!^{n-1}(n-1)!^{m-1}$	$(2^{mn-m-n+2}-2)(m-1)!^{n-1}(n-1)!^{m-1}$
$K_{n,n}$	$(n-1)!^{2n-2}$	$(2^{n^2-2n+2}-1)(n-1)!^{2n-2}$
B_n	$\frac{(2n)!}{2^n n!}$	$(2^{n+1}-1)\frac{(2n)!}{2^n n!}$
Dp_n	$(n-1)!$	$(2^n-1)(n-1)!$
$Dp_n^{k,l}(k\neq l)$	$\frac{(n+k+l)(n+2k-1)!(n+2l-1)!}{2^{k+l-1}n!k!l!}$	$\frac{(2^{n+k+l}-1)(n+k+l)(n+2k-1)!(n+2l-1)!}{2^{k+l-1}n!k!l!}$
$Dp_n^{k,k}$	$\frac{(n+2k)(n+2k-1)!^2}{2^{2k}n!k!^2}$	$\frac{(2^{n+2k}-1)(n+2k)(n+2k-1)!^2}{2^{2k}n!k!^2}$

Table 2

Apply the Burnside Lemma in permutation groups, we got the numbers of unrooted maps of complete graph K_n on orientable or non-orientable surfaces by calculating the stabilizer of each automorphism of complete maps.

Theorem 2.15(Mao, Liu and Tian, [23]) *The number $n^O((K_n)$ of complete maps of order $n \geq 5$ on orientable surfaces is*

$$n^O(K_n) = \frac{1}{2}(\sum_{k|n} + \sum_{k|n,k\equiv 0(mod2)})\frac{(n-2)!^{\frac{n}{k}}}{k^{\frac{n}{k}}(\frac{n}{k})!} + \sum_{k|(n-1),k\neq 1}\frac{\phi(k)(n-2)!^{\frac{n-1}{k}}}{n-1}.$$

and $n(K_4) = 3$.

Theorem 2.16(Mao, Liu and Tian, [23]) *The number $n^N(K_n)$ of complete maps of order $n, n \geq 5$ on non-orientable surfaces is*

$$\begin{aligned} n^N(K_n) &= \frac{1}{2}(\sum_{k|n} + \sum_{k|n,k\equiv 0(mod2)})\frac{(2^{\alpha(n,k)}-1)(n-2)!^{\frac{n}{k}}}{k^{\frac{n}{k}}(\frac{n}{k})!} \\ &+ \sum_{k|(n-1),k\neq 1}\frac{\phi(k)(2^{\beta(n,k)}-1)(n-2)!^{\frac{n-1}{k}}}{n-1}, \end{aligned}$$

and $n^N(K_4) = 8$, *where,*

$$\alpha(n,k) = \begin{cases} \frac{n(n-3)}{2k}, & \text{if} \quad k \equiv 1(mod2); \\ \frac{n(n-2)}{2k}, & \text{if} \quad k \equiv 0(mod2), \end{cases} \qquad \beta(n,k) = \begin{cases} \frac{(n-1)(n-2)}{2k}, & \text{if} \quad k \equiv 1(mod2); \\ \frac{(n-1)(n-3)}{2k}, & \text{if} \quad k \equiv 0(mod2). \end{cases}$$

§3. Complex Networks with Reality

A network is a directed graph G associated with a non-negative integer-valued function c on edges and conserved at each vertex, which are abstracting of practical networks, for instance, the electricity, communication and transportation networks such as those shown in Fig.11 for the high-speed rail network in China planed a few years ago.

Fig.11

Clearly, a network is nothing else but a labeled graph G^L with $L : E(G) \to \mathbb{Z}^+$. generally, a *labeled graph* on a graph $G = (V, E; I)$ is a mapping $\theta_L : V \cup E \to L$ for a label set L, denoted by G^L. If $\theta_L : E \to \emptyset$ or $\theta_L : V \to \emptyset$, then G^L is called a *vertex labeled graph* or an *edge labeled* *graph*, denoted by G^V or G^E, respectively. Otherwise, it is called a *vertex-edge labeled graph*. Similarly, two networks $G_1^{L_1}, G_2^{L_2}$ are equivalent, if there is an isomorphism $\phi : G_1 \to G_2$ such that $\phi(L_1(x)) = L_2(\phi(x))$ for $x \in V(G_1)\bigcup E(G_2)$.

It should be noted that labeled graphs are more useful in understanding matters in the universe. For example, there is a famous story, i.e., the blind men with an elephant. In this story, 6 blind men were be asked to determine what is an elephant looks like. The man touched the elephant's leg, tail, trunk, ear, belly or tusk respectively claims it's like a pillar, a rope, a tree branch, a hand fan, a wall or a solid pipe. Each of them insisted on his own and not accepted others.

Fig.12

They then entered into an endless argument. *All of you are right*! A wise man explained to them: why are you telling it differently is because each one of you touched the different part of the elephant. *What is the meaning of the wise man*? He claimed nothing else but the looks like of an elephant, i.e.,

$$\begin{aligned}\text{An elephant} &= \{4 \text{ pillars}\}\bigcup\{1 \text{ rope}\}\bigcup\{1 \text{ tree branch}\}\\ &\bigcup \{2 \text{ hand fans}\}\bigcup\{1 \text{ wall}\}\bigcup\{1 \text{ solid pipe}\}.\end{aligned}$$

Usually, a thing T is identified with known characters on it at one time, and this process is advanced gradually by ours. For example, let $\mu_1, \mu_2, \cdots, \mu_n$ be the known and $\nu_i, i \geq 1$ the unknown characters at time t. Then, the thing T is understood by

$$T = \left(\bigcup_{i=1}^{n}\{\mu_i\}\right)\bigcup\left(\bigcup_{k\geq 1}\{\nu_k\}\right)$$

in logic and with an approximation $T^\circ = \bigcup\limits_{i=1}^{n}\{\mu_i\}$ at time t, which are both Smarandache multispace ([7],[26]).

What is the implications of this story for understanding matters in the universe? It lies in the situation that humans knowing matters in the universe is analogous to these blind men. However, if the wise man were L.Euler, a mathematician he would tell these blind men that an elephant looks like nothing else but a tree labeled by sets as shown in Fig.13, where, $\{a\}$ =tusk, $\{b_1, b_2\}$ =ears, $\{c\}$ =head, $\{d\}$ =belly, $\{e_1, e_2, e_3, e_4\}$ =legs and $\{f\}$ =tail with their intersection sets labeled on edges.

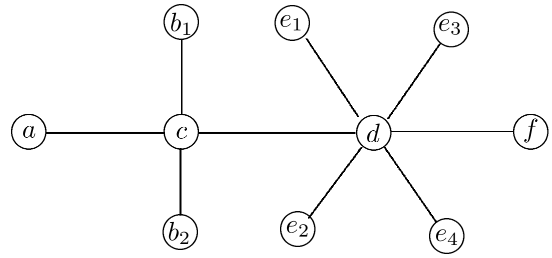


Fig.13

For the case of Euclidean space with dimension≥ 3, the intuition tells us that to embed a graph in a of dimension is not difficult by the result following. However, it is obvious but a universal skeleton inherited in all matters.

Theorem 3.1 *A simple graph G can be rectilinear embedded, i.e., all edge are segment of straight line in a Euclidean space $\mathbb{R}^n$ with $n \geq 3$.*

In fact, we can choose n district points in curve (t, t^2, t^3) of Euclidean space $\mathbb{R}^3$ on n different values of t. Then, it can be easily show that all these straight lines are never intersecting. Whence, it is a trivial problem on embedding graphs of $\mathbb{R}^3$. However, all matters are in 3-dimensional Euclidean space in the eyes of humans, i.e., the reality of a matter in the universe should be understanding on its 1-dimensional skeleton in the space.

Then, *what is the reality of a matter*? The word *reality* of a matter T is its state as it actually exist, including everything that is and has been, no matter it is observable or comprehensible by humans. *How can we hold on the reality of matters*? Usually, a matter T is multilateral, i.e., Smarandache multispace or complex one and so, hold on its reality is difficult for humans in logic, such as the meaning in the story of the blind men with an elephant.

For hold on the reality of matters, a general notion is

$$\text{Matter} \overset{Decompose}{\longrightarrow} \text{Microcosmic Particles} \overset{Abstract}{\longrightarrow} \text{Complex Network.}$$

For example, the physics determine the nature of matters by subdividing a matter to an irreducibly smallest detectable particle ([28]), i.e., elementary particles, which is essentially transfer the matter to a complex network such as those meson's and baryon's composition by quarks.

Similarly, the basic unit of life or the basic unit of heredity are cells and genes in biology which also enables us to get the life networks of cell or genes. This notion can be found in all modern science with an conclusion that *a matter = a complex network*. Its essence of this notion is to determine the nature of irreducibly smallest detectable units and then, holds on reality of the matter. However, a matter can be always divided into submatters, then sub-submatters and so on. A natural question on this notion is whether it has a terminal point or not. On the other hand, it is a very large complex network in general. For example, the complex network of a human body consists of $5 \times 10^{14} - -6 \times 10^{14}$ cells. *Are we really need such a large and complex network for the reality of matters*? Certainly not! *How can we hold on the reality of matters* *by such a complex network*? And *do we have mathematical theory on complex network*? The answer is not certain because although we have established a theory on complex network but it is only a local theory by combination of the graphs and statistics with the help of computer ([3]), can not be used for the reality of matters.

However, we find a beacon light inspired by the *traditional Chinese medicine*. There are 12 meridians which completely reflects the physical condition of human body in traditional Chinese medicine: LU, LI, ST, SP, HT, SI, BL, KI, PC, SJ, GB, LR. For example, the LI and GB meridians are shown in Fig.14.

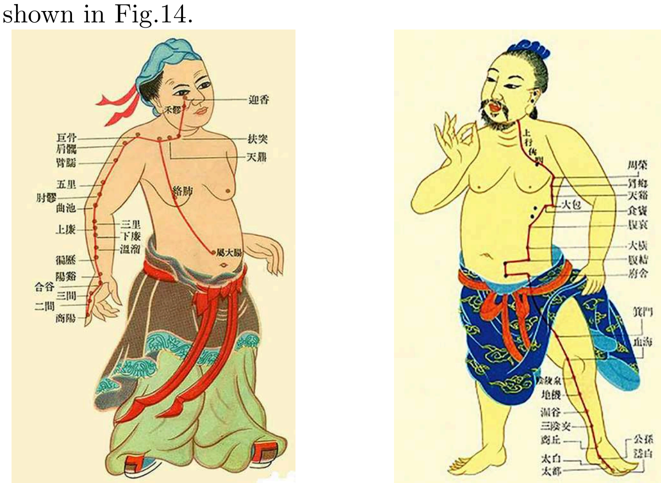


Fig.14

All of these 12 meridians can be classified into 3 classes following:

Class 1. *Paths*, including LU, LI, SP, HT, SI, KI, PC and LR meridians;

Class 2. *Trees*, including GB, ST and SJ meridians;

Class 3. *Gluing Product of circuit with paths* $C_n \bigodot P_{m_1} \bigodot P_{m_2}$, including BL meridian.

According to the Standard China National Standard (GB 12346-90), the inherited graph of the 12 meridians on a human body is shown in Fig.15.

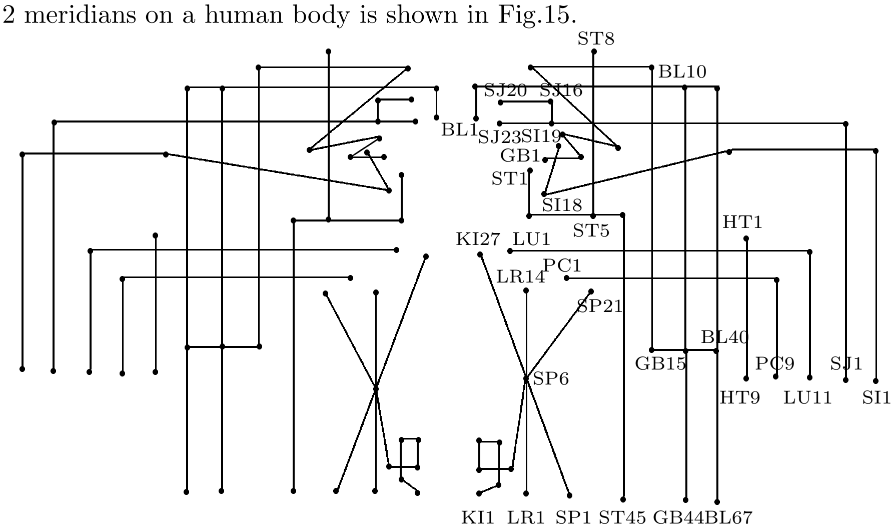


Fig.15 12 Meridian graph on a human body

By the traditional Chinese medicine ([28]), if there exists an imbalanced acupoint on one of the 12 meridians, this person must has illness and in turn, there must be imbalance acupoints on the 12 meridians for a patient. Thus, finding out which acupoint on which meridian is in imbalance with Yin more than Yang or Yang more than Yin is the main duty of a Chinese doctor. Then, the doctor regulates the meridian by acupuncture or drugs so that the balance on the imbalance acupoints recovers again, and then the patient recovers.

Then, *what is the significance of the treatment theory in traditional Chinese medicine to science*? It implies we are not need a large complex network for holding on the body of human. *Whether or not classically mathematical elements enough for understanding complex networks, i.e., matters in the universe*? The answer is negative because all of them are local. Then, *could we establish a mathematics over elements underlying combinatorial structures*? The answer is affirmative, i.e., *mathematical combinatorics* discussed in this paper. Certainly, we can introduce 2 kinds of elements respectively on living of non-living matters.

Element 1(Non-Living Body). A *continuity flow* $\overrightarrow{G}^L$ is an oriented embedded graph $\overrightarrow{G}$ in a topological space $\mathscr{S}$ associated with a mapping $L : v \to L(v),\ (v,u) \to L(v,u)$, 2 end-operators $A^+_{vu} : L(v,u) \to L^{A^+_{vu}}(v,u)$ and $A^+_{uv} : L(u,v) \to L^{A^+_{uv}}(u,v)$ on a Banach space $\mathscr{B}$ over a field $\mathscr{F}$ such as those shown in Fig.16,

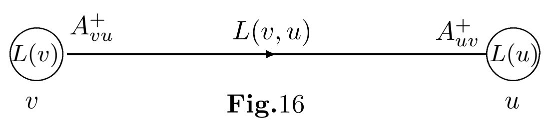


Fig.16

with $L(v,u) = -L(u,v)$, $A^+_{vu}(-L(v,u)) = -L^{A^+_{vu}}(v,u)$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ and holding with continuity equation

$$\sum_{u \in N_G(v)} L^{A^+_{vu}}(v,u) = L(v) \quad \text{for} \quad \forall v \in V\left(\overrightarrow{G}\right).$$

Element 2(Living Body). A *harmonic flow* $\overrightarrow{G}^L$ is an oriented embedded graph $\overrightarrow{G}$ in a topological space $\mathscr{S}$ associated with a mapping $L : v \to L(v) - iL(v)$ for $v \in E\left(\overrightarrow{G}\right)$ and $L : (v,u) \to L(v,u) - iL(v,u)$, 2 end-operators $A^+_{vu} : L(v,u) - iL(v,u) \to L^{A^+_{vu}}(v,u) - iL^{A^+_{vu}}(v,u)$ and $A^+_{uv} : L(v,u) - iL(v,u) \to L^{A^+_{uv}}(v,u) - iL^{A^+_{uv}}(v,u)$ on a Banach space $\mathscr{B}$ over a field $\mathscr{F}$ such as those shown in Fig.17,

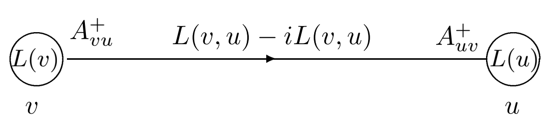


Fig.17

where $i^2 = -1$, $L(v,u) = -L(u,v)$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ and holding with continuity equation

$$\sum_{u \in N_G(v)} \left(L^{A^+_{vu}}(v,u) - iL^{A^+_{vu}}(v,u)\right) = L(v) - iL(v)$$

for $\forall v \in V\left(\overrightarrow{G}\right)$.

Let $\mathscr{G}$ be a closed family of graphs $\overrightarrow{G}$ under the union operation and let $\mathscr{B}$ be a linear space $(\mathscr{B};+,\cdot)$, or furthermore, a commutative ring, a Banach or Hilbert space $(\mathscr{B};+,\cdot)$ over a field $\mathscr{F}$. Denoted by $(\mathscr{G}_{\mathscr{B}};+,\cdot)$ and $\left(\mathscr{G}^{\pm}_{\mathscr{B}};+,\cdot\right)$ the respectively elements 1 and 2 form by graphs $G\in\mathscr{G}$. Then, elements 1 and 2 can be viewed as vectors underlying an embedded graph G in space, which enable us to establish mathematics on complex networks and get results following.

Theorem 3.2([9-10,14-18]) *If $\mathscr{G}$ is a closed family of graphs $\overrightarrow{G}$ under the union operation and $\mathscr{B}$ a linear space $(\mathscr{B};+,\cdot)$, then, $(\mathscr{G}_{\mathscr{B}};+,\cdot)$ and $\left(\mathscr{G}^{\pm}_{\mathscr{B}};+,\cdot\right)$ with linear operators A^{+}_{vu}, A^{+}_{uv} for $\forall(v,u)\in E\left(\bigcup\limits_{G\in\mathscr{G}}\overrightarrow{G}\right)$ under operations $+$ and $\cdot$ form respectively a linear space, and furthermore, a commutative ring if $\mathscr{B}$ is a commutative ring $(\mathscr{B};+,\cdot)$ over a field $\mathscr{F}$.*

Theorem 3.3([9-10,14-18]) *If $\mathscr{G}$ is a closed family of graphs under the union operation and $\mathscr{B}$ a Banach space $(\mathscr{B};+,\cdot)$, then, $(\mathscr{G}_{\mathscr{B}};+,\cdot)$ and $\left(\mathscr{G}^{\pm}_{\mathscr{B}};+,\cdot\right)$ with linear operators A^{+}_{vu}, A^{+}_{uv} for $\forall(v,u)\in E\left(\bigcup\limits_{G\in\mathscr{G}}\overrightarrow{G}\right)$ under operations $+$ and $\cdot$ form respectively a Banach or Hilbert space respect to that $\mathscr{B}$ is a Banach or Hilbert space.*

A few well-known results such as those of Banach theorem, closed graph theorem and Hahn-Banach theorem are also generalized on elements 1 and 2. For example, we obtained results following.

Theorem 3.4(Taylor, [15]) *Let $\overrightarrow{G}^{L}\in\left\langle\overrightarrow{G}_i,1\leq i\leq n\right\rangle^{\mathbb{R}\times\mathbb{R}^n}$ and there exist kth order derivative of L to t on a domain $\mathscr{D}\subset\mathbb{R}$, where $k\geq 1$. If A^{+}_{vu}, A^{+}_{uv} are linear for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, then*

$$\overrightarrow{G}^{L}=\overrightarrow{G}^{L(t_0)}+\frac{t-t_0}{1!}\overrightarrow{G}^{L'(t_0)}+\cdots+\frac{(t-t_0)^k}{k!}\overrightarrow{G}^{L^{(k)}(t_0)}+o\left((t-t_0)^{-k}\,\overrightarrow{G}\right),$$

for $\forall t_0\in\mathscr{D}$, where $o\left((t-t_0)^{-k}\,\overrightarrow{G}\right)$ denotes such an infinitesimal term $\widehat{L}$ of L that

$$\lim_{t\to t_0}\frac{\widehat{L}(v,u)}{(t-t_0)^k}=0\quad for\quad \forall(v,u)\in E\left(\overrightarrow{G}\right).$$

Particularly, if $L(v,u)=f(t)c_{vu}$, where c_{vu} is a constant, denoted by $f(t)\overrightarrow{G}^{L_C}$ with $L_C:(v,u)\to c_{vu}$ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$ and

$$f(t)=f(t_0)+\frac{(t-t_0)}{1!}f'(t_0)+\frac{(t-t_0)^2}{2!}f''(t_0)+\cdots+\frac{(t-t_0)^k}{k!}f^{(k)}(t_0)+o\left((t-t_0)^k\right),$$

then

$$f(t)\overrightarrow{G}^{L_C}=f(t)\cdot\overrightarrow{G}^{L_C}.$$

Theorem 3.5(Hahn-Banach, [19]) *Let $\mathscr{H}^{\pm}_{\mathscr{B}}$ be an element 2 subspace of $\mathscr{G}^{\pm}_{\mathscr{B}}$ and let $F:\mathscr{H}^{\pm}_{\mathscr{B}}\to\mathbb{C}$ be a linear continuous functional on $\mathscr{H}^{\pm}_{\mathscr{B}}$. Then, there is a linear continuous functional $\widetilde{F}:\mathscr{G}^{\pm}_{\mathscr{B}}\to\mathbb{C}$ hold with*

(1) $\widetilde{F}\left(\overrightarrow{G}^{L^2}\right) = F\left(\overrightarrow{G}^{L^2}\right)$ *if* $\overrightarrow{G}^{L^2} \in \mathscr{H}_{\mathscr{B}}^{\pm}$;

(2) $\left\|\widetilde{F}\right\| = \|F\|$.

For applications of elements 1 and 2 to physics and other sciences such as those of elementary particles, gravitations, ecological system, $\cdots$ etc., the reader is refereed to references [11]-[13] and [18]-[20] for details.

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Some Subclasses of
Meromorphic with p-Valent q-Spirallike Functions

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Abstract: In this paper, we introduce and investigate two new subclasses of the function class $\Sigma\mathcal{MS}(p,\lambda,\beta,q)$ and $\Sigma\mathcal{MC}(p,\lambda,\beta,q)$ of q-spirallike meromorphic functions defined in the punctured open unit disc.

Key Words: Univalent functions, meromorphic functions, meromorphic q-spirallike functions.

AMS(2010): 30C45.

§1. Introduction

Let Σ_p be the class of functions f of the form

$$f(z) = \frac{1}{z^p} + \sum_{n=1-p}^{\infty} a_n\ z^n,\ (p \in N = 1, 2, 3, \cdots), \tag{1}$$

which are analytic in the open disc $E^* = \{z : z \in \mathbb{C}\ and\ 0 < |z| < 1\}$. Let $\mathcal{S}$ be the subclass of functions in Σ_p which are univalent in E. Let $\mathcal{P}$ be the class of functions p given by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n\ z^n\ (z \in E), \tag{2}$$

which are analytic in the open disc E and satisfy the condition:

$$\Re\{p(z)\} > 0\ (z \in E). \tag{3}$$

If $f \in \Sigma_p$ and satisfies

$$-\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta\ (z \in E, 0 \leq \beta < p), \tag{4}$$

then we say that f is meromorphic p-valent starlike of order β $(0 \leq \beta < p)$ and we denote this class by $\Sigma\mathcal{MS}(p,\beta)$.

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If $f \in \Sigma_p$ and satisfies

$$-\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \beta \ (z \in E, 0 \le \beta < p), \tag{5}$$

then we say that f is meromorphic p-valent convex of order β and we denote this class by $f \in \Sigma\mathcal{MC}(p,\beta)$.

A function $f \in \Sigma_p$ is said to be λ-spirallike of order β in the unit disk E if

$$-\Re\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}\right\} > \beta, \ (z \in E, \ 0 \le \beta < p, \ |\lambda| < \frac{\pi}{2}).$$

In [8] Jackson introduced and studied the concept of the q-derivative operator ∂_q as follows

$$\partial_q f(z) = \frac{f(z)-f(qz)}{z\ (1-q)}, \quad (z \neq 0, \quad 0<q<1, \quad \partial_q f(0) = f'(0)). \tag{6}$$

Equivalently (6) may be written as

$$\partial_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q \ a_n \ z^{n-1}, \quad z \neq 0, \tag{7}$$

where $[n]_q = \frac{1-\ q^n}{1-\ q}$. Note that as $q \to 1^-$, $[n]_q \to n$.

Definition 1.1 *A function $f \in \Sigma_p$ is said to be meromorphic p-valent λ-q-spirallike functions of order β if it satisfies the following:*

$$-\Re\left\{e^{i\lambda}\frac{z\partial_q f(z)}{f(z)}\right\} > \beta \ (z \in E, \ |\lambda| < \frac{\pi}{2}, \ 0 \le \beta > p\cos\lambda, \ 0 < q \le 1), \tag{8}$$

we denote this class by $\Sigma\mathcal{MS}(p,\lambda,\beta,q)$.

Definition 1.2 *A function $f \in \Sigma_p$ is said to be meromorphic p-valent convex λ-q-spirallike functions of order β if it satisfies the following*

$$-\Re\left\{e^{i\lambda}\frac{\partial_q(z\partial_q f(z))}{\partial_q f(z)}\right\} > \beta \ (z \in E, 0 \le \beta < 1), \tag{9}$$

we denote this class by $\Sigma\mathcal{MC}(p,\lambda,\beta,q)$.

Remark 1.1 *$f \in \mathcal{MS}(p,\lambda,\beta,q)$ iff*

$$-e^{i\lambda}\frac{z\partial_q f(z)}{f(z)} \prec \frac{pe^{i\lambda}-(2\beta - pe^{-i\lambda})z}{1-z}, \tag{10}$$

and $f \in \mathcal{MC}(p,\lambda,\beta,q)$ iff

$$-e^{i\lambda}\left(\frac{\partial_q(z\partial_q f(z))}{\partial_q f(z)}\right) \prec \frac{pe^{i\lambda}-(2\beta - pe^{-i\lambda})z}{1-z}. \tag{11}$$

§2. Main Results

Theorem 2.1 *If the sequence* $\{A_{p+m}\}_0^\infty$ *defined by*

$$\begin{cases} A_p = \frac{2(\beta - p\cos\lambda)}{p+[p]_q}, & m=0, \\ A_{p+m} = \frac{2(\beta - p\cos\lambda)}{p+[p+m]_q}\left(1+\sum_{k=0}^{m-1}|a_{p+k}|\right), & m \in N, \end{cases} \tag{12}$$

and $p \in N$. *Then*

$$A_{p+m} = \frac{2(\beta - p\cos\lambda)}{2\beta + [m+p]_q + p - 2p\cos\lambda}\prod_{k=0}^{m}\frac{2\beta + [k+p]_q + p - 2p\cos\lambda}{p+[p+k]_q}, \tag{13}$$

where $m \in N_0 = N \setminus \{0\}$.

Proof By virtue of (12), we have

$$p + [p+m+1]_q A_{p+m+1} = 2(\beta - p\cos\lambda)\left(1+\sum_{k=0}^{m}A_{p+k}\right), \tag{14}$$

and

$$p + [p+m]_q A_{p+m} = 2(\beta - p\cos\lambda)\left(1+\sum_{k=0}^{m-1}A_{p+k}\right). \tag{15}$$

From (14) and (15), we have

$$\frac{A_{p+m+1}}{A_{p+m}} = \frac{2\beta + [m+p]_q + p - 2p\cos\lambda}{p+[p+m+1]_q} \quad m \in N_0. \tag{16}$$

$$\begin{aligned} A_{p+m} &= \frac{A_{p+m}}{A_{p+m-1}}.\frac{A_{p+m-1}}{A_{p+m-2}}\cdots\frac{A_{p+1}}{A_p}.A_p \\ &= \frac{2\beta+[m+p-1]_q+p-2p\cos\lambda}{p+[p+m]_q}\cdots\frac{2\beta+[p]_q+p-2p\cos\lambda}{p+[p+1]_q}.\frac{2\beta-2p\cos\lambda}{p+[p]_q} \\ &= \frac{2(\beta-p\cos\lambda)}{2\beta+[m+p]_q+p-2p\cos\lambda}\prod_{k=0}^{m}\frac{2\beta+[k+p]_q+p-2p\cos\lambda}{p+[p+k]_q}\ (m\in N). \end{aligned} \tag{17}$$

The proof of Theorem 2.1 is completed. □

As $q \to 1^-$, we get the following result proved by Shi.et al. [13].

Corollary 2.1 *If* $\{A_{p+m}\}_0^\infty$ *defined by*

$$\begin{cases} A_p = \frac{\beta - p\cos\lambda}{p}, & m=0, \\ A_{p+m} = \frac{2(\beta - p\cos\lambda)}{2p+m}\left(1+\sum_{k=0}^{m-1}|a_{p+k}|\right), & m \in N, \end{cases} \tag{18}$$

and $p \in N$. *Then*

$$A_{p+m} \leq \frac{2(\beta - p\cos\lambda)}{2\beta + m + 2p - 2p\cos\lambda} \prod_{k=0}^{m} \frac{2\beta + k + 2p - 2p\cos\lambda}{2p + k}, \tag{19}$$

where, $m \in N_0 = N \setminus \{0\}$.

Theorem 2.2 *Let* $f(z) = \frac{1}{z^p} + \sum_{m=0}^{\infty} a_{p+m} z^{p+m} \in \mathcal{MS}(p, \lambda, \beta, q)$. *Then*

$$|a_{p+m}| \leq \frac{2(\beta - p\cos\lambda)}{2\beta + [m+p]_q + p - 2p\cos\lambda} \prod_{k=0}^{m} \frac{2\beta + [k+p]_q + p - 2p\cos\lambda}{p + [p+k]_q} \ (m \in N_0). \tag{20}$$

Proof Let

$$L(z) = \frac{\beta + e^{i\lambda} \frac{z\partial_q f(z)}{f(z)} + ip\sin\lambda}{\beta - p\cos\lambda} \ (z \in E, \ f \in \mathcal{MS}(p, \lambda, \beta, q)). \tag{21}$$

We know that $L \in \mathcal{P}$. It follows that

$$e^{i\lambda} z\partial_q f(z) = (\beta - p\cos\lambda) f(z) L(z) - (\beta - ip\sin\lambda) f(z). \tag{22}$$

Let

$$L(z) = 1 + l_1 z + l_2 z^2 + \cdots . \tag{23}$$

Then

$$\begin{aligned} & e^{i\beta}\left(\frac{-[p]_q}{z^p} + [p]_q a_p z^p + [p+1]_q a_{p+1} z^{p+1} + \cdots + [p+m]_q a_{p+m} z^{p+m} + \cdots\right) \\ & = (\beta - p\cos\lambda)\left(\frac{1}{z^p} + a_p z^p + a_{p+1} z^{p+1} + \cdots\right) \times (1 + l_1 z + l_2 z^2 + \cdots) \\ & \quad - (\beta - ip\sin\lambda)\left(\frac{1}{z^p} + a_p z^p + a_{p+1} z^{p+1} + \cdots + a_{p+m} z^{p+m} + \cdots\right). \end{aligned} \tag{24}$$

We have from sides (24)

$$\begin{aligned} e^{i\lambda}[p+m]_q a_{p+m} &= (\beta - p\cos\lambda)\,(l_{2p+m} + a_p l_m + a_{p+m} l_{m-1} \\ & \quad + \cdots + a_{p+m}) - (\beta + ip\sin\lambda) a_{p+m}. \end{aligned} \tag{25}$$

Moreover, we know that

$$|l_n| \leq 2 \ (n \in N). \tag{26}$$

From (25) and(26) we have

$$|a_p| \leq \frac{2(\beta - p\cos\lambda)}{p + [p]_q} \tag{27}$$

and

$$|a_{p+m}| \leq \frac{2(\beta - p\cos\lambda)}{p+[p+m]_q}\left(1+\sum_{k=0}^{m-1}|a_{p+k}|\right) \tag{28}$$

with supposing $p \in N$. We define $\{A_{p+m}\}_{m=0}^{\infty}$ by

$$\begin{cases} A_p = \frac{2(\beta-p\cos\lambda)}{p+[p]_q}, & m=0; \\ A_{p+m} = \frac{2(\beta-p\cos\lambda)}{p+[p+m]_q}\left(1+\sum_{k=0}^{m-1}|a_{p+k}|\right), & m \geq 1. \end{cases} \tag{29}$$

Now by the mathematical induction principle we will prove that

$$|a_{p+m}| \leq A_{p+m}(m \in N_0). \tag{30}$$

We can easily verify that

$$|a_p| \leq A_p = \frac{2(\beta - p\cos\lambda)}{p+[p]_q}. \tag{31}$$

Thus, assuming that

$$|a_{p+j}| \leq A_{p+j}(j = 0, 1, \cdots, m.m \in N_0), \tag{32}$$

from (28)) and (32) we have

$$\begin{aligned} |a_{p+m+1}| \leq & \frac{2(\beta - p\cos\lambda)}{p+[p+m+1]_q}\left(1+\sum_{k=0}^{m}|a_{p+k}|\right) \\ \leq & \frac{2(\beta - p\cos\lambda)}{p+[p+m+1]_q}\left(1+\sum_{k=0}^{m}|A_{p+k}|\right) \\ = & A_{p+m+1} \ (m \in N_0). \end{aligned} \tag{33}$$

Therefore, by the principle of mathematical induction, we have

$$|a_{p+m}| \leq A_{p+m} \ (m \leq N_0). \tag{34}$$

By means of Theorem 2.1 and (29), we know that

$$A_{p+m} = \frac{2(\beta - p\cos\lambda)}{2\beta + [m+p]_q + p - 2p\cos\lambda}\prod_{k=0}^{m}\frac{2\beta + [k+p]_q + p - 2p\cos\lambda}{p+[p+k]_q}(m \in n_0). \tag{35}$$

So, from (34) and (35) we get proof of the Theorem 2.2. □

As $q \to 1^-$ we get the following result proved by Shi.et al. [13].

Corollary 2.2 *Let* $f(z) = \frac{1}{z^p} + \sum_{m=0}^{\infty} a_{p+m}z^{p+m} \in \mathcal{MS}(p, \lambda, \beta)$. *Then*

$$A_{p+m} = \frac{2(\beta - p\cos\lambda)}{2\beta + m + 2p - 2p\cos\lambda}\prod_{k=0}^{m}\frac{2\beta + k + 2p - 2p\cos\lambda}{2p+k}(m \in n_0). \tag{36}$$

From Theorem 2.2 we get the following result.

Corollary 2.3 *Let* $f(z) = \frac{1}{z^p} + \sum_{m=0}^{\infty} a_{p+m} z^{p+m} \in \mathcal{MC}(p, \lambda, \beta, q)$. *Then*

$$A_{p+m} = \frac{2p(\beta - p\cos\lambda)}{[p+m](2\beta + [m+p]_q + p - 2p\cos\lambda)} \prod_{k=0}^{m} \frac{2\beta + [k+p]_q + p - 2p\cos\lambda}{p + [p+k]_q} (m \in n_0). \quad (37)$$

Theorem 2.3 *Let* $f(z) = \frac{1}{z^p} + \sum_{m=0}^{\infty} a_{p+m} z^{p+m} \in \mathcal{MS}(p, \lambda, \beta, q)$. *Then*

$$\frac{p\cos\lambda - 2(\beta - p\cos\lambda)r}{1-r} \leq \Re\left(-e^{i\vartheta} \frac{z\partial_q f(z)}{f(z)}\right) \leq \frac{p\cos\lambda + 2(\beta - p\cos\lambda)r}{1+r} \quad (38)$$

for $|z| = r < 1$.

Proof Suppose the function ϕ defined by

$$\phi(z) = \frac{pe^{i\lambda} - (2\beta - pe^{-i\lambda}z)}{1-z} \ (z \in E). \quad (39)$$

Let $z = re^{i\lambda}$ $(0 < r < 1)$. We have

$$\Re\{\phi(z)\} = p\cos\lambda - \frac{2(\beta - p\cos\lambda)r(\cos\vartheta - r)}{1 + r^2 - 2r\cos\vartheta}. \quad (40)$$

Let

$$\varphi(\tau) = p\cos\lambda - \frac{2(\beta - p\cos\lambda)r(\tau - r)}{1 + r^2 - 2\tau r} \ (\tau = \cos\vartheta). \quad (41)$$

Then

$$\partial_q\varphi(\tau) = \frac{-2r(\beta - p\cos\lambda)[(1 + r^2 - 2\tau r) - r[2r]_q(\tau - r)]}{(1 + r^2 - 2qr\tau)(1 + r^2 - 2r\tau)}. \quad (42)$$

This means that

$$p\cos\lambda - \frac{2(\beta - p\cos\lambda)r}{1-r} \leq \Re(\phi(z)) \leq p\cos\lambda + \frac{2(\beta - p\cos\lambda)r}{1+r}, \quad (43)$$

which is equivalent to

$$\frac{p\cos\lambda - 2(\beta - p\cos\lambda)r}{1-r} \leq \Re\{\phi(z)\} \leq \frac{p\cos\lambda + 2(\beta - p\cos\lambda)r}{1+r}. \quad (44)$$

We know that

$$-e^{i\lambda} \frac{z\partial_q f(z)}{f(z)} \prec \phi(z)$$

and $\phi(z)$ is univalent in E, this is prove the inequality (38). □

As $q \to 1^-$ we get the following result proved by Shi.et al. [13].

Corollary 2.4 *Let* $f(z)=\frac{1}{z^p}+\sum_{m=0}^{\infty}a_{p+m}z^{p+m}\in\mathcal{MS}(p,\lambda,\beta)$. *Then*

$$\frac{p\cos\lambda-2(\beta-p\cos\lambda)r}{1-r}\le\Re\left(-e^{i\lambda}\frac{zf'(z)}{f(z)}\right)\le\frac{p\cos\lambda+2(\beta-p\cos\lambda)r}{1+r},\tag{45}$$

for $|z|=r<1$.

Corollary 2.5 *Let* $f(z)=\frac{1}{z^p}+\sum_{m=0}^{\infty}a_{p+m}z^{p+m}\in\mathcal{MC}(p,\lambda,\beta,q)$. *Then*

$$\frac{p\cos\lambda-2(\beta-p\cos\lambda)r}{1-r}\le\Re\left(-e^{i\lambda}\frac{\partial_q(z\partial_q f(z))}{\partial_q f(z)}\right)\le\frac{p\cos\lambda+2(\beta-p\cos\lambda)r}{1+r}\tag{46}$$

for $|z|=r<1$.

Theorem 2.4 *If* $f\in\Sigma_p$ *satisfies*

$$q\sum_{n=1-p}^{\infty}\left(|[n]_qe^{i\lambda}+\gamma|+|[n]_qe^{i\lambda}+2\beta-\gamma|\right)|a_n|\le|[p]_qe^{i\lambda}-2q\beta+q\gamma|-|[p]_qe^{i\lambda}-q\gamma|\tag{47}$$

for some real λ,β *and* γ $(0\le\gamma\le p\cos\lambda)$, *then* $f\in\mathcal{MS}(p,\lambda,\beta,q)$

Proof To prove $f\in\mathcal{MS}(p,\lambda,\beta,q)$, it suffices to show that

$$\left|\frac{e^{i\lambda}\frac{z\partial_q f(z)}{f(z)}+\gamma}{e^{i\lambda}\frac{z\partial_q f(z)}{f(z)}+(2\beta-\gamma)}\right|<1\ (z\in E,0\le\gamma\le p\cos\lambda).\tag{48}$$

From (47), we know that

$$\begin{aligned}\left|[p]_qe^{i\lambda}-2q\beta+q\gamma\right|+q\sum_{n=1-p}^{\infty}\left|[n]_qe^{i\lambda}+2\beta-\gamma\right||a_n| \quad&\ge\quad \left|[p]_qe^{i\lambda}-q\gamma\right|\\ &+q\sum_{n=1-p}^{\infty}|[n]_qe^{i\lambda}+\gamma||a_n|>0.\end{aligned}\tag{49}$$

Now, by the maximum modulus principle, we deduce from (1) and (49) that

$$\begin{aligned}\left|\frac{e^{i\lambda}\frac{z\partial_q f(z)}{f(z)}+\gamma}{e^{i\lambda}\frac{z\partial_q f(z)}{f(z)}+(2\beta-\gamma)}\right|&=\left|\frac{e^{i\lambda}\left(\frac{-[p]_q}{qz^p}+\sum_{n=1-p}^{\infty}[n]_qa_nz^n\right)+\frac{\gamma}{z^p}+\gamma\sum_{n=1-p}^{\infty}a_nz^n}{e^{i\lambda}\left(\frac{-[p]_q}{qz^p}+\sum_{n=1-p}^{\infty}[n]_qa_nz^n\right)+(2\beta-\gamma)\left(\frac{1}{z^p}+\sum_{n=1-p}^{\infty}a_nz^n\right)}\right|\\&=\left|\frac{(-[p]_qe^{i\lambda}+q\gamma)+q\sum_{n=1-p}^{\infty}([n]_qe^{i\lambda}+\gamma)a_nz^n}{(-[p]_qe^{i\lambda}+2q\beta-q\gamma)+q\sum_{n=1-p}^{\infty}([n]_qe^{i\lambda}+2\beta-\gamma)a_nz^n}\right|\\&<\frac{|[p]_qe^{i\lambda}-q\gamma|+q\sum_{n=1-p}^{\infty}|[n]_qe^{i\lambda}+\gamma||a_n|}{|[p]_qe^{i\lambda}-2q\beta+q\gamma|-q\sum_{n=1-p}^{\infty}|[n]_qe^{i\lambda}+2\beta-\gamma||a_n|}\\&\le1.\end{aligned}\tag{50}$$

This completes the proof. □

As $q \to 1^-$ we get the following result proved by Shi.et al.[13].

Corollary 2.6 *If $f \in \Sigma_p$ satisfies the*

$$\sum_{n=1-p}^{\infty} \left(|ne^{i\lambda}+\gamma| + |ne^{i\lambda}+2\beta-\gamma|\right)|a_n| \leq |pe^{i\lambda}-2\beta+\gamma| - |pe^{i\lambda}-\gamma| \tag{51}$$

for some real λ, β and γ $(0 \leq \gamma \leq p\cos\lambda)$, then $f \in \mathcal{MC}(p,\lambda,\beta)$.

Corollary 2.7 *If $f \in \sum @_p$ satisfies the*

$$\begin{aligned} q\sum_{n=1-p}^{\infty} |[n]_q| \left(|[n]_q e^{i\lambda}+\gamma| + |[n]_q e^{i\lambda}+2\beta-\gamma|\right)|a_n| \quad &\leq \quad [p]_q \left(|[p]_q e^{i\lambda} - 2q\beta + q\gamma|\right. \\ &\left. - |[p]_q e^{i\lambda} - q\gamma|\right) \end{aligned} \tag{52}$$

for some real λ, β and γ $(0 \leq \gamma \leq p\cos\lambda)$, then $f \in \mathcal{MC}(p,\lambda,\beta,q)$.

Lemma 2.1([7]) *If $|\phi|$ attains its maximum value on the circle $|z| = r < 1$ at z_0 and ϕ is a nonconstant regular function in E then*

$$z_0\phi'(z_0) = k\phi(z_0), \ \ k \geq 1, \ \ k \in R.$$

Theorem 2.5 *If $f \in \Sigma_p$ satisfies*

$$\left| \frac{f(z)}{f(qz)} + \frac{zf(z)\partial_q^2 f(z)}{f(qz)\partial_q f(z)} - \frac{z\partial_q f(z)}{f(qz)} \right| < \frac{\beta - p}{2\beta} \tag{53}$$

for some real $\beta > p$, than $f \in \mathcal{MS}(p,0,\beta,q)$.

Proof Define the function φ by

$$\varphi(z) = \frac{\frac{z\partial_q f(z)}{f(z)} + p}{\frac{z\partial_q f(z)}{f(z)} + (2\beta - p)} \ (z \in E). \tag{54}$$

Note that φ is analytic in E and $\varphi(0) = 0$. From (54), we have

$$\frac{z\partial_q f(z)}{f(z)} = \frac{-p + (2\beta - p)\varphi(z)}{1 - \varphi(z)}. \tag{55}$$

Taking q-differentiating of (55) logarithmically, we get

$$\frac{f(z)}{f(qz)} + \frac{zf(z)\partial_q^2 f(z)}{f(qz)\partial_q f(z)} - \frac{z\partial_q f(z)}{f(qz)} = \frac{z(1-\varphi(z))(2\beta-p)\partial_q\varphi(z)}{(-p+(2\beta-p)\varphi(z))(1-\varphi(qz))} + \frac{z\partial_q\varphi(z)}{(1-\varphi(qz))}. \tag{56}$$

From (53) and (56), we get that

$$\left|\frac{f(z)}{f(qz)}+\frac{zf(z)\partial_q^2 f(z)}{f(qz)\partial_q f(z)}-\frac{z\partial_q f(z)}{f(qz)}\right|=\left|\frac{2(\beta-p)\partial_q\varphi(z)}{[-p+(2\beta-p)\varphi(z)](1-\varphi(qz))}\right|<\frac{\beta-p}{2\beta}. \tag{57}$$

Consider $z_0 \in E$ such that

$$\max_{|z|\le|z_0|}|\varphi(z)|=|\varphi(z_0)|=1.$$

By Lemma 2.1, let $\varphi(z_0)=e^{i\vartheta}$ and $z_0\partial_q\varphi(z_0)=Le^{i\vartheta}$ $(L\ge 1)$. For such a point z_0, we have that

$$\begin{aligned}&\left|\frac{f(z_0)}{f(qz_0)}+\frac{z_0 f(z_0)\partial_q^2 f(z_0)}{f(qz_0)\partial_q f(z_0)}-\frac{z_0\partial_q f(z_0)}{f(qz_0)}\right|\\ &=\left|\frac{2(\beta-p)Le^{i\vartheta}}{[-p+(2\beta-p)e^{i\vartheta}](1-e^{i\vartheta})}\right|\\ &=\frac{2(\beta-p)L}{\sqrt{p^2+(2\beta-p)^2-2p(2\beta-p)\cos\vartheta}\sqrt{2(1-\cos\vartheta)}}\\ &\ge\frac{\beta-p}{2\beta}.\end{aligned} \tag{58}$$

This contradicts our condition (53). Therefore ,there is no $z_0 \in E$ such that $|\varphi(z_0)|=1$. This implies that $|\varphi(z)|<1$ $(z\in E^*)$, that is,

$$\left|\frac{\frac{z\partial_q f(z)}{f(z)}+p}{\frac{z\partial_q f(z)}{f(z)}+(2\beta-p)}\right|<1,\ (z\in E)$$

thus, we conclude that $f\in\mathcal{MS}(p,0,\beta,q)$. □

As $q\to 1^-$ we get the following result proved by Shi.et al. [13].

Corollary 2.8 *If $f\in\Sigma_p$ satisfies*

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right|<\frac{\beta-p}{2\beta} \tag{59}$$

for some real $\beta>p$, than $f\in\mathcal{MS}(p,0,\beta)$.

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On the Complexity of Some Classes of Circulant Graphs and Chebyshev Polynomials

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Abstract: Deriving closed formulae of the number of spanning trees for various graphs has attracted the attention of a lot of researchers. In this paper we derive simple and explicit formulas for the number of spanning trees in many classes of circulant graphs using the properties of Chebyshev polynomials. Deriving closed formulae of the number of spanning trees for various graphs has attracted the attention of a lot of researchers. In this paper we derive simple and explicit formulas for the number of spanning trees in many classes of circulant graphs using the properties of Chebyshev polynomials.

Key Words: Number of spanning trees, circulant graphs, Chebyshev polynomials.

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§1. Introduction

The number of spanning trees $\tau(G)$ in a graph G (networks) is an important invariant. We call $\tau(G)$ the complexity of G. The evaluation of this number is not only interesting from a mathematical (computational) perspective, but also, it is an important measure of reliability of a network and designing electrical circuits. Some computationally hard problems such as the travelling salesman problem can be solved approximately by using spanning trees. In this work we consider finite undirected graph with no loops or multiple edges. Let G be such a graph of n vertices. A spanning tree for a graph G is a subgraph of G that is a tree and contains all vertices of $\tau(G)$. The number of spanning trees of G, is the total number of distinct spanning subgraphs of G that are trees. A classic result of Kirchhoff [**?**] can be used to determine the number $\tau(G)$ for $G(V,E)$. Let $V = \{v_1, v_2, \cdots, v_n\}$. The Kirchhoff matrix H is defined as $n \times n$ characteristic matrix $H = D - A$, where D is the diagonal matrix of the degrees of G and A is the adjacency matrix of G. Then the matrix $H - [a_{ij}]$ is defined as follows:

(1) a_{ij}, when v_i and v_j are adjacent and $i \neq j$;

(*ii*) a_{ij}, is equal to the degree of vertex v_i if $i = j$;

(*iii*) $a_{ij} = 0$, otherwise. All of co-factors of H are equal to $\tau(G)$.

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There are more than one method for calculating $\tau(G)$. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p$ denote the eigenvalues of matrix of a p point graph. It can be easily shown that $\mu_p = 0$. Kelmans and Chelnokov [2] proved that The formula for the number of spanning trees in a d-regular graph can be expressed as

$$\tau(G) = \frac{1}{p}\Big[\prod_{k=1}^{p-1}(d-\mu_k)\Big]$$

where $\mu_0 = d, \mu_1, \mu_2, \cdots, \mu_{p-1}$ are the eigenvalues of the corresponding adjacency matrix of the graph. Many works have conceived techniques to derive the number of spanning tree of a graph can be found at [3-12]. The circulant graphs are an important class of graphs. Among other applications, they are used in the design of local area networks, see [13-19].

Let $1 \leq a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_k \leq \frac{n}{2}$, where n and $a_i (i = 1, 2, \ldots, k)$ are positive integers. An undirected circulant graph $C_n(a_1, a_2, a_3, \cdots, a_k)$ is a regular graph whose set of vertices is $V = \{0, 1, 2, [dots, n-1\}$ and whose set of edges is $E = \{i, i+a_i (\text{mod } n)/i = 0, 1, 2, \cdots, n-1, j = 1, 2, \cdots, k\}$. If $a_k \leq \frac{n}{2}$, then $C_n(a_1, a_2, a_3, \cdots, a_k)$ is a $2k$-regular graph; if $a_k = \frac{n}{2}$, then it is a $2k-1$-regular one, see Nikolopoulos [20] and Papadopoulos [21]. The well known formula $\tau(C_n(1, 2)) = nF_n^2$, where F_n is the n^{th} Fibonacci number, see Kleiman, and Golden [22]. We have obtained another proof for this formula in Theorem 3.3. The formulas of $\tau(C_{2n}(1, n))$, $\tau(C_{3n}(1, n))$, $\tau(C_{4n}(1, n))$ can be found in Yuanping, et. al.[23].

§2. Chebyshev Polynomial

In this section we introduce some relations concerning Chebyshev polynomials of the first and second kind which we use it in our computations. We begin from their definitions, see Yuanping, et. al.[24]. Let $A_n(x)$ be $n \times n$ matrix such that:

$$A_n(x) = \begin{pmatrix} 2x & -1 & 0 & \dots & \dots \\ -1 & 2x & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2x & -1 \\ \vdots & \ddots & 0 & -1 & 2x \end{pmatrix}$$

where all other elements are zeros.

Further, we recall that the Chebyshev polynomials of the first kind are defined by:

$$T_n(x) = \cos(n \arccos). \tag{1}$$

The Chebyshev polynomials of the second kind are defined by

$$U_{n-1}(x) = \frac{1}{n}\frac{d}{dx}T_n(x) = \frac{\sin(n \arccos)}{\sin(\arccos)}. \tag{2}$$

It is easily verified that

$$U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0. \tag{3}$$

It can then be shown from this recursion that by expanding one gets

$$U_n(x) = det(A_n(x)), \quad n \geq 1. \tag{4}$$

Furthermore by using standard methods for solving the recursion (3), one obtains the explicit formula

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\Big[\big(x+\sqrt{x^2-1}\big)^{n+1} - \big(x-\sqrt{x^2-1}\big)^{n+1}\Big], \quad n \geq 1, \tag{5}$$

where the identity is true for all complex (except at $x = \pm 1$, where the function can be taken as the limit). The definition of easily yields its zeros and it can therefore be verified that

$$U_{n-1}(x) = 2^{n-1}\Big[\prod_{j=1}^{n-1}(x - \cos\frac{j\pi}{n})\Big]. \tag{6}$$

One further notes that

$$U_{n-1}(-x) = (-1)^{n-1}U_{n-1}(x). \tag{7}$$

These two results yield another formula for $U_n(x)$

$$U_{n-1}^2(x) = 4^{n-1}\Big[\prod_{j=1}^{n-1}(x^2 - \cos^2\frac{j\pi}{n})\Big]. \tag{8}$$

Finally, a simple manipulation of the above formula yields the following formula (9), which is extremely useful to us latter:

$$U_{n-1}^2\Big(\sqrt{\frac{x+2}{4}}\Big) = \Big[\prod_{j=1}^{n-1}(x - 2\cos\frac{j\pi}{n})\Big] \tag{9}$$

Furthermore, one can show that

$$U_{n-1}^2(x) = \frac{1}{2(1-x^2)}\big[1 - T_{2n}\big] = \frac{1}{2(1-x^2)}\big[1 - T_{2n}(2-2x^2)\big] \tag{10}$$

and

$$T_n(x) = \frac{1}{2}\Big[\big(x+\sqrt{x^2-1}\big)^n + \big(x-\sqrt{x^2-1}\big)^n\Big]. \tag{11}$$

§3. Main Results

In our main results, i.e., Theorems 3.1 - 3.6 we use the following conclusion.

Lemma 3.1([25]) *The Kirchhoff matrix of the circulant graph* $C_n(s_1, s_2, s_3, \cdots, s_k)$ *has* n

eigenvalues, namely: 0 and the value $2k-\varepsilon^{-s_1 j}-\cdots-\varepsilon^{-s_k j}-\varepsilon^{-s_1 j}-\cdots-\varepsilon^{-s_k j}$ *with* $\varepsilon = e^{\frac{2\pi j}{n}}$ *for any* $j=\{1,2,\cdots,n-1\}$

Corollary 3.2 *For the circulant graph* $C_n(s_1,s_2,s_3,\cdots,s_k)$,

$$\begin{aligned}\tau(C_n(s_1,s_2,s_3,\cdots,s_k)) &= \frac{1}{n}\left[\prod_{j=1}^{n-1}\left(2k-\varepsilon^{-s_1 j}-\cdots-\varepsilon^{-s_k j}-\varepsilon^{-s_1 j}-\cdots-\varepsilon^{-s_k j}\right)\right]\\ &= \frac{1}{n}\left[\prod_{j=1}^{n-1}\left(\sum_{i=1}^{k}(2-2\cos\frac{j s_i\pi}{n})\right)\right].\end{aligned}$$

Proof The proof follows immediately from Lemma 3.1. □

Theorem 3.3 *For the spanning trees of* C_{12n} *with three jumps* $1,2n,3n$ *, we have:*

$$\begin{aligned}\tau(C_{12n}(1,2n,3n)) &= \frac{n}{12}\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{4n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{4n}-1\right]^2\\ &\times\frac{n}{12}\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{2n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{2n}+1\right]^2\\ &\times\frac{n}{12}\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{2n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{2n}\right]^2\\ &\times\frac{n}{12}\left[\left(\sqrt{2}+1\right)^{n}+\left(\sqrt{2}-1\right)^{n}\right]^2\\ &\times\frac{n}{12}\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{2n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{2n}-1\right]^2\end{aligned}$$

Proof Let $\varepsilon = e^{\frac{2\pi i}{12n}}$. Applying Lemma 3.1, we have

$$\begin{aligned}\tau(C_{12n}(n,2n,3n)) &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(6-\varepsilon^{-j}-\varepsilon^{-2nj}-\varepsilon^{-3nj}-\varepsilon^{j}-\varepsilon^{2nj}-\varepsilon^{3nj}\right)\\ &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{4\pi jn}{12n}-2\cos\frac{6\pi jn}{12n}\right)\\ &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}-2\cos\frac{\pi j}{2}\right)\end{aligned}$$

$$= \frac{1}{12n}\prod_{j=1,2\nmid j}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}\right)$$

$$\times \prod_{j=1,2|j}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}-2\cos\frac{\pi j}{2}\right)$$

$$= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}\right)$$

$$\times \prod_{j=1}^{12n-1}\frac{6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}-2\cos\frac{\pi j}{2}}{6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}}$$

If we put $j=2j'$ in the second term for some integer j' we get

$$\tau(C_{12n}(n,2n,3n)) = \frac{1}{12n}\prod_{j=1}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}\right)$$

$$\times \prod_{j=1}^{6n-1}\frac{6-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-2\cos \pi j}{6-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}}$$

$$= \frac{1}{12n}\prod_{j=1,2\nmid j,3\nmid j}^{12n-1}\left(5-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1,2|j,3|j}^{12n-1}\left(4-2\cos\frac{2\pi j}{12n}\right)$$

$$\times \prod_{j=1,2|j,3|j}^{12n-1}\left(8-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1,2|j,3|j}^{12n-1}\left(7-2\cos\frac{2\pi j}{12n}\right)$$

$$\times\frac{\prod_{j=1,2\nmid j}^{6n-1}\left(8-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}\right)\prod_{j=1,2\nmid j}^{6n-1}\left(4-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}\right)}{\prod_{j=1}^{6n-1}\left(6-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}\right)}.$$

Thus,

$$\tau(C_{12n}(n,2n,3n)) = \frac{1}{12n}\prod_{j=1}^{12n-1}\left(5-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1}^{2n-1}\frac{\left(5-2\cos\frac{2\pi j}{2n}\right)\left(4-2\cos\frac{2\pi j}{2n}\right)}{\left(8-2\cos\frac{2\pi j}{2n}\right)\left(7-2\cos\frac{2\pi j}{2n}\right)}$$

$$\times \prod_{j=1}^{6n-1}\frac{8-2\cos\frac{2\pi j}{6n}}{5-2\cos\frac{2\pi j}{6n}}\prod_{j=1}^{4n-1}\frac{7-2\cos\frac{2\pi j}{4n}}{5-2\cos\frac{2\pi j}{4n}}$$

$$\times\frac{\prod_{j=1}^{6n-1}\left(8-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}\right)\prod_{j=1}^{3n-1}\left(4-2\cos\frac{2\pi j}{3n}-2\cos\frac{4\pi j}{3}\right)}{\prod_{j=1}^{6n-1}\left(6-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}\right)\prod_{j=1}^{3n-1}\left(8-2\cos\frac{2\pi j}{3n}-2\cos\frac{4\pi j}{3}\right)}.$$

So, we get that

$$\tau(C_{12n}(n,2n,3n)) = \frac{1}{12n}\prod_{j=1}^{12n-1}\left(5-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1}^{2n-1}\frac{\left(5-2\cos\frac{2\pi j}{2n}\right)\left(4-2\cos\frac{2\pi j}{2n}\right)}{\left(8-2\cos\frac{2\pi j}{2n}\right)\left(7-2\cos\frac{2\pi j}{2n}\right)}$$

$$\times\prod_{j=1}^{6n-1}\frac{8-2\cos\frac{2\pi j}{6n}}{5-2\cos\frac{2\pi j}{6n}}\prod_{j=1}^{4n-1}\frac{7-2\cos\frac{2\pi j}{4n}}{5-2\cos\frac{2\pi j}{4n}}$$

$$\times\frac{\prod\limits_{j=1,3|j}^{6n-1}\left(9-2\cos\frac{2\pi j}{6n}\right)\prod\limits_{j=1,3|j}^{6n-1}\left(6-2\cos\frac{2\pi j}{6n}\right)}{\prod\limits_{j=1,3|j}^{6n-1}\left(7-2\cos\frac{2\pi j}{6n}\right)\prod\limits_{j=1,3|j}^{6n-1}\left(4-2\cos\frac{2\pi j}{6n}\right)}$$

$$\times\frac{\prod\limits_{j=1}^{3n-1}\left(6-2\cos\frac{2\pi j}{3n}-4\cos^2\frac{2\pi j}{3}\right)}{\prod\limits_{j=1}^{3n-1}\left(10-2\cos\frac{2\pi j}{3n}-4\cos^2\frac{2\pi j}{3}\right)},$$

i.e.,

$$\tau(C_{12n}(n,2n,3n)) = \frac{1}{12n}\prod_{j=1}^{12n-1}\left(5-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1}^{2n-1}\frac{\left(5-2\cos\frac{2\pi j}{2n}\right)\left(4-2\cos\frac{2\pi j}{2n}\right)}{\left(8-2\cos\frac{2\pi j}{2n}\right)\left(7-2\cos\frac{2\pi j}{2n}\right)}$$

$$\times\prod_{j=1}^{6n-1}\frac{8-2\cos\frac{2\pi j}{6n}}{5-2\cos\frac{2\pi j}{6n}}\prod_{j=1}^{4n-1}\frac{7-2\cos\frac{2\pi j}{4n}}{5-2\cos\frac{2\pi j}{4n}}$$

$$\times\frac{\prod\limits_{j=1}^{6n-1}\left(9-2\cos\frac{2\pi j}{6n}\right)\prod\limits_{j=1}^{6n-1}\frac{\left(6-2\cos\frac{2\pi j}{6n}\right)}{\left(9-2\cos\frac{2\pi j}{6n}\right)}}{\prod\limits_{j=1}^{6n-1}\left(7-2\cos\frac{2\pi j}{6n}\right)\prod\limits_{j=1}^{6n-1}\frac{\left(4-2\cos\frac{2\pi j}{6n}\right)}{\left(7-2\cos\frac{2\pi j}{6n}\right)}}$$

$$\times\frac{\prod\limits_{j=1,3|j}^{3n-1}\left(5-2\cos\frac{2\pi j}{3n}\right)\prod\limits_{j=1,3|j}^{3n-1}\left(2-2\cos\frac{2\pi j}{3n}\right)}{\prod\limits_{j=1,3|j}^{3n-1}\left(9-2\cos\frac{2\pi j}{3n}\right)\prod\limits_{j=1,3|j}^{3n-1}\left(6-2\cos\frac{2\pi j}{3n}\right)}$$

Furthermore,

$$\tau(C_{12n}(n,2n,3n)) = \frac{1}{12n}\prod_{j=1}^{12n-1}\left(5-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1}^{2n-1}\frac{\left(5-2\cos\frac{2\pi j}{2n}\right)\left(4-2\cos\frac{2\pi j}{2n}\right)}{\left(8-2\cos\frac{2\pi j}{2n}\right)\left(7-2\cos\frac{2\pi j}{2n}\right)}$$

$$\times\prod_{j=1}^{6n-1}\frac{8-2\cos\frac{2\pi j}{6n}}{5-2\cos\frac{2\pi j}{6n}}\prod_{j=1}^{4n-1}\frac{7-2\cos\frac{2\pi j}{4n}}{5-2\cos\frac{2\pi j}{4n}}$$

$$\times\frac{\prod_{j=1}^{6n-1}\left(9-2\cos\frac{2\pi j}{6n}\right)\prod_{j=1}^{2n-1}\frac{\left(6-2\cos\frac{2\pi j}{6n}\right)}{\left(9-2\cos\frac{2\pi j}{6n}\right)}}{\prod_{j=1}^{6n-1}\left(7-2\cos\frac{2\pi j}{6n}\right)\prod_{j=1}^{2n-1}\frac{\left(4-2\cos\frac{2\pi j}{6n}\right)}{\left(7-2\cos\frac{2\pi j}{6n}\right)}}$$

$$\times\frac{\prod_{j=1}^{3n-1}\left(5-2\cos\frac{2\pi j}{3n}\right)\prod_{j=1}^{3n-1}\frac{\left(2-2\cos\frac{2\pi j}{3n}\right)}{\left(5-2\cos\frac{2\pi j}{3n}\right)}}{\prod_{j=1}^{3n-1}\left(9-2\cos\frac{2\pi j}{3n}\right)\prod_{j=1}^{3n-1}\frac{\left(6-2\cos\frac{2\pi j}{3n}\right)}{\left(9-2\cos\frac{2\pi j}{3n}\right)}}.$$

We get that

$$\tau(C_{12n}(n,2n,3n)) = \frac{1}{12n}\prod_{j=1}^{12n-1}\left(5-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1}^{2n-1}\frac{\left(5-2\cos\frac{2\pi j}{2n}\right)\left(4-2\cos\frac{2\pi j}{2n}\right)}{\left(8-2\cos\frac{2\pi j}{2n}\right)\left(7-2\cos\frac{2\pi j}{2n}\right)}$$

$$\times\prod_{j=1}^{6n-1}\frac{8-2\cos\frac{2\pi j}{6n}}{5-2\cos\frac{2\pi j}{6n}}\prod_{j=1}^{4n-1}\frac{7-2\cos\frac{2\pi j}{4n}}{5-2\cos\frac{2\pi j}{4n}}$$

$$\times\frac{\prod_{j=1}^{6n-1}\left(9-2\cos\frac{2\pi j}{6n}\right)\prod_{j=1}^{2n-1}\frac{\left(6-2\cos\frac{2\pi j}{6n}\right)}{\left(9-2\cos\frac{2\pi j}{6n}\right)}}{\prod_{j=1}^{6n-1}\left(7-2\cos\frac{2\pi j}{6n}\right)\prod_{j=1}^{2n-1}\frac{\left(4-2\cos\frac{2\pi j}{6n}\right)}{\left(7-2\cos\frac{2\pi j}{6n}\right)}}$$

$$\times\frac{\prod_{j=1}^{3n-1}\left(5-2\cos\frac{2\pi j}{3n}\right)\prod_{j=1}^{n-1}\frac{\left(2-2\cos\frac{2\pi j}{n}\right)}{\left(5-2\cos\frac{2\pi j}{n}\right)}}{\prod_{j=1}^{3n-1}\left(9-2\cos\frac{2\pi j}{n}\right)\prod_{j=1}^{n-1}\frac{\left(6-2\cos\frac{2\pi j}{n}\right)}{\left(9-2\cos\frac{2\pi j}{n}\right)}}.$$

Thus,

$$\tau(C_{12n}(n,2n,3n)) = \frac{1}{12n}\times U^2_{12n-1}\left(\sqrt{\frac{7}{4}}\right)\times\frac{U^2_{2n-1}\left(\sqrt{\frac{7}{4}}\right)\times U^2_{2n-1}\left(\sqrt{\frac{3}{2}}\right)}{U^2_{2n-1}\left(\frac{3}{2}\right)\times U^2_{2n-1}\left(\sqrt{\frac{5}{2}}\right)}$$

$$\times\frac{U^2_{4n-1}\left(\frac{3}{2}\right)\times U^2_{6n-1}\left(\sqrt{\frac{5}{2}}\right)}{U^2_{6n-1}\left(\sqrt{\frac{7}{2}}\right)\times U^2_{6n-1}\left(\sqrt{\frac{7}{2}}\right)}$$

$$\times\frac{U^2_{6n-1}\left(\sqrt{\frac{11}{4}}\right)\times U^2_{2n-1}\left(\frac{3}{2}\right)\times U^2_{2n-1}\left(\sqrt{2}\right)\times U^2_{3n-1}\left(\frac{7}{4}\right)\times U^2_{n-1}\left(\frac{11}{4}\right)n^2}{U^2_{6n-1}\left(\frac{3}{2}\right)\times U^2_{2n-1}\left(\sqrt{\frac{3}{2}}\right)\times U^2_{n-1}\left(\sqrt{\frac{7}{4}}\right)\times U^2_{3n-1}\left(\sqrt{\frac{11}{4}}\right)\times U^2_{n-1}\left(\sqrt{2}\right)},$$

which implies that

$$\tau(C_{12n}(n,2n,3n)) = \frac{n}{12}\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{4n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{4n}-1\right]^2$$

$$\times\frac{n}{12}\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{2n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{2n}+1\right]^2$$
$$\times\frac{n}{12}\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{2n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{2n}\right]^2$$
$$\times\frac{n}{12}\left[\left(\sqrt{2}+1\right)^{n}+\left(\sqrt{2}-1\right)^{n}\right]^2$$
$$\times\frac{n}{12}\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{2n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{2n}-1\right]^2,$$

where (6), (8) , (9) and (10) are used to derive the last two steps. □

Theorem 3.4 *For the spanning trees of C_{6n} with three jumps $1, 3n, 6n$, we have*

$$\tau(C_{12n}(1,3n,6n)) = \frac{3n}{4}\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{6n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{6n}\right]^2 \times\frac{3n}{4}\left[\left(\sqrt{2}+1\right)^{3n}+\left(\sqrt{2}-1\right)^{3n}\right]^2$$

Proof Let $\varepsilon = e^{\frac{2\pi i}{12n}}$. Applying Lemma 3.1, we have

$$\begin{aligned}\tau(C_{12n}(1,3n,6n)) &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(6-\varepsilon^{-j}-\varepsilon^{-3nj}-\varepsilon^{-6nj}-\varepsilon^{j}-\varepsilon^{3nj}-\varepsilon^{6nj}\right)\\ &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{6\pi jn}{12n}-2\cos\frac{12\pi jn}{12n}\right)\\ &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{2}-2\cos\pi j\right)\\ &= \frac{1}{12n}\prod_{j=1,2\nmid j}^{12n-1}\left(8-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1,2\mid j}^{12n-1}\left(4-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{2}\right).\end{aligned}$$

Whence,

$$\begin{aligned}\tau(C_{12n}(1,3n,6n)) &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(8-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1}^{6n-1}\frac{4-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{2}}{8-2\cos\frac{2\pi j}{6n}}\\ &= \frac{1}{12n}\frac{\prod_{j=1}^{12n-1}\left(8-2\cos\frac{2\pi j}{12n}\right)}{\prod_{j=1}^{6n-1}\left(8-2\cos\frac{2\pi j}{6n}\right)}\times\prod_{j=1,2\nmid j}^{6n-1}\left(6-2\cos\frac{2\pi j}{6n}\right)\prod_{j=1,2\mid j}^{6n-1}\left(6-2\cos\frac{2\pi j}{6n}\right).\end{aligned}$$

Thus,

$$\tau(C_{12n}(1,3n,6n)) = \frac{1}{12n}\frac{\prod_{j=1}^{12n-1}\left(8-2\cos\frac{2\pi j}{12n}\right)}{\prod_{j=1}^{6n-1}\left(8-2\cos\frac{2\pi j}{6n}\right)}\prod_{j=1}^{6n-1}\left(6-2\cos\frac{2\pi j}{6n}\right)$$

$$\times\prod_{j=1}^{6n-1}\frac{2-2\cos\frac{2\pi j}{6n}}{6-2\cos\frac{2\pi j}{6n}}$$

$$= \frac{1}{12n}\frac{\prod_{j=1}^{12n-1}\left(8-2\cos\frac{2\pi j}{12n}\right)}{\prod_{j=1}^{6n-1}\left(8-2\cos\frac{2\pi j}{6n}\right)}\prod_{j=1}^{6n-1}\left(6-2\cos\frac{2\pi j}{6n}\right)$$

$$\times\prod_{j=1}^{3n-1}\frac{2-2\cos\frac{2\pi j}{3n}}{6-2\cos\frac{2\pi j}{3n}},$$

which implies that,

$$\tau(C_{12n}(1,3n,6n)) = \frac{1}{12n}\times\frac{U^2_{12n-1}\left(\sqrt{\frac{5}{2}}\right)\times(3n)^2\times U^2_{6n-1}(\sqrt{2})}{U^2_{6n-1}\left(\sqrt{\frac{5}{2}}\right)\times U^2_{3n-1}(\sqrt{2})}$$

$$= \frac{3n}{4}\times\frac{U^2_{12n-1}\left(\sqrt{\frac{5}{2}}\right)\times U^2_{6n-1}(\sqrt{2})}{U^2_{6n-1}\left(\sqrt{\frac{5}{2}}\right)\times U^2_{3n-1}(\sqrt{2})},$$

i.e.,

$$\tau(C_{12n}(1,3n,6n)) = \frac{3n}{4}\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{6n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{6n}\right]^2$$

$$\times\frac{3n}{4}\left[\left(\sqrt{2}+1\right)^{3n}+\left(\sqrt{2}-1\right)^{3n}\right]^2,$$

where (6), (8) , (9) and (10) are used to derive the last two steps. □

Theorem 3.5 *For the spanning trees of C_{12n} with three jumps $1, 3n, 4n$, we have:*

$$\tau(C_{12n}(1,3n,4n)) = \frac{n}{12}\left[\left(\sqrt{\frac{9}{4}}+\sqrt{\frac{5}{4}}\right)^{4n}+\left(\sqrt{\frac{9}{4}}-\sqrt{\frac{5}{4}}\right)^{4n}-1\right]^2$$

$$\times\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{2n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{2n}+1\right]^2$$

$$\times\left[\left(\sqrt{\frac{3}{2}}+\sqrt{\frac{1}{2}}\right)^{2n}+\left(\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}}\right)^{2n}\right]^2$$

$$\times\left[\left(\sqrt{2}+1\right)^{n}+\left(\sqrt{2}-1\right)^{n}\right]^2$$

$$\times\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{2n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{2n}-1\right]^2$$

Proof Let $\varepsilon = e^{\frac{2\pi i}{12n}}$. Applying Lemma 3.1, we have

$$\begin{aligned}
\tau(C_{12n}(1,3n,4n)) &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(6-\varepsilon^{-j}-\varepsilon^{-3nj}-\varepsilon^{-4nj}-\varepsilon^{j}-\varepsilon^{3nj}-\varepsilon^{4nj}\right)\\
&= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{2}-2\cos\frac{2\pi j}{3}\right)\\
&= \frac{1}{12n}\prod_{j=1,2\nmid j}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}\right)\\
&\times \prod_{j=1,2|j}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}-2\cos\frac{\pi j}{2}\right)\\
&= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}\right)\\
&\quad\times\prod_{j=1}^{6n-1}\frac{6-2\cos\frac{2\pi j}{6n}-2\cos\pi j-2\cos\frac{4\pi j}{3}}{6-2\cos\frac{2\pi j}{6n}-2\cos\frac{4\pi j}{3}}\\
&= \frac{1}{12n}\prod_{j=1,3\nmid j}^{12n-1}\left(7-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1,3|j}^{12n-1}\left(4-2\cos\frac{2\pi j}{12n}\right)\\
&\quad\times\prod_{j=1}^{6n-1}\frac{8-2\cos\frac{2\pi j}{6n}-2\cos\pi j-2\cos^2\frac{2\pi j}{3}}{\prod_{j=1}^{6n-1}8-2\cos\frac{2\pi j}{6n}-2\cos^2\frac{2\pi j}{3}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\tau(C_{12n}(1,3n,4n)) &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(7-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1}^{4n-1}\frac{4-2\cos\frac{2\pi j}{4n}}{7-2\cos\frac{2\pi j}{4n}}\\
&\times\frac{\prod_{j=1,2\nmid j}^{6n-1}\left(10-2\cos\frac{2\pi j}{6n}-4\cos^2\frac{2\pi j}{3}\right)\prod_{j=1,2|j}^{6n-1}\left(6-2\cos\frac{2\pi j}{6n}-4\cos^2\frac{2\pi j}{3}\right)}{\prod_{j=1,3\nmid j}^{6n-1}\left(7-2\cos\frac{2\pi j}{6n}\right)\prod_{j=1,3|j}^{6n-1}\left(4-2\cos\frac{2\pi j}{6n}\right)}.
\end{aligned}$$

We therefore get that

$$\tau(C_{12n}(1,3n,4n)) = \frac{1}{12n}\prod_{j=1}^{12n-1}\left(7-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1}^{4n-1}\frac{4-2\cos\frac{2\pi j}{4n}}{7-2\cos\frac{2\pi j}{4n}} \times\frac{\prod_{j=1}^{6n-1}\left(10-2\cos\frac{2\pi j}{6n}-4\cos^2\frac{2\pi j}{3}\right)\prod_{j=1}^{3n-1}\frac{6-2\cos\frac{2\pi j}{3n}-4\cos^2\frac{2\pi j}{3}}{10-2\cos\frac{2\pi j}{3n}-4\cos^2\frac{2\pi j}{3}}}{\prod_{j=1}^{6n-1}\left(7-2\cos\frac{2\pi j}{6n}\right)\prod_{j=1}^{2n-1}\frac{4-2\cos\frac{2\pi j}{2n}}{7-2\cos\frac{2\pi j}{2n}}},$$

i.e.,

$$\tau(C_{12n}(1,3n,4n)) = \frac{1}{12n}\prod_{j=1}^{12n-1}\left(7-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1}^{4n-1}\frac{4-2\cos\frac{2\pi j}{4n}}{7-2\cos\frac{2\pi j}{4n}} \times\frac{\prod_{j=1,3\nmid j}^{6n-1}\left(9-2\cos\frac{2\pi j}{6n}\right)\prod_{j=1,3\mid j}^{6n-1}\left(6-2\cos\frac{2\pi j}{6n}\right)}{\prod_{j=1}^{6n-1}\left(7-2\cos\frac{2\pi j}{6n}\right)\prod_{j=1}^{2n-1}\frac{4-2\cos\frac{2\pi j}{2n}}{7-2\cos\frac{2\pi j}{2n}}} \times\frac{\prod_{j=1,3\nmid j}^{6n-1}\left(5-2\cos\frac{2\pi j}{3n}\right)\prod_{j=1,3\mid j}^{6n-1}\left(7-2\cos\frac{2\pi j}{3n}\right)}{\prod_{j=1,3\nmid j}^{6n-1}\left(9-2\cos\frac{2\pi j}{3n}\right)\prod_{j=1,3\mid j}^{6n-1}\left(6-2\cos\frac{2\pi j}{3n}\right)},$$

which implies that

$$\tau(C_{12n}(1,3n,4n)) = \frac{1}{12n}\prod_{j=1}^{12n-1}\left(7-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1}^{4n-1}\frac{4-2\cos\frac{2\pi j}{4n}}{7-2\cos\frac{2\pi j}{4n}} \times\frac{\prod_{j=1}^{6n-1}\left(9-2\cos\frac{2\pi j}{6n}\right)\prod_{j=1}^{6n-1}\frac{6-2\cos\frac{2\pi j}{6n}}{9-2\cos\frac{2\pi j}{6n}}}{\prod_{j=1}^{6n-1}\left(7-2\cos\frac{2\pi j}{6n}\right)\prod_{j=1}^{2n-1}\frac{4-2\cos\frac{2\pi j}{2n}}{7-2\cos\frac{2\pi j}{2n}}} \times\frac{\prod_{j=1}^{3n-1}\left(5-2\cos\frac{2\pi j}{3n}\right)\prod_{j=1}^{3n-1}\frac{7-2\cos\frac{2\pi j}{3n}}{5-2\cos\frac{2\pi j}{3n}}}{\prod_{j=1}^{3n-1}\left(9-2\cos\frac{2\pi j}{3n}\right)\prod_{j=1}^{n-1}\frac{6-2\cos\frac{2\pi j}{n}}{9-2\cos\frac{2\pi j}{n}}}.$$

Thus,

$$\tau(C_{12n}(1,3n,4n)) = \frac{1}{12n}\times\frac{\frac{U_{4n-1}^2\left(\sqrt{\frac{3}{2}}\right)\times U_{12n-1}^2\left(\frac{3}{2}\right)}{U_{4n-1}^2\left(\sqrt{\frac{3}{2}}\right)}}{\frac{U_{2n-1}^2\left(\sqrt{\frac{3}{2}}\right)\times U_{6n-1}^2\left(\frac{3}{2}\right)}{U_{2n-1}^2\left(\sqrt{\frac{3}{2}}\right)}}$$

$$\times \frac{\frac{U_{6n-1}^{2}\left(\sqrt{\frac{11}{4}}\right)\times U_{2n-1}^{2}\left(\sqrt{2}\right)}{U_{2n-1}^{2}\left(\sqrt{\frac{11}{4}}\right)} \times \frac{n^{2}\times U_{3n-1}^{2}\left(\sqrt{\frac{7}{4}}\right)}{U_{3n-1}^{2}\left(\sqrt{\frac{11}{4}}\right)}}{\frac{U_{n-1}^{2}\left(\sqrt{2}\right)\times U_{3n-1}^{2}\left(\frac{11}{4}\right)}{U_{n-1}^{2}\left(\sqrt{\frac{11}{4}}\right)}}.$$

We have

$$\begin{aligned}
\tau(C_{12n}(1,3n,4n)) &= \frac{n}{12}\left[\left(\sqrt{\frac{9}{4}}+\sqrt{\frac{5}{4}}\right)^{4n}+\left(\sqrt{\frac{9}{4}}-\sqrt{\frac{5}{4}}\right)^{4n}-1\right]^{2} \\
&\times\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{2n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{2n}+1\right]^{2} \\
&\times\left[\left(\sqrt{\frac{3}{2}}+\sqrt{\frac{1}{2}}\right)^{2n}+\left(\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}}\right)^{2n}\right]^{2} \\
&\times\left[\left(\sqrt{2}+1\right)^{n}+\left(\sqrt{2}-1\right)^{n}\right]^{2} \\
&\times\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{2n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{2n}-1\right]^{2},
\end{aligned}$$

where (6), (8) , (9) and (10) are used to derive the last two steps. □

Theorem 3.6 *For the spanning trees of* C_{12n} *with three jumps* $1, 2n, 3n, 6n$ *, we have*

$$\begin{aligned}
\tau(C_{12n}(1,2n,3n,6n)) &= \frac{n}{12}\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{4n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{4n}-1\right]^{2} \\
&\times\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{2n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{2n}+1\right]^{2} \\
&\times\left[\left(\sqrt{\frac{13}{4}}+\sqrt{\frac{9}{4}}\right)^{4n}+\left(\sqrt{\frac{13}{4}}-\sqrt{\frac{9}{4}}\right)^{4n}+1\right]^{2} \\
&\times\left[\left(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}}\right)^{2n}+\left(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}}\right)^{2n}\right]^{2} \\
&\times\left[\left(\sqrt{2}+1\right)^{n}+\left(\sqrt{2}-1\right)^{n}\right]^{2} \\
&\times\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{2n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{2n}+1\right]^{2}.
\end{aligned}$$

Proof Let $\varepsilon = e^{\frac{2\pi i}{12n}}$. Applying Lemma 3.1, we get the required result. □

Theorem 3.7 *For the spanning trees of C_{12n} with four jumps $1, 2n, 3n, 4n$, we have*

$$\begin{aligned}\tau(C_{12n}(1,2n,3n,4n)) &= \frac{n}{12}\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{6n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{6n}\right]^2\\ &\times\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{2n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{2n}+1\right]^2\\ &\times\left[\left(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}}\right)^{2n}+\left(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}}\right)^{2n}-1\right]^2\\ &\times\left[\left(\sqrt{2}+1\right)^{n}+\left(\sqrt{2}-1\right)^{n}\right]^2.\end{aligned}$$

Proof Let $\varepsilon = e^{\frac{2\pi i}{12n}}$. Applying Lemma 3.1, we have

$$\begin{aligned}&\tau(C_{12n}(1,2n,3n,4n))\\ &=\frac{1}{12n}\prod_{j=1}^{12n-1}\left(8-\varepsilon^{-j}-\varepsilon^{-2nj}-\varepsilon^{-3nj}-\varepsilon^{-4nj}-\varepsilon^{j}-\varepsilon^{2nj}-\varepsilon^{3nj}-\varepsilon^{4nj}\right)\\ &=\frac{1}{12n}\prod_{j=1}^{12n-1}\left(8-2\cos\frac{2\pi jn}{12n}-2\cos\frac{6\pi jn}{12n}-2\cos\frac{8\pi jn}{12n}\right)\\ &=\frac{1}{12n}\prod_{j=1}^{12n-1}\left(8-2\cos\frac{\pi j}{3}-2\cos\frac{\pi j}{2}-2\cos\frac{2\pi j}{3}\right)\\ &=\frac{1}{12n}\prod_{j=1,2\nmid j}^{12n-1}\left(8-2\cos\frac{\pi j}{3}-2\cos\frac{\pi j}{2}-2\cos\frac{2\pi j}{3}\right)\\ &\quad\times\prod_{j=1,2|j}^{12n-1}\left(8-2\cos\frac{\pi j}{3}-2\cos\frac{\pi j}{2}-2\cos\frac{2\pi j}{3}\right)\\ &=\frac{1}{12n}\prod_{j=1}^{12n-1}\left(8-2\cos\frac{\pi j}{3}-2\cos\frac{\pi j}{2}-2\cos\frac{2\pi j}{3}\right)\\ &\quad\times\prod_{j=1}^{6n-1}\frac{8-2\cos\frac{\pi j}{3}-2\cos\frac{\pi j}{2}-2\cos\frac{2\pi j}{3}}{8-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-2\cos\frac{4\pi j}{3}}.\end{aligned}$$

Thus,

$$\tau(C_{12n}(1,2n,3n,4n)) = \frac{1}{12n}\prod_{j=1,3\nmid j}^{12n-1}\left(9-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}\right)$$

$$\times \prod_{j=1,3|j}^{12n-1} \left(6 - 2\cos\frac{2\pi j}{12n} - 2\cos\frac{\pi j}{3}\right)$$

$$\times \frac{\prod_{j=1}^{6n-1} \left(10 - 2\cos\frac{2\pi j}{6n} - 2\cos\frac{2\pi j}{3} - 4\cos^2\frac{2\pi j}{3} - 2\cos\pi j\right)}{\prod_{j=1}^{6n-1} \left(10 - 2\cos\frac{2\pi j}{6n} - 2\cos\frac{2\pi j}{3} - 4\cos^2\frac{2\pi j}{3}\right)},$$

i.e.,

$$\begin{aligned}\tau(C_{12n}(1,2n,3n,4n)) &= \frac{1}{12n} \prod_{j=}^{12n-1} \left(9 - 2\cos\frac{2\pi j}{12n} - 2\cos\frac{\pi j}{3}\right) \\ &\times \prod_{j=1}^{4n-1} \frac{6 - 2\cos\frac{2\pi j}{4n} - 2\cos\frac{\pi j}{3}}{9 - 2\cos\frac{2\pi j}{4n} - 2\cos\frac{\pi j}{3}} \\ &\times \frac{\prod_{j=1}^{6n-1} \left(10 - 2\cos\frac{2\pi j}{6n} - 2\cos\frac{2\pi j}{3} - 4\cos^2\frac{2\pi j}{3} - 2\cos\pi j\right)}{\prod_{j=1}^{6n-1} \left(10 - 2\cos\frac{2\pi j}{6n} - 2\cos\frac{2\pi j}{3} - 4\cos^2\frac{2\pi j}{3}\right)}.\end{aligned}$$

We get that

$$\begin{aligned}\tau(C_{12n}(1,2n,3n,4n)) &= \frac{1}{12n} \prod_{j=1}^{12n-1} \left(9 - 2\cos\frac{2\pi j}{12n} - 2\cos\frac{\pi j}{3}\right) \\ &\times \prod_{j=1}^{4n-1} \frac{6 - 2\cos\frac{2\pi j}{4n} - 2\cos\frac{\pi j}{3}}{9 - 2\cos\frac{2\pi j}{4n} - 2\cos\frac{\pi j}{3}} \\ &\times \frac{\prod_{j=1,2\nmid j}^{6n-1} \left(12 - 2\cos\frac{2\pi j}{6n} - 2\cos\frac{2\pi j}{3} - 4\cos^2\frac{2\pi j}{3}\right)}{\prod_{j=1,3\nmid j}^{6n-1} \left(10 - 2\cos\frac{2\pi j}{6n}\right)} \\ &\times \frac{\prod_{j=1,2|j}^{6n-1} \left(8 - 2\cos\frac{2\pi j}{6n} - 2\cos\frac{2\pi j}{3} - 4\cos^2\frac{2\pi j}{3}\right)}{\prod_{j=1,3|j}^{6n-1} \left(4 - 2\cos\frac{2\pi j}{6n}\right)},\end{aligned}$$

which implies that

$$\tau(C_{12n}(1,2n,3n,4n)) = \frac{1}{12n} \prod_{j=1,2\nmid j,3\nmid j}^{12n-1} \left(8 - 2\cos\frac{2\pi j}{12n}\right) \prod_{j=1,2|j,3|j}^{12n-1} \left(7 - 2\cos\frac{2\pi j}{12n}\right)$$

$$\times \prod_{j=1,2\nmid j,3|j}^{12n-1}\left(11-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1,2\nmid j,3|j}^{12n-1}\left(10-2\cos\frac{2\pi j}{12n}\right)$$

$$\times \frac{\prod_{j=1,2\nmid j}^{4n-1}\left(8-2\cos\frac{2\pi j}{4n}\right)\prod_{j=1,2|j}^{4n-1}\left(4-2\cos\frac{2\pi j}{4n}\right)}{\prod_{j=1,2\nmid j}^{4n-1}\left(11-2\cos\frac{2\pi j}{4n}\right)\prod_{j=1,2|j}^{4n-1}\left(7-2\cos\frac{2\pi j}{4n}\right)}$$

$$\times \frac{\prod_{j=1}^{6n-1}\left(12-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}\right)}{\prod_{j=1}^{6n-1}\left(10-2\cos\frac{2\pi j}{6n}\right)}$$

$$\times \frac{\prod_{j=1}^{6n-1}\frac{8-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}}{12-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}}}{\prod_{j=1}^{6n-1}\frac{4-2\cos\frac{2\pi j}{6n}}{10-2\cos\frac{2\pi j}{6n}}}.$$

Thus,

$$\begin{aligned}\tau(C_{12n}(1,2n,3n,4n)) &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(8-2\cos\frac{2\pi j}{12n}\right)\\
&\times \prod_{j=1}^{12n-1}\frac{\left(8-2\cos\frac{2\pi j}{12n}\right)\left(7-2\cos\frac{2\pi j}{12n}\right)}{\left(11-2\cos\frac{2\pi j}{12n}\right)\left(10-2\cos\frac{2\pi j}{12n}\right)}\\
&\times \prod_{j=1}^{12n-1}\frac{10-2\cos\frac{2\pi j}{12n}]}{8-2\cos\frac{2\pi j}{12n}}\times\prod_{j=1}^{12n-1}\frac{11-2\cos\frac{2\pi j}{12n}}{8-2\cos\frac{2\pi j}{12n}}\\
&\times \frac{\prod_{j=1}^{4n-1}\left(8-2\cos\frac{2\pi j}{4n}\right)\prod_{j=1}^{4n-1}\frac{4-2\cos\frac{2\pi j}{4n}}{8-2\cos\frac{2\pi j}{4n}}}{\prod_{j=1}^{4n-1}\left(11-2\cos\frac{2\pi j}{4n}\right)\prod_{j=1}^{4n-1}\frac{7-2\cos\frac{2\pi j}{4n}}{11-2\cos\frac{2\pi j}{4n}}}\\
&\times \frac{\prod_{j=1}^{6n-1}\left(12-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}\right)}{\prod_{j=1}^{6n-1}\left(10-2\cos\frac{2\pi j}{6n}\right)}\\
&\times \frac{\prod_{j=1}^{6n-1}\frac{8-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}}{12-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}}}{\prod_{j=1}^{6n-1}\frac{4-2\cos\frac{2\pi j}{6n}}{10-2\cos\frac{2\pi j}{6n}}}.\end{aligned}$$

We get that

$$\begin{aligned}\tau(C_{12n}(1,2n,3n,4n)) &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(8-2\cos\frac{2\pi j}{12n}\right)\\ &\times\prod_{j=1}^{12n-1}\frac{\left(8-2\cos\frac{2\pi j}{12n}\right)\left(7-2\cos\frac{2\pi j}{12n}\right)}{\left(11-2\cos\frac{2\pi j}{12n}\right)\left(10-2\cos\frac{2\pi j}{12n}\right)}\\ &\times\prod_{j=1}^{12n-1}\frac{10-2\cos\frac{2\pi j}{12n}}{8-2\cos\frac{2\pi j}{12n}}\prod_{j=1}^{12n-1}\frac{11-2\cos\frac{2\pi j}{12n}}{8-2\cos\frac{2\pi j}{12n}}\\ &\times\frac{\prod_{j=1}^{4n-1}\left(8-2\cos\frac{2\pi j}{4n}\right)\prod_{j=1}^{4n-1}\frac{4-2\cos\frac{2\pi j}{4n}}{8-2\cos\frac{2\pi j}{4n}}}{\prod_{j=1}^{4n-1}\left(11-2\cos\frac{2\pi j}{4n}\right)\prod_{j=1}^{4n-1}\frac{7-2\cos\frac{2\pi j}{4n}}{11-2\cos\frac{2\pi j}{4n}}}\\ &\times\frac{\prod_{j=1}^{6n-1}\left(12-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}\right)}{\prod_{j=1}^{6n-1}\left(10-2\cos\frac{2\pi j}{6n}\right)}\\ &\times\frac{\prod_{j=1}^{2n-1}\frac{8-2\cos\frac{2\pi j}{2n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}}{12-2\cos\frac{2\pi j}{2n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}}}{\prod_{j=1}^{3n-1}\frac{4-2\cos\frac{2\pi j}{3n}}{10-2\cos\frac{2\pi j}{3n}}},\end{aligned}$$

i.e.,

$$\begin{aligned}&\tau(C_{12n}(1,2n,3n,4n))=\frac{1}{12n}\\ &\times\frac{U_{12n-1}^2\left(\sqrt{\frac{11}{4}}\right)\times U_{2n-1}^2\left(\sqrt{\frac{11}{4}}\right)\times U_{2n-1}^2\left(\sqrt{\frac{7}{2}}\right)\times U_{4n-1}^2\left(\sqrt{\frac{7}{2}}\right)\times U_{6n-1}^2\left(\sqrt{\frac{13}{4}}\right)}{U_{2n-1}^2\left(\sqrt{\frac{7}{2}}\right)\times U_{2n-1}^2\left(\sqrt{\frac{13}{4}}\right)\times U_{4n-1}^2\left(\sqrt{\frac{11}{4}}\right)\times U_{6n-1}^2\left(\sqrt{\frac{11}{4}}\right)}\\ &\times\frac{U_{6n-1}^2\left(\sqrt{\frac{11}{4}}\right)\times U_{2n-1}^2\left(\sqrt{2}\right)\times U_{3n-1}^2\left(\sqrt{\frac{7}{4}}\right)\times U_{n-1}^2\left(\sqrt{\frac{11}{4}}\right)\times U_{2n-1}^2\left(\sqrt{\frac{11}{4}}\right)}{U_{2n-1}^2\left(\sqrt{\frac{11}{4}}\right)\times U_{n-1}^2\left(\sqrt{\frac{7}{4}}\right)\times U_{3n-1}^2\left(\sqrt{\frac{11}{4}}\right)\times U_{n-1}^2\left(\sqrt{2}\right)\times U_{6n-1}^2\left(\sqrt{\frac{11}{4}}\right)}\end{aligned}$$

So, we have

$$\begin{aligned}\tau(C_{12n}(1,2n,3n,4n)) &= \frac{n}{12}\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{6n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{6n}\right]^2\\ &\times\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{2n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{2n}+1\right]^2\end{aligned}$$

$$\times\left[\left(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}}\right)^{2n}+\left(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}}\right)^{2n}-1\right]^2$$
$$\times\left[\left(\sqrt{2}+1\right)^{n}+\left(\sqrt{2}-1\right)^{n}\right]^2,$$

where (6), (8), (9) and (10) are used to derive the last two steps. □

Theorem 3.8 *For the spanning trees of C_{6n} with four jumps $1, 3n, 4n, 6n$, we have*

$$\begin{aligned}\tau(C_{12n}(1,3n,4n,6n)) &= \frac{n}{12}\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{8n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{8n}+1\right]^2\\
&\times\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{2n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{2n}-1\right]^2\\
&\times\left[\left(\sqrt{\frac{3}{2}}+\sqrt{\frac{5}{2}}\right)^{2n}+\left(\sqrt{\frac{3}{2}}-\sqrt{\frac{5}{2}}\right)^{2n}\right]^2\\
&\times\left[\left(\sqrt{2}+1\right)^{n}+\left(\sqrt{2}-1\right)^{n}\right]^2\\
&\times\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{2n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{2n}+1\right]^2\\
&\times\left[\left(\sqrt{\frac{13}{4}}+\sqrt{\frac{9}{4}}\right)^{2n}+\left(\sqrt{\frac{13}{4}}-\sqrt{\frac{9}{4}}\right)^{2n}+1\right]^2\end{aligned}$$

Proof Let $\varepsilon = e^{\frac{2\pi i}{12n}}$. Apply Lemma 3.1, we get the required result. □

Theorem 3.9 *For the spanning trees of C_{6n} with four jumps $1, 2n, 3n, 4n, 6n$, we have*

$$\begin{aligned}\tau(C_{12n}(1,2n,3n,4n,6n)) &= \frac{n}{12}\times\left[\left(\sqrt{2}+1\right)^{n}+\left(\sqrt{2}-1\right)^{n}\right]^2\\
&\times\left[\left(\sqrt{2}+\sqrt{3}\right)^{n}+\left(\sqrt{2}-\sqrt{3}\right)^{n}-1\right]^2\\
&\times\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{2n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{2n}+1\right]^2\\
&\times\left[\left(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}}\right)^{2n}+\left(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}}\right)^{2n}-1\right]^2\\
&\times\left[\left(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}}\right)^{2n}+\left(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}}\right)^{2n}\right]^2\end{aligned}$$

$$\times \left[\left(\sqrt{\frac{7}{2}} + \sqrt{\frac{5}{2}} \right)^{4n} + \left(\sqrt{\frac{7}{2}} - \sqrt{\frac{5}{2}} \right)^{4n} + 1 \right]^2$$

Proof Let $\varepsilon = e^{\frac{2\pi i}{12n}}$. Applying Lemma 3.1, We get the required result. □

§4. Conclusions

The number of spanning trees in graphs (networks) is an important invariant. The evaluation of this number is not only interesting from a mathematical (computational) perspective, but also, it is an important measure of reliability of a network and designing electrical circuits. Some computationally hard problems such as the travelling salesman problem can be solved approximately by using spanning trees. Due to the high dependence of the network design and reliability on the graph theory we prove our results in Section 3.

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Some Fixed Point Results for Multivalued Nonexpansive Mappings in Banach Spaces

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Abstract: In this paper, we deal with the approximation of fixed point for multivalued nonexpansive mappings via a new three-step algorithm which is independent and faster than the iterative process discussed by Agarwal et al. [3] and establish some strong convergence theorems and a weak convergence theorem in the setting of Banach spaces. Our results extend and generalize the previous works from the existing literature.

Key Words: Multivalued nonexpansive mapping, three-step iteration scheme, fixed point, weak and strong convergence, Banach space.

AMS(2010): 47H10, 54H25.

§1. Introduction

Throughout this paper, let $\mathcal{X}$ be a real Banach space with the norm $\|.\|$. Let $\mathbb{N}$ denotes the set of all positive integers and let $F(\mathcal{T})$ denotes the set of all fixed points of the mapping $\mathcal{T}$.

Let $\mathcal{K}$ be a subset of $\mathcal{X}$. A subset $\mathcal{K}$ is called proximal if for each $x \in \mathcal{X}$, there exists an element $k \in \mathcal{K}$ such that $d(x,k) = \inf\{\|x-y\| : y \in \mathcal{K}\} = d(x,\mathcal{K})$. It is well known that a weakly compact convex subset of a Banach space and closed convex subsets of a uniformly convex Banach space are Proximal.

We shall denote $CB(\mathcal{K})$, $C(\mathcal{K})$ and $P(\mathcal{K})$ by the families of all nonempty closed and bounded subsets, nonempty compact subsets and nonempty proximal subsets of $\mathcal{K}$, respectively. Let H denote the Hausdorff metric induced by the metric d of $\mathcal{X}$, that is,

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\}$$

for every $A, B \in CB(\mathcal{X})$, where $d(x,B) = \inf\{\|x-y\| : y \in B\}$.

A multivalued mapping $\mathcal{T}\colon \mathcal{K} \to CB(\mathcal{K})$ is said to be a *contraction* if there exists a constant $b \in [0,1)$ such that for any $x, y \in \mathcal{K}$,

$$H(\mathcal{T}x, \mathcal{T}y) \le b\,\|x-y\|,$$

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and $\mathcal{T}$ is said to be *nonexpansive* if

$$H(\mathcal{T}x, \mathcal{T}y) \leq \|x - y\|,$$

for all $x, y \in \mathcal{K}$. A point $x \in \mathcal{K}$ is called a fixed point of $\mathcal{T}$ if $x \in \mathcal{T}x$.

Later, an interesting and rich fixed point theory for such maps was developed which has applications in control theory, convex optimization, differential inclusion and economics (see [5] and references cited therein). Moreover, the existence of fixed points for multivalued non-expansive mappings in uniformly convex Banach spaces was proved by Lim [7]. Many authors have studied the fixed point for multivalued mappings (e.g., see [4, 6, 8, 10, 15, 16, 19]).

In 2005, Sastry and Babu [11] obtained the convergence results from single valued mappings to multivalued mappings by defining Ishikawa and Mann iterates for multivalued mappings with a fixed point. They considered the following:

Let $\mathcal{K}$ be a nonempty convex subset of $\mathcal{X}$, $\mathcal{T}\colon \mathcal{K} \to P(\mathcal{K})$ is a multivalued mapping with $p \in F(\mathcal{T})$.

(i) The *Mann iteration* is defined by

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n s_n,\ n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ is a real sequence in $(0,1)$ and $s_n \in \mathcal{T}x_n$ such that $\|s_n - p\| = d(p, \mathcal{T}x_n)$.

(ii) The *Ishikawa iteration* is defined by

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n r_n, \\ y_n = (1-\beta_n)x_n + \beta_n s_n,\ n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$, $\|s_n - r_n\| = d(\mathcal{T}x_n, \mathcal{T}y_n)$ and $\|r_n - p\| = d(\mathcal{T}y_n, \mathcal{T}p)$ for $s_n \in \mathcal{T}x_n$ and $r_n \in \mathcal{T}y_n$. They established some strong and weak convergence results of the above iterates for multivalued nonexpansive mappings $\mathcal{T}$ under some appropriate conditions.

In 2007, Panyanak [10] extended the results of Sastry and Babu [11] to a uniformly convex Banach space and also modified the above Ishikawa iterative scheme as follows:

Let $\mathcal{T}\colon \mathcal{K} \to P(\mathcal{K})$ be a multi-valued mapping.

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ y_n = (1-\beta_n)x_n + \beta_n z_n,\ n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\{\beta_n\}$ is a real sequence in $[0,1]$, $z_n \in \mathcal{T}x_n$ and $u_n \in \mathcal{T}x_n$ are such that $\|z_n - u_n\| =$

$d(u_n, \mathcal{T}x_n)$ and $\|x_n - u_n\| = d(x_n, F(\mathcal{T}))$, respectively, and

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n z'_n, \ n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$, $z'_n \in \mathcal{T}x_n$ and $v_n \in \mathcal{T}x_n$ are such that $\|z'_n - v_n\| = d(v_n, \mathcal{T}x_n)$ and $\|y_n - v_n\| = d(y_n, F(\mathcal{T}))$, respectively and proved a convergence theorem of Mann iterates for a mapping defined on a noncompact domain. Later in 2008, Song and Wang [14] proved strong convergence theorems of Mann and Ishikawa iterates for multivalued nonexpansive mappings under some appropriate control conditions. Furthermore, they also gave an affirmative answer to Panyanak's open question in [10].

Recently, Abbas and Nazir [1] introduced and studied the following iteration scheme: let $\mathcal{K}$ be a nonempty subset of a Banach space $\mathcal{X}$ and $\mathcal{T}$ be a nonlinear mapping of $\mathcal{K}$ into itself. Then the sequence $\{x_n\}$ in $\mathcal{K}$ is defined by

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1-\alpha_n)\mathcal{T}y_n + \alpha_n \mathcal{T}z_n, \\ y_n = (1-\beta_n)\mathcal{T}x_n + \beta_n \mathcal{T}z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n \mathcal{T}x_n, \ n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$. They showed that this process converges faster than both Picard and the Agarwal et al. ([3]) and in support gave analytic proof by a numerical example (for more details, see [1]).

Motivated by Sastry and Babu [11], Panyanak [10] and Song and Wang [14], we first give a multivalued version of the iteration scheme (1.5) of Abbas and Nazir [1] and then study its convergence analysis in the setting of Banach spaces. We define our iteration scheme as follows:

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1-\alpha_n)v_n + \alpha_n w_n, \\ y_n = (1-\beta_n)u_n + \beta_n w_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n u_n, \ n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$, $u_n \in \mathcal{T}x_n$, $v_n \in \mathcal{T}y_n$ and $w_n \in \mathcal{T}z_n$ such that $\|w_n - u_n\| = d(\mathcal{T}z_n, \mathcal{T}x_n)$, $\|v_n - w_n\| = d(\mathcal{T}y_n, \mathcal{T}z_n)$, $\|v_n - u_n\| = d(\mathcal{T}y_n, \mathcal{T}x_n)$, $\|u_{n+1} - v_n\| = d(\mathcal{T}x_{n+1}, \mathcal{T}y_n)$ and $\|u_{n+1} - w_n\| = d(\mathcal{T}x_{n+1}, \mathcal{T}z_n)$, respectively.

Now, we recall the following definitions.

Definition 1.1 *A Banach space X is said to satisfy Opial condition [9] if for any sequence $\{x_n\}$ in X, x_n converges to x weakly it follows that*

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|,$$

for all $y \in X$ with $y \neq x$.

Examples of Banach spaces satisfying Opial condition are Hilbert spaces and all spaces $l^p(1 < p < \infty)$. On the other hand, $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial condition.

Definition 1.2 *A multivalued mapping $\mathcal{T}\colon \mathcal{K} \to P(\mathcal{X})$ is called demiclosed at $y \in \mathcal{K}$ if for any sequence $\{x_n\}$ in $\mathcal{K}$ weakly convergent to an element x and $y_n \in \mathcal{T}x_n$ strongly convergent to y, we have $y \in \mathcal{T}x$.*

The following is the multivalued version of condition (I) of Senter and Dotson [13].

Definition 1.3 *A multivalued nonexpansive mapping $\mathcal{T}\colon \mathcal{K} \to CB(\mathcal{K})$ where $\mathcal{K}$ a subset of $\mathcal{X}$ is said to satisfy condition (I) if there exists a nondecreasing function $f\colon [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$ such that $d(x, \mathcal{T}x) \geq f(d(x, F(\mathcal{T})))$ for all $x \in \mathcal{K}$, where $F(\mathcal{T}) \neq \emptyset$ is the fixed point set of the multivalued mapping $\mathcal{T}$.*

We need the following Lemmas to prove our main results.

Lemma 1.4 ([18]) *Let $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ be three sequences of nonnegative real numbers satisfying the following conditions:*

$$p_{n+1} \leq (1 + q_n)p_n + r_n,\ n \geq 0, \quad \sum_{n=0}^{\infty} q_n < \infty, \quad \sum_{n=0}^{\infty} r_n < \infty.$$

Then,

(1) $\lim_{n\to\infty} p_n$ *exists;*
(2) *In addition, if* $\liminf_{n\to\infty} p_n = 0$, *then* $\lim_{n\to\infty} p_n = 0$.

Lemma 1.5 ([12]) *Let E be a uniformly convex Banach space and $0 < \alpha \leq t_n \leq \beta < 1$ for all $n \in \mathbb{N}$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$ hold for some $a \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 1.6 ([16]) *Let $\mathcal{T}\colon \mathcal{K} \to P(\mathcal{K})$ be a multivalued mapping and $P_{\mathcal{T}}(x) = \{y \in \mathcal{T}x : \|x - y\| = d(x, \mathcal{T}x)\}$. Then the following are equivalent:*

(1) $x \in F(\mathcal{T})$;
(2) $P_{\mathcal{T}}(x) = \{x\}$;
(3) $x \in F(P_{\mathcal{T}})$.

Moreover, $F(\mathcal{T}) = F(P_{\mathcal{T}})$.

§2. Main Results

In this section we prove some strong and a weak convergence theorems using iteration scheme (1.6). First, we need the following lemmas to prove main results.

Lemma 2.1 *Let $\mathcal{X}$ be a real Banach space and $\mathcal{K}$ be a nonempty closed and convex subset of*

$\mathcal{X}$. Let $\mathcal{T}\colon \mathcal{K} \to P(\mathcal{K})$ be a multivalued mapping such that $F(\mathcal{T}) \neq \emptyset$ and $P_{\mathcal{T}}$ is a nonexpansive mapping. Let $\{x_n\}$ be the sequence defined by (1.6), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(\mathcal{T})$.

Proof Let $p \in F(\mathcal{T})$. Then $p \in P_{\mathcal{T}}(p) = \{p\}$ by Lemma 1.6. It follows from (1.6) that

$$\begin{aligned} \|z_n - p\| &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|u_n - p\| \\ &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n H(P_{\mathcal{T}}(x_n), P_{\mathcal{T}}(p)) \\ &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{2.1}$$

Again using (1.6) and (2.1), we obtain

$$\begin{aligned} \|y_n - p\| &\leq (1-\beta_n)\|u_n - p\| + \beta_n\|w_n - p\| \\ &\leq (1-\beta_n)H(P_{\mathcal{T}}(x_n), P_{\mathcal{T}}(p)) + \beta_n H(P_{\mathcal{T}}(z_n), P_{\mathcal{T}}(p)) \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n\|z_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{2.2}$$

Now using (1.6), (2.1) and (2.2), we obtain

$$\begin{aligned} \|x_{n+1} - p\| &\leq (1-\alpha_n)\|v_n - p\| + \alpha_n\|w_n - p\| \\ &\leq (1-\alpha_n)H(P_{\mathcal{T}}(y_n), P_{\mathcal{T}}(p)) + \alpha_n H(P_{\mathcal{T}}(z_n), P_{\mathcal{T}}(p)) \\ &\leq (1-\alpha_n)\|y_n - p\| + \alpha_n\|z_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{2.3}$$

It follows from Lemma 1.4 that $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F(\mathcal{T})$. This completes the proof. □

Lemma 2.2 *Let $\mathcal{X}$ be a uniformly convex Banach space and $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$. Let $\mathcal{T}\colon \mathcal{K} \to P(\mathcal{K})$ be a multivalued mapping such that $F(\mathcal{T}) \neq \emptyset$ and $P_{\mathcal{T}}$ is a nonexpansive mapping. Let $\{x_n\}$ be the sequence defined by (1.6), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$. Then $\lim_{n\to\infty} d(x_n, \mathcal{T}x_n) = 0$, $\lim_{n\to\infty} d(x_n, \mathcal{T}y_n) = 0$ and $\lim_{n\to\infty} d(x_n, \mathcal{T}z_n) = 0$.*

Proof From Lemma 2.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F(\mathcal{T})$. We suppose that $\lim_{n\to\infty} \|x_n - p\| = l$ for some $l \geq 0$.

Since $\limsup_{n\to\infty} \|u_n - p\| \leq \limsup_{n\to\infty} H(P_{\mathcal{T}}(x_n), P_{\mathcal{T}}(p)) \leq \limsup_{n\to\infty}$ $\|x_n - p\| = l$, so

$$\limsup_{n\to\infty} \|u_n - p\| \leq l. \tag{2.4}$$

Again, since $\limsup_{n\to\infty} \|v_n - p\| \leq \limsup_{n\to\infty} H(P_{\mathcal{T}}(y_n), P_{\mathcal{T}}(p)) \leq \limsup_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = l$, so

$$\limsup_{n\to\infty} \|v_n - p\| \leq l. \tag{2.5}$$

Similarly,

$$\limsup_{n\to\infty} \|w_n - p\| \leq l. \tag{2.6}$$

Applying Lemma 1.5, we get

$$\lim_{n\to\infty} \|u_n - v_n\| = 0 \tag{2.7}$$

$$\lim_{n\to\infty} \|w_n - u_n\| = 0 \tag{2.8}$$

and

$$\lim_{n\to\infty} \|v_n - w_n\| = 0. \tag{2.9}$$

Taking lim sup on both sides of (2.1) and (2.2), we obtain

$$\limsup_{n\to\infty} \|z_n - p\| \leq l \tag{2.10}$$

and

$$\limsup_{n\to\infty} \|y_n - p\| \leq l. \tag{2.11}$$

Also,

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1-\alpha_n)v_n + \alpha_n w_n - p\| \\
&= \|(v_n - p) + \alpha_n(w_n - v_n)\| \\
&\leq \|v_n - p\| + \alpha_n\|w_n - v_n\| \\
&\leq \|v_n - p\| + \|w_n - v_n\|
\end{aligned}$$

implies that

$$l \leq \liminf_{n\to\infty} \|v_n - p\|. \tag{2.12}$$

Combining (2.5) and (2.12), we obtain

$$\lim_{n\to\infty} \|v_n - p\| = l. \tag{2.13}$$

Thus,

$$\begin{aligned}
\|v_n - p\| &\leq \|v_n - w_n\| + \|w_n - p\| \\
&\leq \|v_n - w_n\| + H(P_{\mathcal{T}}(z_n), P_{\mathcal{T}}(p)) \\
&\leq \|v_n - w_n\| + \|z_n - p\|
\end{aligned}$$

gives

$$l \leq \liminf_{n\to\infty} \|z_n - p\| \tag{2.14}$$

and by virtue of (2.10), we obtain

$$\lim_{n\to\infty} \|z_n - p\| = l. \tag{2.15}$$

Similarly,

$$\begin{aligned}
\|w_n - p\| &\leq \|w_n - v_n\| + \|v_n - p\| \\
&\leq \|w_n - v_n\| + H(P_{\mathcal{T}}(y_n), P_{\mathcal{T}}(p)) \\
&\leq \|w_n - v_n\| + \|y_n - p\|
\end{aligned}$$

gives

$$l \leq \liminf_{n\to\infty} \|y_n - p\| \tag{2.16}$$

and by virtue of (2.11), we obtain

$$\lim_{n\to\infty} \|y_n - p\| = l. \tag{2.17}$$

Applying Lemma 1.5 once again, we obtain

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{2.18}$$

Notice that

$$\|x_n - v_n\| \leq \|x_n - u_n\| + \|u_n - v_n\|.$$

Using (2.7) and (2.18), we obtain

$$\lim_{n\to\infty} \|x_n - v_n\| = 0 \tag{2.19}$$

and

$$\|x_n - w_n\| \leq \|x_n - u_n\| + \|u_n - w_n\|.$$

Using (2.8) and (2.19), we obtain

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{2.20}$$

Since $d(x_n, \mathcal{T}x_n) \leq \|x_n - u_n\|$, we have

$$\lim_{n\to\infty} d(x_n, \mathcal{T}x_n) = 0. \tag{2.21}$$

Again since $d(x_n, \mathcal{T}y_n) \leq \|x_n - v_n\|$, we have

$$\lim_{n\to\infty} d(x_n, \mathcal{T}y_n) = 0. \tag{2.22}$$

Similarly, since $d(x_n, \mathcal{T}z_n) \leq \|x_n - w_n\|$, we have

$$\lim_{n\to\infty} d(x_n, \mathcal{T}z_n) = 0. \tag{2.23}$$

This completes the proof. □

Now we shall prove some strong convergence theorems using iteration scheme (1.6) in real Banach spaces.

Theorem 2.3 *Let $\mathcal{X}$ be a real Banach space and $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$. Let $\mathcal{T}\colon \mathcal{K} \to P(\mathcal{K})$ be a multivalued mapping such that $F(\mathcal{T}) \neq \emptyset$ and $P_{\mathcal{T}}$ is a nonexpansive mapping. Let $\{x_n\}$ be the sequence defined by (**??**), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$. Then $\{x_n\}$ converges strongly to a fixed point of $\mathcal{T}$ if and only if $\liminf_{n\to\infty} d(x_n, F(\mathcal{T})) = 0$.*

Proof The necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(\mathcal{T})) = 0$. As proved in Lemma 1.6, we have

$$\|x_{n+1} - p\| \leq \|x_n - p\|$$

which gives

$$d(x_{n+1}, F(\mathcal{T})) \leq d(x_n, F(\mathcal{T})).$$

This implies that $\lim_{n\to\infty} d(x_n, F(\mathcal{T}))$ exists by Lemma 1.4 and so by hypothesis, $\liminf_{n\to\infty} d(x_n, F(\mathcal{T})) = 0$. Therefore, we must have $\lim_{n\to\infty} d(x_n, F(\mathcal{T})) = 0$.

Next, we have to show that $\{x_n\}$ is a Cauchy sequence in $\mathcal{K}$. Let $\varepsilon > 0$ be arbitrary chosen. Since $\lim_{n\to\infty} d(x_n, F(\mathcal{T})) = 0$, there exists a constant n_1 such that for all $n \geq n_1$ we have

$$d(x_n, F(\mathcal{T})) < \frac{\varepsilon}{4}.$$

In particular, $\inf\{\|x_{n_1} - p\| : p \in F(\mathcal{T})\} < \frac{\varepsilon}{4}$. There must exists a $q \in F(\mathcal{T})$ such that

$$\|x_{n_1} - q\| < \frac{\varepsilon}{2}.$$

Now for $m, n \geq n_1$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - q\| + \|x_n - q\| \\ &\leq 2\|x_{n_1} - q\| \\ &< 2\Big(\frac{\varepsilon}{2}\Big) = \varepsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset $\mathcal{K}$ of a Banach space $\mathcal{X}$, and so it must converge in $\mathcal{K}$. Let $\lim_{n\to\infty} x_n = q_1$. Now

$$\begin{aligned} d(q_1, P_{\mathcal{T}}(q_1)) &\leq \|x_n - q_1\| + d(x_n, P_{\mathcal{T}}(x_n)) + H(P_{\mathcal{T}}(x_n), P_{\mathcal{T}}(q_1)) \\ &\leq 2\|x_n - q_1\| + \|x_n - u_n\| \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

which gives that $d(q_1, \mathcal{T}q_1) = 0$. But $P_{\mathcal{T}}$ is a nonexpansive mapping and so $F(\mathcal{T})$ is closed. Therefore, $q_1 \in F(P_{\mathcal{T}}) = F(\mathcal{T})$. This shows that $\{x_n\}$ converges strongly to a point of $F(\mathcal{T})$. This completes the proof. □

Theorem 2.4 *Let $\mathcal{X}$ be a real Banach space and $\mathcal{K}$ be a nonempty compact convex subset of $\mathcal{X}$. Let $\mathcal{T}\colon \mathcal{K} \to P(\mathcal{K})$ be a multivalued mapping such that $F(\mathcal{T}) \neq \emptyset$ and $P_{\mathcal{T}}$ is a nonexpansive mapping. Let $\{x_n\}$ be the sequence defined by (**??**), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$. Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converges strongly to a fixed point of $\mathcal{T}$.*

Proof By Lemma 2.2, we have $\lim_{n\to\infty} d(x_n, \mathcal{T}x_n) = 0$. Since by hypothesis $\mathcal{K}$ be a nonempty compact convex subset of $\mathcal{X}$, so there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} \|x_{n_k} - u\| = 0$ for some $u \in \mathcal{K}$. Thus

$$\begin{aligned} d(u, P_{\mathcal{T}}(u)) &\leq \|x_{n_k} - u\| + d(x_{n_k}, P_{\mathcal{T}}(x_{n_k})) + H(P_{\mathcal{T}}(x_{n_k}), P_{\mathcal{T}}(u)) \\ &\leq 2\|x_{n_k} - u\| + \|x_{n_k} - u_{n_k}\| \\ &\to 0 \text{ as } k \to \infty. \end{aligned}$$

This shows that u is a fixed of $\mathcal{T}$. From Lemma 2.1, we get that $\lim_{n\to\infty} \|x_n - u\| = 0$. Again from Lemma 2.2, we get that

$$\begin{aligned} \|y_n - x_n\| &= \|(1-\beta_n)u_n + \beta_n w_n - x_n\| \\ &\leq \|u_n - x_n\| + \beta_n\|w_n - u_n\| \\ &\leq \|u_n - x_n\| + \|w_n - u_n\| \\ &\to 0 \text{ as } n \to \infty, \end{aligned}$$

and

$$\begin{aligned} \|z_n - x_n\| &= \|(1-\gamma_n)x_n + \gamma_n u_n - x_n\| \\ &\leq \gamma_n\|u_n - x_n\| \\ &\leq \|u_n - x_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

It follows that $\lim_{n\to\infty} \|y_n - u\| = 0$ and $\lim_{n\to\infty} \|z_n - u\| = 0$. Thus the conclusion follows. This completes the proof. □

Now, we apply Theorem 2.3 to obtain another strong convergence theorem in uniformly convex Banach spaces satisfies condition (I) of Senter and Dotson [13].

Theorem 2.5 *Let $\mathcal{X}$ be a uniformly convex Banach space and $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$. Let $\mathcal{T}\colon \mathcal{K} \to P(\mathcal{K})$ be a multivalued mapping satisfying condition (I) such that $F(\mathcal{T}) \neq \emptyset$ and $P_{\mathcal{T}}$ is a nonexpansive mapping. Let $\{x_n\}$ be the sequence defined by (1.6), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$. Then $\{x_n\}$ converges strongly to a fixed point of $\mathcal{T}$.*

Proof By Lemma 2.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(\mathcal{T})$ and so the sequence $\{x_n\}$ is bounded . Let $\lim_{n\to\infty} \|x_n - p\| = r$ for some $r > 0$.

Now from Lemma 2.1, we know that

$$\|x_{n+1} - p\| \leq \|x_n - p\|$$

which implies that

$$\inf_{p\in F(\mathcal{T})} \|x_{n+1} - p\| \leq \inf_{p\in F(\mathcal{T})} \|x_n - p\|,$$

and also $d(x_{n+1}, F(\mathcal{T})) \leq d(x_n, F(\mathcal{T}))$. And so, $\lim_{n\to\infty} d(x_n, F(\mathcal{T}))$ exists. By using condition (I) and Lemma 2.2, we have

$$\lim_{n\to\infty} f(d(x_n, F(\mathcal{T}))) \leq \lim_{n\to\infty} d(x_n, \mathcal{T}x_n) = 0.$$

That is,

$$\lim_{n\to\infty} f(d(x_n, F(\mathcal{T}))) = 0.$$

Since f is a nondecreasing function and $f(0) = 0$, it follows that $\lim_{n\to\infty} d(x_n, F(\mathcal{T})) = 0$. The conclusion follows from Theorem 2.3. This completes the proof. $\square$

Now, we prove the weak convergence theorem of the sequence $\{x_n\}$ defined by (1.6).

Theorem 2.6 *Let $\mathcal{X}$ be a uniformly convex Banach space satisfying Opial's condition and $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$. Let $\mathcal{T}\colon \mathcal{K} \to P(\mathcal{K})$ be a multivalued mapping such that $F(\mathcal{T}) \neq \emptyset$ and $P_{\mathcal{T}}$ is a nonexpansive mapping. Let $\{x_n\}$ be the sequence defined by (**??**), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$. Let $I - P_{\mathcal{T}}$ be demiclosed with respect to zero. Then $\{x_n\}$ converges weakly to a fixed point of $\mathcal{T}$.*

Proof Let $z \in F(\mathcal{T})$. From Lemma 2.1, we know that $\lim_{n\to\infty} \|x_n - z\|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in $F(\mathcal{T})$. To prove this, let p_1 and p_2 be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By (2.18), there exists $u_n \in \mathcal{T}x_n$ such that $\lim_{n\to\infty} \|x_n - u_n\| = 0$. Since $I - P_{\mathcal{T}}$ is demiclosed with respect to zero, therefore we obtain $p_1 \in F(\mathcal{T})$. In the same way, we can prove that $p_2 \in F(\mathcal{T})$.

Next, we prove uniqueness. For this, suppose that $p_1 \neq p_2$. Then by Opial's condition, we have

$$\begin{aligned}\lim_{n\to\infty}\|x_n-p_1\| &= \lim_{n_i\to\infty}\|x_{n_i}-p_1\|\\ &< \lim_{n_i\to\infty}\|x_{n_i}-p_2\|\\ &= \lim_{n\to\infty}\|x_n-p_2\|\\ &= \lim_{n_j\to\infty}\|x_{n_j}-p_2\|\\ &< \lim_{n_j\to\infty}\|x_{n_j}-p_1\|\\ &= \lim_{n\to\infty}\|x_n-p_1\|,\end{aligned}$$

which is a contradiction. Hence $\{x_n\}$ converges weakly to a fixed point of $\mathcal{T}$. This completes the proof. □

Remark 2.7 Our results extend, generalize and improve several corresponding results from the existing literature and iterative schemes discussed by Panyanak [10], Sastry and Babu [11], Song and Wang [14] and Song and Cho [16] in the sense of faster iterative scheme.

Suzuki [17] introduced a condition on mappings called condition (C) which is weaker than nonexpansiveness.

Recently, Abkar and Eslamian [2] introduced the definition of condition (C) for multi-valued mapping. The definition is as follows.

Definition 2.8([2]) *A multivalued mapping* $\mathcal{T}\colon \mathcal{X} \to CB(\mathcal{X})$ *is said to satisfy condition* (C) *provided that*

$$\frac{1}{2}d(x,\mathcal{T}x) \le \|x-y\| \Rightarrow H(\mathcal{T}x,\mathcal{T}y) \le \|x-y\|,\ x,y\in\mathcal{X}.$$

The following result can be found in [2].

Lemma 2.9([2]) *Let* $\mathcal{T}\colon \mathcal{X} \to CB(\mathcal{X})$ *be a multi-valued mapping. If* $\mathcal{T}$ *is nonexpansive, then* $\mathcal{T}$ *satisfies the condition* (C).

We mention that there exist single-valued and multi-valued mappings satisfying the condition (C) which are not nonexpansive.

Example 2.10([17]) Define a mapping $\mathcal{T}$ on [0.3] by

$$T(x) = \begin{cases} 0, & \text{if } x \neq 3,\\ 1, & \text{if } x = 3.\end{cases}$$

Then $\mathcal{T}$ is a single-valued mapping satisfying condition (C), but $\mathcal{T}$ is not nonexpansive.

Example 2.11([2]) Define a mapping $\mathcal{T}\colon [0,5] \to [0,5]$ by

$$T(x) = \begin{cases} [0, \frac{x}{5}], & \text{if } x \neq 5, \\ \{1\}, & \text{if } x = 5. \end{cases}$$

Then it is easy to show that $\mathcal{T}$ is a multi-valued mapping satisfying condition (C), but $\mathcal{T}$ is not nonexpansive.

Now, we obtain some strong convergence results using iteration scheme (1.6) and condition (C).

Theorem 2.12 *Let $\mathcal{X}$ be a real Banach space and $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$. Let $\mathcal{T}\colon \mathcal{K} \to P(\mathcal{K})$ be a multivalued mapping such that $F(\mathcal{T}) \neq \emptyset$ and $P_{\mathcal{T}}$ satisfies condition (C). Let $\{x_n\}$ be the sequence defined by (1.6), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$. Then $\{x_n\}$ converges strongly to a fixed point of $\mathcal{T}$ if and only if $\liminf_{n\to\infty} d(x_n, F(\mathcal{T})) = 0$.*

Proof The proof of Theorem 2.12 immediately follows from Lemma 2.1 and Theorem 2.3. This completes the proof. □

Theorem 2.13 *Let $\mathcal{X}$ be a real Banach space and $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$. Let $\mathcal{T}\colon \mathcal{K} \to P(\mathcal{K})$ be a multivalued mapping such that $F(\mathcal{T}) \neq \emptyset$ and $P_{\mathcal{T}}$ satisfies condition (C). Let $\{x_n\}$ be the sequence defined by (1.6), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$. If the following condition is satisfied:*

(c_1) there exists an increasing function $\psi\colon [0,\infty) \to [0,\infty)$ with $\psi(r) > 0$, $\forall\, r > 0$ such that

$$d(x_n, \mathcal{T}x_n) \geq \psi(d(x_n, F(\mathcal{T})), \tag{2.24}$$

then $\{x_n\}$ converges strongly to a fixed point of $\mathcal{T}$.

Proof As in the proof of Lemma 2.2, we know that $\lim_{n\to\infty} d(x_n, \mathcal{T}x_n) = 0$. Hence from (2.21) we obtain $\lim_{n\to\infty} d(x_n, F(\mathcal{T})) = 0$. The conclusion of Theorem 2.13 can be obtained from Theorem 2.12 immediately. This completes the proof. □

§3. Concluding Remarks

In this paper, we study a new three-step iteration scheme for multivalued nonexpansive mappings in Banach spaces and establish some strong convergence theorems and a weak convergence theorem under some appropriate conditions applying on the space. Our results extend and generalize several results from the current existing literature.

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Algebraic Properties of the Path Complexes of Cycles

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Abstract: Let G be a simple graph and $\Delta_t(G)$ be a simplicial complex whose facets correspond to the paths of length t ($t \geq 2$) in G. It is shown that $\Delta_t(C_n)$ is matroid, vertex decomposable, shellable and Cohen-Macaulay if and only if $n = t$ or $n = t+1$, where C_n is an n-cycle. As a consequence we show that if $n = t$ or $t+1$ then $\Delta_t(C_n)$ is partitionable and Stanley's conjecture holds for $K[\Delta_t(C_n)]$.

Key Words: Vertex decomposable, simplicial complex, matroid, path.

AMS(2010): 13F20, 05E40, 13F55.

§1. Introduction

Let $R = K[x_1, \cdots, x_n]$, where K is a field. Fix an integer $n \geq t \geq 2$ and let G be a directed graph. A sequence $x_{i_1}, \cdots, x_{i_t}$ of distinct vertices is called a path of length t if there are $t-1$ distinct directed edges $e_1, \cdots, e_{t-1}$ where e_j is a directed edge from x_{i_j} to $x_{i_{j+1}}$. Then the path ideal of G of length t is the monomial ideal $I_t(G) = (x_{i_1} \cdots x_{i_t} : x_{i_1}, \cdots, x_{i_t}$ is a path of length t in G in the polynomial ring $R = K[x_1, \cdots, x_n]$. The distance $d(x, y)$ of two vertices x and y of a graph G is the length of the shortest path from x to y. The path complex $\Delta_t(G)$ is defined by

$$\Delta_t(G) = \langle \{x_{i_1}, \cdots, x_{i_t}\} : x_{i_1}, \cdots, x_{i_t} \text{ is a path of length t in G } \rangle.$$

Path ideals of graphs were first introduced by Conca and De Negri [3] in the context of monomial ideals of linear type. Recently the path ideal of cycles has been extensively studied by several mathematicians. In [9], it is shown that $I_2(C_n)$ is sequentially Cohen-Macaulay, if and only if, $n = 3$ or $n = 5$. Generalizing this result, in [13], it is proved that $I_t(C_n)$, $(t > 2)$, is sequentially Cohen-Macaulay, if and only if $n = t$ or $n = t+1$ or $n = 2t+1$. Also, the Betti numbers of the ideal $I_t(C_n)$ and $I_t(L_n)$ is computed explicitly in [1]. In particular, it has been shown that

Theorem 1.1(Corollary 5.15, [1]) *Let n, t, p and d be integers such that $n \geq t \geq 2$, $n = (t+1)p + d$, where $p \geq 0$ and $0 \leq d < (t+1)$. Then,*

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(*i*) *The projective dimension of the path ideal of a graph cycle C_n or line L_n is given by*

$$\operatorname{pd}(I_t(C_n)) = \begin{cases} 2p, & d \neq 0 \\ 2p-1, & d = 0 \end{cases} \qquad \operatorname{pd}(I_t(L_n)) = \begin{cases} 2p-1, & d \neq t, \\ 2p, & d = t. \end{cases}$$

(*ii*) *The regularity of the path ideal of a graph cycle C_n or line L_n is given by*

$$\operatorname{reg}(I_t(C_n)) = (t-1)p + d + 1$$

$$\operatorname{reg}(I_t(L_n)) = \begin{cases} p(t-1)+1, & d < t, \\ p(t-1)+t, & d = t. \end{cases}$$

In [8] it has been shown that, $\Delta_t(G)$ is a simplicial tree if G is a rooted tree and $t \geq 2$. One of interesting problems in combinatorial commutative algebra is the Stanley's conjectures. The Stanley's conjectures are studied by many researchers. Let R be a $\mathbb{N}^n$-graded ring and M a $\mathbb{Z}^n$- graded R- module. Then, Stanley [10] conjectured that

$$\operatorname{depth}(M) \leq \operatorname{sdepth}(M)$$

He also conjectured in [11] that each Cohen-Macaulay simplicial complex is partitionable. Herzog, Soleyman Jahan and Yassemi in [7] showed that the conjecture about partitionability is a special case of the Stanley's first conjecture. In this work, we study algebraic properties of $\Delta_t(C_n)$. In Section 1, we recall some definitions and results which will be needed later. In Section 3, we show that the following conditions are equivalent for all $t > 2$:

(*i*) $\Delta_t(C_n)$ is matroid;
(*ii*) $\Delta_t(C_n)$ is vertex decomposable;
(*iii*) $\Delta_t(C_n)$ is shellable;
(*iv*) $\Delta_t(C_n)$ is Cohen-Macaulay;
(*v*) $n = t$ or $t + 1$.
(See Theorem 3.6).

In Section 4 as an application of our results we show that if $n = t$ or $t+1$ then $\Delta_t(C_n)$ is partitionable and Stanley's conjecture holds for $K[\Delta_t(C_n)]$.

§2. Preliminaries

In this section we recall some definitions and results which will be needed later.

Definition 2.1 *A simplicial complex Δ over a set of vertices $V = \{x_1, \cdots, x_n\}$, is a collection of subsets of V, with the property that:*

(*a*) $\{x_i\} \in \Delta$ *for all* i;
(*b*) *If $F \in \Delta$, then all subsets of F are also in Δ (including the empty set).*

An element of Δ is called a *face* of Δ and complement of a face F is $V \setminus F$ and it is denoted by F^c. Also, the complement of the simplicial complex $\Delta = \langle F_1, \cdots, F_r \rangle$ is $\Delta^c = \langle F_1^c, \cdots, F_r^c \rangle$. The *dimension* of a face F of Δ, $\dim F$, is $|F| - 1$ where, $|F|$ is the number of elements of F and $\dim \varnothing = -1$. The faces of dimensions 0 and 1 are called *vertices* and *edges*, respectively. A *non-face* of Δ is a subset F of V with $F \notin \Delta$. we denote by $\mathcal{N}(\Delta)$, the set of all minimal non-faces of Δ. The maximal faces of Δ under inclusion are called *facets* of Δ. The *dimension* of the simplicial complex Δ, $\dim\Delta$, is the maximum of dimensions of its facets. If all facets of Δ have the same dimension, then Δ is called *pure*.

Let $\mathcal{F}(\Delta) = \{F_1, \cdots, F_q\}$ be the facet set of Δ. It is clear that $\mathcal{F}(\Delta)$ determines Δ completely and we write $\Delta = \langle F_1, \cdots, F_q \rangle$. A simplicial complex with only one facet is called a *simplex*. A simplicial complex Γ is called a *subcomplex* of Δ, if $\mathcal{F}(\Gamma) \subset \mathcal{F}(\Delta)$.

For $v \in V$, the subcomplex of Δ obtained by removing all faces $F \in \Delta$ with $v \in F$ is denoted by $\Delta \setminus v$. That is,

$$\Delta \setminus v = \langle F \in \Delta : \ v \notin F \rangle.$$

The *link* of a face $F \in \Delta$, denoted by $\mathrm{link}_\Delta(F)$, is a simplicial complex on V with the faces, $G \in \Delta$ such that, $G \cap F = \varnothing$ and $G \cup F \in \Delta$. The link of a vertex $v \in V$ is simply denoted by $\mathrm{link}_\Delta(v)$.

$$\mathrm{link}_\Delta(v) = \big\{F \in \Delta : \ v \notin F, \ F \cup \{v\} \in \Delta\big\}.$$

Let Δ be a simplicial complex over n vertices $\{x_1, \cdots, x_n\}$. For $F \subset \{x_1, \cdots, x_n\}$, we set:

$$\mathbf{x}_F = \prod_{x_i \in F} x_i.$$

We define the *facet ideal* of Δ, denoted by $I(\Delta)$, to be the ideal of S generated by $\{\mathbf{x}_F : \quad F \in \mathcal{F}(\Delta)\}$. The *non-face ideal* or the *Stanley-Reisner ideal* of Δ, denoted by I_Δ, is the ideal of S generated by square-free monomials $\{\mathbf{x}_F : \quad F \in \mathcal{N}(\Delta)\}$. Also we call $K[\Delta] := S/I_\Delta$ the *Stanley-Reisner ring* of Δ.

Definition 2.2 *A simplicial complex Δ on $\{x_1, \cdots, x_n\}$ is said to be a matroid if, for any two facets F and G of Δ and any $x_i \in F$, there exists a $x_j \in G$ such that $(F \setminus \{x_i\}) \cup \{x_j\}$ is a facet of Δ.*

Definition 2.3 *A simplicial complex Δ is recursively defined to be vertex decomposable, if it is either a simplex, or else has some vertex v so that*

(*a*) *Both $\Delta \setminus v$ and $\mathrm{link}_\Delta(v)$ are vertex decomposable, and*

(*b*) *No face of $\mathrm{link}_\Delta(v)$ is a facet of $\Delta \setminus v$.*

A vertex v which satisfies in condition (b) is called a shedding vertex.

Definition 2.4 *A simplicial complex Δ is shellable, if the facets of Δ can be ordered $F_1, \cdots, F_s$ such that, for all $1 \leq i < j \leq s$, there exists some $v \in F_j \setminus F_i$ and some $l \in \{1, \cdots, j-1\}$ with $F_j \setminus F_l = \{v\}$.*

A simplicial complex Δ is called disconnected, if the vertex set V of Δ is a disjoint union $V = V_1 \cup V_2$ such that no face of Δ has vertices in both V_1 and V_2. Otherwise Δ is connected. It is well-known that

matroid $\Longrightarrow$ vertex decomposable $\Longrightarrow$ shellable $\Longrightarrow$ Cohen-Macaulay

Definition 2.5 *For a given simplicial complex Δ on V, we define $\Delta^\vee$, the Alexander dual of Δ, by*

$$\Delta^\vee = \{V \setminus F : \ F \notin \Delta\}.$$

It is known that for the complex Δ one has $I_{\Delta^\vee} = I(\Delta^c)$. Let $I \neq 0$ be a homogeneous ideal of S and $\mathbb{N}$ be the set of non-negative integers. For every $i \in \mathbb{N} \cup \{0\}$, one defines:

$$t_i^S(I) = \max\{j : \ \beta_{i,j}^S(I) \neq 0\}$$

where $\beta_{i,j}^S(I)$ is the i, j-th graded Betti number of I as an S-module. The *Castelnuovo-Mumford regularity* of I is given by:

$$\operatorname{reg}(I) = \sup\{t_i^S(I) - i : \ i \in \mathbb{Z}\}.$$

We say that the ideal I has a *d-linear resolution*, if I is generated by homogeneous polynomials of degree d and $\beta_{i,j}^S(I) = 0$, for all $j \neq i + d$ and $i \geq 0$. For an ideal which has a d-linear resolution, the Castelnuovo-Mumford regularity would be d. If I is a graded ideal of S, we write (I_d) for the ideal generated by all homogeneous polynomials of degree d belonging to I.

Definition 2.6 *A graded ideal I is componentwise linear if (I_d) has a linear resolution for all d.*

Also, we write $I_{[d]}$ for the ideal generated by the squarefree monomials of degree d belonging to I.

Definition 2.7 *A graded S-module M is called sequentially Cohen-Macaulay (over K), if there exists a finite filtration of graded S-modules,*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each M_i/M_{i-1} is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1}).$$

The Alexander dual, allows us to make a bridge between (sequentially) Cohen-Macaulay ideals and (componetwise) linear ideals.

Definition 2.8(Alexander Duality) *For a square-free monomial ideal $I = (M_1, \cdots, M_q) \subset S =$*

$K[x_1, \cdots, x_n]$, the Alexander dual of I, denoted by $I^\vee$, is defined to be

$$I^\vee = P_{M_1} \cap \cdots \cap P_{M_q}$$

where, P_{M_i} is prime ideal generated by $\{x_j : x_j | M_i\}$.

Theorem 2.9(Proposition 8.2.20, [6]; Theorem 3, [4]) *Let I be a square-free monomial ideal in $S = K[x_1, \cdots, x_n]$.*

(i) The ideal I is componentwise linear ideal if and only if $S/I^\vee$ is sequentially Cohen-Macaulay;

(ii) The ideal I has a q-linear resolution if and only if $S/I^\vee$ is Cohen-Macaulay of dimension $n - q$.

Remark 2.10 Two special cases, we will be considering in this paper, are when G is a cycle C_n, or a line graph L_n on vertices $\{x_1, \cdots, x_n\}$ with edges

$$E(C_n) = \big\{\{x_1, x_2\}, \{x_2, x_3\}, \cdots, \{x_{n-1}, x_n\}, \{x_n, x_1\}\big\};$$
$$E(L_n) = \big\{\{x_1, x_2\}, \{x_2, x_3\}, \cdots, \{x_{n-1}, x_n\}\big\}.$$

§3. Vertex Decomposability Path Complexes of Cycles

As the main result of this section, it is shown that $\Delta_t(C_n)$ is matroid, vertex decomposable, shellable and Cohen-Macaualay if and only if $n = t$ or $n = t + 1$. For the proof we shall need the following lemmas and propositions.

Lemma 3.1 *Let $\Delta_t(L_n)$ be a simplicial complex on $\{x_1, \cdots, x_n\}$ and $2 \leq t \leq n$. Then $\Delta_t(L_n)$ is vertex decomposable.*

Proof If $t = n$, then $\Delta_n(L_n)$ is a simplex which is vertex decomposable. Let $2 \leq t < n$ then one has

$$\Delta_t(L_n) = \langle\{x_1, \cdots, x_t\}, \{x_2, \cdots, x_{t+1}\}, \cdots, \{x_{n-t+1}, \cdots, x_n\}\rangle.$$

So $\Delta_t(L_n) \setminus x_n = \langle\{x_1, \cdots, x_t\}, \{x_2, \cdots, x_{t+1}\}, \cdots, \{x_{n-t}, \cdots, x_{n-1}\}\rangle$. Now we use induction on the number of vertices of L_n and by induction hypothesis $\Delta_t(L_n)\setminus x_n$ is vertex decomposable. On the other hand, it is clear that $\text{link}_{\Delta_t(L_n)}\{x_n\} = \langle\{x_{n-t+1}, \cdots, x_{n-1}\}\rangle$. Thus $\text{link}_{\Delta_t(L_n)}\{x_n\}$ is a simplex which is not a facet of $\Delta_t(L_n) \setminus x_n$. Therefore $\Delta_t(L_n)$ is vertex decomposable. □

Lemma 3.2 *Let $\Delta_2(C_n)$ be a simplicial complex on $\{x_1, \cdots, x_n\}$. Then $\Delta_2(C_n)$ is vertex decomposable.*

Proof Since $\Delta_2(C_n) = \langle\{x_1, x_2\}, \{x_2, x_3\}, \cdots, \{x_{n-1}, x_n\}, \{x_n, x_1\}\rangle$, we have

$$\Delta_2(C_n) \setminus x_n = \langle\{x_1, , x_2\}, \{x_2, x_3\}, \cdots, \{x_{n-2}, x_{n-1}\}\rangle.$$

By lemma 3.1 $\Delta_2(C_n)\setminus x_n$ is vertex decomposable. Also it is trivial that $\mathrm{link}_{\Delta_2(C_n)}\{x_n\} = \langle\{x_{n-1}\},\{x_1\}\rangle$ is vertex decomposable and no face of $\mathrm{link}_{\Delta_2(C_n)}\{x_n\}$ is a facet of $\Delta_2(C_n)\setminus x_n$. Therefore $\Delta_2(C_n)$ is vertex decomposable. □

Lemma 3.3 *Let $\Delta_t(C_n)$ be a simplicial complex on $\{x_1,\cdots,x_n\}$ and $3\le t\le n-2$. Then $\Delta_t(C_n)$ is not Cohen-Macaulay.*

Proof It suffices to show that $I_{\Delta_t(C_n)^\vee}$ has not a linear resolution. Since $I_{\Delta_t(C_n)^\vee} = I(\Delta_t(C_n)^c)$ then one can easily check that $I_{\Delta_t(C_n)^\vee} = I_{n-t}(C_n)$. By Theorem 1.1 we have

$$\mathrm{reg}(I_{\Delta_t(C_n)^\vee}) = (n-t-1)p+d+1.$$

Since $3\le t\le n-2$ then one has $\mathrm{reg}(I_{\Delta_t(C_n)^\vee})\neq n-t$ and by Theorem 2.9 $\Delta_t(C_n)$ is not Cohen-Macaulay. □

Proposition 3.4 *Let $\Delta_t(C_n)$ be a simplicial complex on $\{x_1,\cdots,x_n\}$ and $t\ge 3$. Then $\Delta_t(C_n)$ is vertex decomposable if and only if $n=t$ or $t+1$.*

Proof By Lemma 3.3 it suffices to show that if $n=t$ or $t+1$, then $\Delta_t(C_n)$ is vertex decomposable. If $n=t$, then $\Delta_n(C_n)$ is a simplex which is vertex decomposable. If $t=n-1$, then we have

$$\Delta_{n-1}(C_n) = \langle\{x_1,\cdots,x_{n-1}\},\{x_2,\cdots,x_n\},\{x_3,\cdots,x_n,x_1\},\cdots,\{x_n,x_1,\cdots,x_{n-2}\}\rangle.$$

Now we use induction on the number of vertices of C_n and show that $\Delta_{n-1}(C_n)$ is vertex decomposable. It is clear that $\Delta_{n-1}(C_n)\setminus x_n = \langle\{x_1,\cdots,x_{n-1}\}\rangle$ is a simplex which is vertex decomposable.

On the other hand,

$$\mathrm{link}_{\Delta_{n-1}(C_n)}\{x_n\} = \langle\{x_1,\cdots,x_{n-2}\},\cdots,\{x_{n-1},x_1,\cdots,x_{n-3}\}\rangle = \Delta_{n-2}(C_{n-1}).$$

By induction hypothesis $\mathrm{link}_{\Delta_{n-1}(C_n)}\{x_n\}$ is vertex decomposable. It is easy to see that no face of $\mathrm{link}_{\Delta_{n-1}(C_n)}\{x_n\}$ is a facet of $\Delta_{n-1}(C_n)\setminus x_n$. Therefore $\Delta_{n-1}(C_n)$ is vertex decomposable. □

Proposition 3.5 *$\Delta_2(C_n)$ is a matroid if and only if $n=3$ or 4.*

Proof If $n=3$ or 4, then it is easy to see that $\Delta_2(C_n)$ is a matroid. Now we prove the converse. It suffices to show that $\Delta_2(C_n)$ is not a matroid for all $n\ge 5$. We consider two facets $\{x_1,x_2\}$ and $\{x_{n-1},x_n\}$. Then we have $(\{x_1,x_2\}\setminus\{x_1\})\cup\{x_{n-1}\} = \{x_2,x_{n-1}\}$ and $(\{x_1,x_2\}\setminus\{x_1\})\cup\{x_n\} = \{x_2,x_n\}$. Since $\{x_2,x_{n-1}\}$ and $\{x_2,x_n\}$ are not the facets of $\Delta_2(C_n)$. So $\Delta_2(C_n)$ is not matroid for all $n\ge 5$. □

For the simplicial complexes one has the following implication:

Matroid $\Rightarrow$ *vertex decomposable* $\Rightarrow$ *shellable* $\Rightarrow$ *Cohen-Macaulay*

Note that these implications are strict, but by the following theorem, for path complexes, the reverse implications are also valid.

Theorem 3.6 *Let $t \geq 3$. Then the following conditions are equivalent:*

(i) $\Delta_t(C_n)$ *is matroid;*

(ii) $\Delta_t(C_n)$ *is vertex decomposable;*

(iii) $\Delta_t(C_n)$ *is shellable;*

(iv) $\Delta_t(C_n)$ *is Cohen-Macaulay;*

(v) $n = t$ *or* $t+1$.

Proof $(i) \Longrightarrow (ii)$, $(ii) \Longrightarrow (iii)$ and $(iii) \Longrightarrow (iv)$ is well-known.

$(iv) \Longrightarrow (v)$ follows from Lemma 3.3 and Proposition 3.4.

$(v) \Longrightarrow$ (i): If $n = t$, then $\Delta_t(C_n)$ is a simplex which is a matroid. If $n = t+1$, then

$$\Delta_t(C_n) = \langle \{x_1, \cdots, x_t\}, \{x_2, \cdots, x_{t+1}\}, \{x_3, \cdots, x_{t+1}, x_1\}, \cdots, \{x_{t+1}, x_1, \cdots, x_{t-1}\}\rangle.$$

For any two facets F and G of $\Delta_t(C_n)$ one has $| F \cap G | = t-1$. We claim that for any two facets F and G of $\Delta_t(C_n)$ and any $x_i \in F$, there exists a $x_j \in G$ such that $(F \setminus \{x_i\}) \cup \{x_j\}$ is a facet of $\Delta_t(C_n)$. We have to consider two cases. If $x_i \in F$ and $x_i \notin G$, then we choose $x_j \in G$ such that $x_j \notin F$. Thus $(F \setminus \{x_i\}) \cup \{x_j\} = G$ which is a facet of $\Delta_t(C_n)$.
For other case, if $x_i \in F$ and $x_i \in G$, then we choose $x_j \in G$ such that x_j is the same x_i. Therefore $(F \setminus \{x_i\}) \cup \{x_i\} = F$ is a facet of $\Delta_t(C_n)$ which completes the proof. □

§4. Stanley Decompositions

Let R be any standard graded K- algebra over an infinite field K, *i.e*, R is a finitely generated graded algebra $R = \bigoplus_{i \geq 0} R_i$ such that $R_0 = K$ and R is generated by R_1. There are several characterizations of the depth of such an algebra. We use the one that depth (R) is the maximal length of a regular R- sequence consisting of linear forms. Let $x_F = \sqcap_{i \in F} x_i$ be a squarefree monomial for some $F \subseteq [n]$ and $Z \subseteq \{x_1, \cdots, x_n\}$. The K- subspace $x_F K[Z]$ of $S = K[x_1, \cdots, x_n]$ is the subspace generated by monomials $x_F u$, where u is a monomial in the polynomial ring $K[Z]$. It is called a square free Stanley space if $\{x_i : i \in F\} \subseteq Z$. The dimension of this Stanley space is $|Z|$. Let Δ be a simplicial complex on $\{x_1, \cdots, x_n\}$. A square free Stanley decomposition $\mathcal{D}$ of $K[\Delta]$ is a finite direct sum $\bigoplus_i u_i K[Z_i]$ of squarefree Stanley spaces which is isomorphic as a $\mathbb{Z}^n$- graded K- vector space to $K[\Delta]$, *i.e.*

$$K[\Delta] \cong \bigoplus_i u_i K[Z_i].$$

We denote by sdepth $(\mathcal{D})$ the minimal dimension of a Stanley space in $\mathcal{D}$ and define sdepth $(K[\Delta])$ $= \max\{\text{sdepth}\,(\mathcal{D})\}$, where $\mathcal{D}$ is a Stanley decomposition of $K[\Delta]$. Stanley conjectured in [10] the upper bound for the depth of $K[\Delta]$ holding with

$$\operatorname{depth}(K[\Delta]) \leq \operatorname{sdepth}(K[\Delta]).$$

Also we recall another conjecture of Stanley. Let Δ be again a simplicial complex on $\{x_1, \cdots, x_n\}$ with facets $G_1, \cdots, G_t$. The complex Δ is called partitionable if there exists a partition $\Delta = \bigcup_{i=1}^{t}[F_i, G_i]$ where $F_i \subseteq G_i$ are suitable faces of Δ. Here the interval $[F_i, G_i]$ is the set of faces $\{H \in \Delta : F_i \subseteq H \subseteq G_i\}$. In [11] and [12] respectively Stanley conjectured each Cohen-Macaulay simplicial complex is partitionable. This conjecture is a special case of the previous conjecture. Indeed, Herzog, Soleyman Jahan and Yassemi [7] proved that for Cohen-Macaulay simplicial complex Δ on $\{x_1, \ldots, x_n\}$ we have that $\operatorname{depth}(K[\Delta]) \leq \operatorname{sdepth}(K[\Delta])$ if and only if Δ is partitionable. Since each vertex decomposable simplicial complex is shellable and each shellable complex is partitionable. Then as a consequence of our results, we obtain

Corollary 3.1 *If $n = t$ or $t + 1$ then $\Delta_t(C_n)$ is partitionable and Stanley's conjecture holds for $K[\Delta_t(C_n)]$.*

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Some Classes of Analytic Functions with q-Calculus

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Abstract: We study the estimates for the Second q-Hankel determinant of analytic functions in a class which unifies a number of classes studied previously by Darus, Ramreddy, Ravichandran, Yang and others. Our class includes q-convex and q-starlike functions. Also we study the estimate for q-Toeplitz determinants whose elements are the coefficients a_n for f in close-to-q-convex functions.

Key Words: Second Hankel determinant, subordination, q-starlike and q-convex functions, close-to-q-convex, Toeplitz matrices.

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§1. Introduction

The Hankel determinants $H_q(n)$ of Taylor's coefficients of function $f \in \mathcal{A}$ where $\mathcal{A}$ denotes the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \ \ and \ \ |z| < 1\}$. is defined by

$$\mathbf{H_q(n)} := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}$$

where $(a_1 = 1, \ \ n = q \in \mathbb{N})$. Hankel matrices (and determinants) play an important role in several branches of mathematics and have many applications [11]. $H_2(1)$ is the classical Fekete-Szegö functional. Fekete-Szegö in [4] found the maximum value of $H_2(1)$. Pommerenke in [16] proved that the Hankel determinant of univalent functions satisfy

$$|H_q(n)| < Kn^{-(\frac{1}{2}+\beta)q+\frac{3}{2}} \quad (n = 1, 2, \cdots, q = 2, 3, \cdots),$$

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where $\beta > \frac{1}{4000}$ and K depends only on q.

Hayman [8] showed that

$$|H_2(n)| = |a_n a_{n+2} - a_{n+1}^2| < An^{\frac{1}{2}} \quad , n = 2, 3, \cdots,$$

where A is an absolute constant for a really mean univalent functions. Hankel determinants are useful in showing that a function of bounded characteristic in $\mathcal{U}$, i.e, a function which is ratio of tow bounded analytic functions with its Laurent series around the origin having integral coefficients, is rational. In recent years, several authors investigated bounds for the Hankel determinant belonging tow various subclasses of univalent and multivalent functions in a class which unifies a number of classes studied earlier by Deepak Bansal, K. I. Noor, T. Yavuz, Sarika Verma, Shigeyoshi Owa and others. Closely related to Hankel determinants are the Toepliz determinants. A Toeplitz matrix $T_q(n)$ defined by

$$\mathbf{T_q(n)} := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_n \end{vmatrix}$$

A Toeplitz matrix can be thought of as an upside-down Hankel matrix, in that Hankel matrices have constant entries along the reverse diagonal, whereas Toeplitz matrices have constant entries along the diagonal. A good summary of the applications of Toeplitz matrices to a wide range of areas of pure and applied mathematics can also be found in [11].

We aim to define q-starlike, q-convex functions and Ma-Minda starlike and convex functions. We use the concept of principle of subordination and q-calculus to define our classes. Recently in the second half of the twentieth century q-calculus aroused interest due to lot of applications in the various mathematical fields such as combinatorics, number theory, quantum theory and the theory of relativity. The q-derivative of a function is defined in the following.

Definition 1.1([9]) *The q-derivative of f is given by*

$$\partial_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{z(1-q)}, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \quad , \quad \textit{where } 0 < q < 1. \tag{1.2}$$

Equivalently, (1.2) may be written as

$$\partial_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad z \neq 0$$

where

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & \text{q} \neq 1 \\ n, & \text{q} = 1. \end{cases}$$

Note that as $q \to 1^-$, $[n]_q \to n$.

Definition 1.2([7]) *Let f be analytic in $\mathcal{U}$ and be given by (1.1). Then a function f is starlike if and only if,* $\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0$. *We denote the class of starlike functions by S^*.*

The class of functions with positive real part plays a significant role in complex function theory. Using principle of subordination we define the functions with positive real part.

Definition 1.3([17]) *Let f and g be analytic in $\mathcal{U}$, then f is said to be subordinate to the function g, written $f(z) \prec g(z)$, if there exists an analytic function $\omega : \mathcal{U} \to \mathcal{U}$ satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$, $z \in \mathcal{U}$.*

Definition 1.4([3]) *Let $\mathcal{P}$ denote the class of analytic functions $p : \mathcal{U} \to \mathbb{C}$, $p(0) = 1$, and $\Re\{p(z)\} > 0$, then $p(z) \prec \frac{1+z}{1-z}$.*

The class $\mathcal{P}$ can be completely characterized in terms of subordination. We need the following lemmas to derive our results.

Lemma 1.5([3]) *If the function $p \in \mathcal{P}$ is given by the series*

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \tag{1.3}$$

then the following sharp estimate holds:

$$|c_n| \le 2 \quad (n = 1, 2, \cdots).$$

Lemma 1.6([6]) *If the function $p \in \mathcal{P}$ is given by the series (1.3), then*

$$2c_2 = c_1^2 + x(4 - c_1^2), \tag{1.4}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z, \tag{1.5}$$

for some x, z with $|x| \le 1$ and $|z| \le 1$.

§2. Main Results

Definition 2.1 *Let $\varphi : U \to \mathbb{C}$ be analytic, and let the Maclaurin series of φ is given by*

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1, B_2 \in \mathbb{R}, B_1 > 0). \tag{2.1}$$

Let $0 \le \gamma \le 1$ and $\tau \in \mathbb{C}\backslash\{0\}$. A function $f \in \mathcal{A}$ is in the class $\mathcal{R}^{\tau}_{q,\gamma}(\varphi)$ if it satisfies the following subordination:

$$1 + \frac{1}{\tau}(\partial_q f(z) + \gamma z \partial_q^2 f(z) - 1) \prec \varphi(z).$$

Theorem 2.2 *Let $0 \leq \gamma \leq 1$, $\tau \in \mathbb{C}\backslash\{0\}$ and let the function f as in (1.1) be in the class $\mathcal{R}^{\tau}_{q,\gamma}(\varphi)$. Also let $p = \frac{[2]_q[4]_q(1+\gamma)(1+[3]_q\gamma)}{([3]_q)^2(1+[2]_q\gamma)^2}$.*

(1) *If B_1, B_2 and B_3 satisfy the conditions $2|B_2|(1-p) + B_1(1-2p) \leq 0$, $|B_1B_3 - pB_2^2| - pB_1^2 \leq 0$, then the second Hankel determinant satisfies*

$$|a_2a_4 - a_3^2| \leq \frac{|\tau|^2 B_1^2}{([3]_q)^2(1+[2]_q\gamma)^2}.$$

(2) *If B_1, B_2 and B_3 satisfy the conditions $2|B_2|(1-p) + B_1(1-2p) \geq 0$, $2|B_1B_3 - pB_2^2| - 2(1-p)B_1|B_2| - B_1 \geq 0$, or the conditions $2|B_2|(1-p)+B_1(1-2p) \leq 0$, $|B_1B_3 - pB_2^2| - pB_1^2 \geq 0$, then the second Hankel determinant satisfies*

$$|a_2a_4 - a_3^2| \leq \frac{|\tau|^2}{[2]_q[4]_q(1+\gamma)(1+[3]_q\gamma)}|B_1B_3 - pB_2^2|.$$

(3) *If B_1, B_2 and B_3 satisfy the conditions $2|B_2|(1-p) + B_1(1-2p) > 0$, $|B_1B_3 - pB_2^2| - pB_1^2 \leq 0$, then the second Hankel determinant satisfies*

$$\begin{aligned} |a_2a_4 - a_3^2| &\leq \frac{|\tau|^2 B_1^2}{4[4]_q[2]_q(1+\gamma)(1+[3]_q\gamma)} \\ &\times \left[\frac{4p|B_1B_3 - pB_2^2| - 4B_1(1-p)[|B_2|(3-2p) + B_1] - 4B_2^2(1-p)^2 - B_1^2(1-2p)^2}{|B_1B_3 - pB_2^2| - B_1(1-p)(2|B_2| + B_1)}\right]. \end{aligned}$$

Proof Since $f \in \mathcal{R}^{\tau}_{q,\gamma}(\varphi)$, there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ in $\mathcal{U}$ such that

$$1 + \frac{1}{\tau}(\partial_q f(z) + \gamma z \partial_q^2 f(z) - 1) = \varphi(w(z)). \tag{2.2}$$

Define the function p_1 by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1z + c_2z^2 + \cdots,$$

or equivalently,

$$w(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left(c_1z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)z^3 + \cdots\right) \tag{2.3}$$

Then p_1 is analytic in $\mathcal{U}$ with $p_1(0) = 0$ and has a positive real part in $\mathcal{U}$. By using (2.3) together with (2.1), it is evident that

$$\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}B_1c_1z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right)z^2 + \cdots \tag{2.4}$$

since f has the Maclaurin series given by (1.1), a computation shows that

$$\begin{aligned}1+\frac{1}{\tau}(\partial_q f(z)+\gamma z\partial_q^2 f(z)-1) &= 1+\frac{[2]_q a_2(1+\gamma)}{\tau}z+\frac{[3]_q a_3(1+[2]_q\gamma)}{\tau}z^2\\ &\quad +\frac{[4]_q a_4(1+[3]_q\gamma)}{\tau}z^3+\cdots. \end{aligned}\tag{2.5}$$

It follows from (2.2), (2.4) and (2.5) that

$$\begin{aligned}a_2 &= \frac{\tau B_1 c_1}{2[2]_q(1+\gamma)},\\ a_3 &= \frac{\tau B_1}{4[3]_q(1+[2]_q\gamma)}\left[2c_2+c_1^2(\frac{B_2}{B_1}-1)\right],\\ a_4 &= \frac{\tau}{8[4]_q(1+[3]_q\gamma)}\left[B_1(4c_3-4c_1c_2+c_1^3)+2B_2c_1(2c_2-c_1^2)+B_3c_1^3\right].\end{aligned}$$

Therefore,

$$\begin{aligned}& a_2a_4-a_3^2\\ &= \frac{\tau^2 B_1 c_1}{16([4]_q[2]_q)(1+\gamma)(1+[3]_q\gamma)}\left[B_1(4c_3-4c_1c_2+c_1^3)+2B_2c_1(2c_2-c_1^2)+B_3c_1^3\right]\\ &\quad -\frac{\tau^2 B_1^2}{16([3]_q)^2(1+[2]_q\gamma)^2}\left[4c_2^2+c_1^4(\frac{B_2}{B_1}-1)^2+4c_2c_1^2(\frac{B_2}{B_1}-1)\right]\\ =\ & \frac{\tau^2 B_1 c_1}{16([4]_q[2]_q)(1+\gamma)(1+[3]_q\gamma)}\left\{\left[\left(4c_1c_3-4c_2c_1^2+c_1^4\right)+\frac{2B_2c_1^2}{B_1}\left(2c_2-c_1^2\right)+\frac{B_3}{B_1}c_1^4\right]\right.\\ &\quad \left.-\frac{[4]_q[2]_q(1+\gamma)(1+[3]_q\gamma)}{([3]_q)^2(1+[2]_q\gamma)^2}\left[4c_2^2+c_1^4(\frac{B_2}{B_1}-1)^2+4c_2c_1^2(\frac{B_2}{B_1}-1)\right]\right\},\end{aligned}$$

which yields

$$\begin{aligned}|a_2a_4-a_3^2| &= T\left|4c_1c_3+c_1^4\left[1-2\frac{B_2}{B_1}-p(\frac{B_2}{B_1}-1)^2+\frac{B_3}{B_1}\right]\right.\\ &\quad \left.-4pc_2^2-4c_1^2c_2\left[1-\frac{B_2}{B_1}+p(\frac{B_2}{B_1}-1)\right]\right|,\end{aligned}\tag{2.6}$$

where,

$$T=\frac{|\tau|^2 B_1^2}{16([4]_q[2]_q)(1+\gamma)(1+[3]_q\gamma)} \quad \text{and} \quad p=\frac{[4]_q[2]_q(1+\gamma)(1+[3]_q\gamma)}{([3]_q)^2(1+[2]_q\gamma)^2}.$$

It can be easily verified that $p\in[\frac{64}{81},\frac{8}{9}]$ for $0\le\gamma\le1$ and $0\le q\le1$. Let

$$\begin{aligned}&d_1=4, \qquad d_2=-4\left[1-\tfrac{B_2}{B_1}+p(\tfrac{B_2}{B_1}-1)\right],\\ &d_3=-4p, \qquad d_4=\left[1-2\tfrac{B_2}{B_1}-p(\tfrac{B_2}{B_1}-1)^2+\tfrac{B_3}{B_1}\right].\end{aligned}\tag{2.7}$$

Then (2.6) becomes

$$\left|a_2a_4-a_3^2\right|=T\left|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4\right|.\tag{2.8}$$

Since the function $p(e^{i\theta}z)(\theta \in \mathbb{R})$ is in the class $\mathcal{P}$ for any $p \in \mathcal{P}$, there is no loss of generality in assuming $c_1 > 0$. Write $c_1 = c, c \in [0,2]$. Substituting the values of c_2 and c_3 respectively from (1.6) and (1.5) in (2.8), we obtain

$$\begin{aligned} |a_2a_4 - a_3^2| &= \frac{T}{4}\left|c^4\left(d_1 + 2d_2 + d_3 + 4d_4\right) + 2xc^2(4-c^2)\left(d_1 + d_2 + d_3\right)\right. \\ &\left. + (4-c^2)x^2\left(-d_1c^2 + d_3(4-c^2)\right) + 2d_1c(4-c^2)(1-|x|^2z)\right|. \end{aligned}$$

Replacing $|x|$ by μ and substituting the values of d_1, d_2, d_3 and d_4 from (2.7) yields

$$\begin{aligned} |a_2a_4 - a_3^2| &\leq \frac{T}{4}\left[4c^4\left|\frac{B_3}{B_1} - p\frac{B_2^2}{B_1^2}\right| + 8\left|\frac{B_2}{B_1}\right|\mu c^2(4-c^2)(1-p)\right. \\ &\left. +(4-c^2)\mu^2(4c^2 + 4p(4-c^2)) + 8c(4-c^2)(1-\mu^2)\right] \\ &= T\left[c^4\left|\frac{B_3}{B_1} - p\frac{B_2^2}{B_1^2}\right| + 2c(4-c^2) + 2\mu\left|\frac{B_2}{B_1}\right|c^2(4-c^2)(1-p)\right. \\ &\left. +\mu^2(4-c^2)(1-p)(c-\alpha)(c-\beta)\right] \equiv F(c,\mu), \end{aligned} \tag{2.9}$$

where $\alpha = 2,\ \beta = 2p/(1-p) > 2$.

Note that for $(c,\mu) \in [0,2] \times [0,1]$, differentiating $F(c,\mu)$ in (2.9) partially with respect to μ yields

$$\frac{\partial F}{\partial \mu} = T\left[2\left|\frac{B_2}{B_1}\right|c^2(4-c^2)(1-p) + 2\mu(4-c^2)(1-p)(c-\alpha)(c-\beta)\right]. \tag{2.10}$$

Then, for $0 < \mu < 1,\ 0 < q < 1$ and any fixed c with $0 < c < 2$, it is clear from (2.10) that $\frac{\partial F}{\partial \mu} > 0$, that is, $F(c,\mu)$ is an increasing function of μ. Hence, for fixed $c \in [0,2]$, the maximum of $F(c,\mu)$ occurs at $\mu = 1$, and

$$\max F(c,\mu) = F(c,1) \equiv G(c),$$

which is

$$G(c) = T\left\{c^4\left[\left|\frac{B_3}{B_1} - p\frac{B_2^2}{B_1^2}\right| - (1-p)\left(2\left|\frac{B_2}{B_1}\right| + 1\right)\right] + 4c^2\left[2\left|\frac{B_2}{B_1}\right|(1-p) + 1 - 2p\right] + 16p\right\}.$$

Let

$$\begin{aligned} X &= \left|\frac{B_3}{B_1} - p\frac{B_2^2}{B_1^2}\right| - (1-p)\left(2\left|\frac{B_2}{B_1}\right| + 1\right), \\ Y &= 4\left[2\left|\frac{B_2}{B_1}\right|(1-p) + 1 - 2p\right], \\ Z &= 16p. \end{aligned} \tag{2.11}$$

Since

$$\max(Xt^2 + Yt + Z) = \begin{cases} Z, & Y \leq 0, X \leq \frac{-Y}{4}; \\ 16X + 4Y + Z, & Y \geq 0, X \geq \frac{-Y}{8} or Y \leq 0, X \geq \frac{-Y}{4}; \\ \frac{4XZ - Y^2}{4X}, & Y > 0, X \leq \frac{-y}{8}, \end{cases} \tag{2.12}$$

where $0 \le t \le 4$. Then we have

$$|a_2a_4 - a_3^2| \le \frac{B_1}{16([4]_q-1)([3]_q-1)([2]_q-1)} \times \begin{cases} Z, & Y \le 0, X \le \frac{-Y}{4}; \\ 16X+4Y+Z, & Y \ge 0, X \ge \frac{-Y}{8} \text{ or } Y \le 0, X \ge \frac{-Y}{4}; \\ \frac{4XZ-Y^2}{4X}, & Y > 0, X \le \frac{-y}{8}, \end{cases}$$

where X, Y and Z are given by (2.11). □

Remark 2.3 notice that

(1) As $q \to 1^-$ Theorem 2.2 reduces to Theorem 3 in [12].

(2) As $q \to 1^-$ for the choice $\varphi(z) := (1+Az)/(1+Bz)$ with $-1 \le B < A \le 1$ Theorem 2.2 reduces to Theorem 2.1 in [12].

Definition 2.4 *An analytic function f is close-to-q-convex in $\mathcal{U}$, if and only if, there exists $g \in S_q^*$ such that*

$$\Re\left\{\frac{z\partial_q f(z)}{g(z)}\right\} > 0.$$

We denote the class of close-to-q-convex functions by K_q.

For $f \in S^*$, we can write $z\partial_q f(z) = f(z)h(z)$, where $h \in P$, the class of function satisfying $\Re h(z) > 0$ for $z \in \mathcal{U}$ and

$$h(z) = 1 + \sum_{n=2}^{\infty} c_n z^n.$$

For $f \in K_q$, we can write $z\partial_q f(z) = g(z)p(z)$, where $p \in P$ and

$$p(z) = 1 + \sum_{n=2}^{\infty} p_n z^n.$$

Theorem 2.5 *Let $f \in K_q$ and given by* (1.1) *with associated starlike function g define by*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

Then

$$T_2(2) = |a_3^2 - a_2^2| \le [5]_q, \qquad (b_2 \in \mathbb{R})$$

and the inequality is sharp.

Proof Write $z\partial_q f(z) = g(z)h(z)$ and $zg'(z) = g(z)p(z)$, with

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

and

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n.$$

Then equating the coefficients in $z\partial_q f(z) = g(z)h(z)$ where coefficients' relations from $zg'(z) = g(z)p(z)$ is also used, we obtain

$$\begin{aligned} a_2 &= \frac{c_1 + p_1}{[2]_q}, \\ a_3 &= \frac{p_1^2 + p_2 + 2p_1c_1 + 2c_2}{2[3]_q} \end{aligned}$$

so that

$$\begin{aligned} |a_3^2 - a_2^2| &= \left| \frac{-1}{[2]_q^2}c_1^2 + \frac{1}{[3]_q^2}c_2^2 - \frac{2}{[2]_q^2}c_1p_1 + \frac{2}{[3]_q^2}c_1c_2p_1 - \frac{1}{[2]_q^2}p_1^2 + \frac{1}{[3]_q^2}c_1^2p_1^2 + \frac{1}{[3]_q^2}c_2p_1^2 \right. \\ &\quad \left. + \frac{1}{[3]_q^2}c_1p_1^3 + \frac{1}{4[3]_q^2}p_1^4 + \frac{1}{[3]_q^2}c_2p_2 + \frac{1}{[3]_q^2}c_1p_1p_2 + \frac{1}{2[3]_q^2}p_1^2p_2 + \frac{1}{4[3]_q^2}p_2^2 \right|. \end{aligned}$$

We now use Lemma 1.6 to express c_2 and p_2 in terms of c_1 and p_1 and writing$X = 4 - c_1^2$ and $Y = 4 - p_1^2$ for simplicity to get

$$\begin{aligned} &|a_3^2 - a_2^2| \\ &= \left| \frac{-1}{[2]_q^2}c_1^2 + \frac{1}{4[3]_q^2}c_1^4 - \frac{2}{[2]_q^2}c_1p_1 + \frac{1}{[3]_q^2}c_1^3p_1 - \frac{1}{[2]_q^2}p_1^2 + \frac{7}{4[3]_q^2}c_1^2p_1^2 \right. \\ &\quad + \frac{3}{2[3]_q^2}c_1p_1^3 + \frac{9}{16[3]_q^2}p_1^4 + \frac{1}{2[3]_q^2}c_1^2xX + \frac{1}{[3]_q^2}c_1p_1xX + \frac{3}{4[3]_q^2}p_1^2xX \\ &\quad + \frac{1}{4[3]_q^2}x^2X^2 + \frac{1}{4[3]_q^2}c_1^2yY + \frac{1}{2[3]_q^2}c_1p_1yY \\ &\quad \left. + \frac{3}{8[3]_q^2}p_1^2yY + \frac{1}{4[3]_q^2}xXyY + \frac{1}{16[3]_q^2}y^2Y^2 \right|. \end{aligned}$$

Without loss in generality we can assume that $c_1 = c$ where $0 \leq c \leq 2$. Also since we are assuming $b_2 = p_1$ to be real, we can write $p_1 = r$, with $0 \leq |r| \leq 2$, and write $|r| = p$. We note at this point a further normalisation of p_1 to be real would remove the requirement that $p_1 = b_2$ is real, but such normalisation does not appear to be justified. It follows from Lemma 1.6 that with now $X = 4 - c^2$ and $Y = 4 - p^2$. So,

$$\begin{aligned} &|a_3^2 - a_2^2| \\ &\leq \left| \frac{-1}{[2]_q^2}c^2 + \frac{1}{4[3]_q^2}c^4 - \frac{2}{[2]_q^2}cp + \frac{1}{[3]_q^2}c^3p - \frac{1}{[2]_q^2}p^2 + \frac{7}{4[3]_q^2}c^2p^2 + \frac{3}{2[3]_q^2}cp^3 + \frac{9}{16[3]_q^2}p^4 \right| \\ &\quad + \frac{1}{2[3]_q^2}c^2|x|X + \frac{1}{[3]_q^2}cp|x|X + \frac{3}{4[3]_q^2}p^2|x|X + \frac{1}{4[3]_q^2}|x|^2X^2 + \frac{1}{4[3]_q^2}c^2|y|Y \\ &\quad + \frac{1}{2[3]_q^2}cp|y|Y + \frac{3}{8[3]_q^2}p^2|y|Y + \frac{1}{4[3]_q^2}|x|X|y|Y + \frac{1}{16[3]_q^2}|y|^2Y^2. \end{aligned}$$

Now we assume $|x| \leq 1$ and $|y| \leq 1$ and simplify to obtain

$$\begin{aligned}|a_3^2 - a_2^2| \leq & \left| \frac{-1}{[2]_q^2}c^2 + \frac{1}{4[3]_q^2}c^4 - \frac{2}{[2]_q^2}cp + \frac{1}{[3]_q^2}c^3p - \frac{1}{[2]_q^2}p^2 + \frac{7}{4[3]_q^2}c^2p^2 + \frac{3}{2[3]_q^2}cp^3 + \frac{9}{16[3]_q^2}p^4 \right| \\ & + \frac{9}{[3]_q^2} - \frac{1}{4[3]_q^2}c^4 + \frac{6}{[3]_q^2}cp - \frac{1}{[3]_q^2}c^3p + \frac{3}{[3]_q^2}p^2 - \frac{3}{4[3]_q^2}c^2p^2 - \frac{1}{2[3]_q^2}cp^3 - \frac{5}{16[3]_q^2}p^4.\end{aligned}$$

Suppose that the expression between the modulus signs is positive, then

$$\begin{aligned}|a_3^2 - a_2^2| \leq \psi_1(c,p) = & \frac{9}{[3]_q^2} - \frac{1}{[2]_q^2}c^2 + \frac{2(3[2]_q^2 - [3]_q^2)}{[2]_q^2[3]_q^2}cp \\ & + \frac{2(3[2]_q^2 - [3]_q^2)}{[2]_q^2[3]_q^2}p^2 + \frac{1}{[3]_q^2}c^2P^2 + \frac{1}{[3]_q^2}cp^3 + \frac{1}{4[3]_q^2}p^4.\end{aligned}$$

Then for $0 \leq c \leq 2$ and $0 \leq p \leq 2$ and fixed q with $0 < q < 1$ and calculus we get that $\psi_1(c,p)$ has a maximum value of $[5]_q$ at [0,2].

If the expression between the modulus signs is negative, then

$$\begin{aligned}|a_3^2 - a_2^2| \leq \psi_2(c,p) = & \frac{9}{[3]_q^2} + \frac{1}{[2]_q^2}c^2 - \frac{1}{2[3]_q^2}c^4 + \frac{2(3[2]_q^2 + [3]_q^2)}{[2]_q^2[3]_q^2}cp \\ & - \frac{2}{[3]_q^2}c^3p\frac{3[2]_q^2 + [3]_q^2}{[2]_q^2[3]_q^2}p^2 - \frac{5}{2[3]_q^2}c^2P^2 - \frac{2}{[3]_q^2}cp^3 - \frac{7}{8[3]_q^2}p^4.\end{aligned}$$

Then for $0 \leq c \leq 2$ and $0 \leq p \leq 2$ and fixed q with $0 < q < 1$ and calculus we get that $\psi_2(c,p)$ has a maximum value less than $[3]_q$. Thus the proof is complete. □

As $q \to 1^-$, we have following result due to D. K. Thomas and S. Abdul Halim [18].

Corollary 2.6 *Let $f \in K$ and be given by (**??**) with the associated starlike function g be defined by*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

Then

$$T_2(2) = |a_3^2 - a_2^2| \leq 5,$$

provided b_2 is real. The inequality is sharp.

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On Obic Algebras

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Abstract: In this paper, obic algebras are introduced and their properties are investigated. Homomorphisms and krib maps as well as monics of obic algebras are studied. Properties of implicative obic algeras are also investigated.

Key Words: Obic algebras, krib maps, monics.

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§1. Introduction

Algebras of type (2,0) are well known types of algebraic structures. They comprise non-empty sets, some constant element together with a binary operation. In [1], Kim and Kim introduced the notion of BE-algebras. Ahn and so, in [2] and [3] introduced the notions of ideals and upper sets in BE-algebras and investigated related properties. In this paper, a new class of algebras called obic algebras are introduced. Their properties are investigated. Homomorphisms and krib maps as well as monics of obic algebras are studied. Moreover, translations in obic algebras are investigated as well as properties of implicative obic algebras.

Definition 1.1 *A non-empty set X together with a binary operation $*$ defined on X is called a groupoid.*

Definition 1.2 *A triple $(X;*,0)$, where X is a non-empty set, $*$ a binary operation on X and 0 a constant element of X is called an obic algebra if the following axioms hold for all $x,y,z \in X$:*

(1) $x*0=x$;
(2) $[x*(y*z)]*x = x*[y*(z*x)]$;
(3) $x*x=0$.

Example 1.1 Consider the multiplicative group $G=\{1,-1,i,-i\}$. Define a binary operation $*$ on G by $a*b=ab^{-1}$. Then $(G;*,1)$ is an obic algebra.

Example 1.2 Let $\mathbb{Z}$ denote the set of integers. Then $(\mathbb{Z};-,0)$ is an obic algebra.

Example 1.3 Let $X=\{0,1\}$. Define a binary operation $*$ on X in Table 1.

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$*$	0	1
0	0	1
1	1	0

Table 1

Then, $(X, *, 0)$ is an obic algebra.

We shall adopt the notation X for an obic algebra $(X; *, 0)$ unless stated otherwise.

Definition 1.3 *An obic algebra is called simple if* $y * (z * x) = x * (y * z)$ *for all* $x, y, z \in X$.

Definition 1.4 *An obic algebra is called plain if* $0 * (y * z) = (0 * y) * z$ *for all* $y, z \in X$.

Definition 1.5 *An obic algebra* X *is said to have the weak property (WP) if* $x * y = 0$ *and* $y * x = 0$ *imply that* $x = y$.

Definition 1.6 *An obic algebra* X *is called prime if* $0 * x = 0$ *for all* $x \in X$.

Lemma 1.1 *Let* X *be an obic algebra. Then for all* $x, y \in X$, *the following hold:*
$x * y = [x * (y * x)] * x$.

Definition 1.7 *A non-empty subset* S *of an obic algebra* X *is called a subalgebra if* S *is an obic algebra with respect to the binary operation in* X.

Example 1.4 Let X be an obic algebra. Then X and $\{0\}$ are subalgebras of X.

Example 1.5 Let X be the obic algebra in example 1.1. Then the subset $\{1, -1\}$ is a subalgebra of X.

The following results are immediately obtained by the definition.

Proposition 1.1 *A non-empty subset* S *of an obic algebra is a subalgebra if and only if the following hold:*

(1) $0 \in S$;

(2) $x * y \in S$ *for all* $x, y \in S$.

Proposition 1.2 *Let* X *be a plain obic algebra. Then, the subset* $S = \{x \in X : 0 * x = 0\}$ *is a subalgebra of* X.

§2. Obic Homomorphisms

Definition 2.1 *Let* $(X; *, 0)$ *and* $(Y; \circ, 0')$ *be obic algebras. A function* $f: X \to Y$ *is called an obic homomorphism if* $f(a * b) = f(a) \circ f(b)$ *for all* $a, b \in X$.

Definition 2.2 *Let* $f : X \to Y$ *be an obic homomorphism. The set* $\{x \in X : f(x) = 0'\}$ *is called the kernel of* f.

Proposition 2.1 *Let $f: X \to Y$ be an obic homomorphism. Then the kernel of f is a subalgebra of X.*

Then, we get conclusions following by definition.

Proposition 2.2 *Let $f: X \to Y$ be an obic homomorphism. Then,*

(1) $f(0) = 0'$;
(2) $x * y = 0 \Rightarrow f(x) \circ f(y) = 0'$ *for all* $x, y \in X$.

Let $f: X \to Y$ be an obic homomorphism. Define a relation $\sim$ by $(x \sim y) \Leftrightarrow f(x) = f(y)$. Then, we know

Lemma 2.1 *Let $f: X \to Y$ be an obic homomorphism. The relation $\sim$ defined by $(x \sim y) \Leftrightarrow f(x) = f(y)$ is an equivalence relation.*

Definition 2.3 *An equivalence relation $\sim$ on an obic algebra X is called a congruence if $(x \sim y)$ and $(u \sim v) \Rightarrow (x * u) \sim (y * v)$.*

We have the following result by definition.

Lemma 2.2 *Let $f: X \to Y$ be an obic homomorphism. The equivalence relation $\sim$ defined by $(x \sim y) \Rightarrow f(x) = f(y)$ is a congruence.*

Let $[x]$ be the equivalence class of $x \in X$ and let $\overline{X}$ denote the collection of equivalence classes in the equivalence relation $\sim$. Define a binary operation $\diamond$ on $\overline{X}$ by $[x] \diamond [y] = [x * y]$.

Theorem 2.1 *Let $f: X \to Y$ be an obic homomorphism. Then $(\overline{X}; \diamond, [0])$ is an obic algebra.*

Proof By Lemma 2.2, the binary operation $\diamond$ is well-defined. Now, let $[x], [y], [z] \in \overline{X}$. Consider $[x] \diamond [0] = [x * 0] = [x]$. Also,

$$\begin{aligned}([x] \diamond ([y] \diamond [z])) \diamond [x] &= ([x] \diamond [y * z]) \diamond [x] = ([x * (y * z)]) \diamond [x] \\ &= [(x * (y * z)) * x] = [x * (y * (z * x))] \\ &= [x] \diamond ([y] \diamond ([z] \diamond [x])).\end{aligned}$$

Also, $[x] \diamond [x] = [x * x] = [0]$. □

Theorem 2.2 *Let $f: X \to X$ be an endomorphism. Then $f(X)$ is isomorphic to $\overline{X}$.*

Proof Consider the map $\phi: f(X) \to \overline{X}$ such that $\phi(y) = [y]$. Let $y_1, y_2 \in f(X)$. Then $\phi(y_1 * y_2) = [y_1 * y_2] = [y_1] \diamond [y_2] = \phi(y_1) \diamond \phi(y_2)$. Also, ϕ is one to one and onto. □

Theorem 2.3 *Let $\phi: X \to X$ be an obic homomorphism; where X has the weak property. Then ϕ is one to one if and only if $ker(\phi) = \{0\}$.*

Proof Suppose ϕ is one to one. Let $x \in ker(\phi)$. Then $\phi(x) = 0 = \phi(0)$. So, $ker(\phi) = \{0\}$.

Conversely, suppose $ker(\phi) = \{0\}$. Let $x, y \in X$ such that $\phi(x) = \phi(y)$. Then $\phi(x * y) = \phi(x) * \phi(y) = 0$. Also, $\phi(y * x) = 0$. So, $(x * y), (y * x) \in ker(\phi)$. Hence ϕ is one to one. □

Definition 2.4 *An obic homomorphism $f : X \to X$ is called idempotent if $f(f(x)) = f(x)$ for all $x \in X$.*

Theorem 2.4 *Let X be an obic algebra with weak property. Let ϕ be an idempotent endomorphism on X. Then ϕ is one to one if and only if ϕ is the identity map.*

Proof Suppose ϕ is one to one. Let $x \in X$. Then $\phi((x * \phi(x))) = \phi(x) * \phi(\phi(x)) = \phi(x) * \phi(x) = 0 = \phi(0)$.So, $x * \phi(x) = 0$. Similarly argument gives $\phi(x) * x = 0$. And so $\phi(x) = x$. Hence ϕ is the identity map.

The converse is obvious. □

§3. Implicative Obic Algebras

Definition 3.1 *An obic algebra X is called implicative if $x * (y * x) = x$ for all $x, y \in X$.*

The following conclusion can be obtained by the definition.

Lemma 3.1 *Let X be an implicative obic algebra. Then the following hold:*

(1) $0 * 0 = 0$;

(2) $x * y = (x * y) * (0 * y)$;

(3) $x * y = (x * (y * x)) * y$;

(4) $y * x = y * (x * (y * x))$.

Definition 3.2 *Let X be an obic algebra. Let x be a fixed element of X. The map $L_x : X \to X$ such that $L_x(a) = x * a$ for all $a \in X$ is called a left translation on X. Similarly, the map $R_x : X \to X$ such that $R_x(a) = a * x$ for all $a \in X$ is called a right translation on X.*

Theorem 3.1 *Let $L_x : X \to X$ be an endomorphism. Then $x = 0$. Moreover, if X is implicative, then $x = x * (x * y)$.*

Proof Consider $x = x * 0 = L_x(0) = L_x(0 * 0) = L_x(0) * L_x(0) = (x * 0) * (x * 0) = 0$. Now suppose X is implicative. Let $y \in X$. Then $x = x * 0 = L_x(0) = L_x(0 * y) = L_x(0) * L_x(y) = x * (x * y)$. □

Denote the collection of left translations on an obic algebra X by $L(X)$ and define a binary operation $\odot$ on $L(X)$ by $(L_a \odot L_b)(x) = L_a(x) * L_b(x)$ for all $x \in X$.

Theorem 3.2 *Let X be an implicative obic algebra. Then $(L(X); \odot, L_0)$ is an obic algebra.*

Proof Let $L_a, L_b, L_c \in L(X)$. For every $x \in X$, consider $(L_a \odot L_0)(x) = L_a(x) * L_0(x) = (a * x) * (0 * x) = (a * x) = L_a(x)$. So, $L_a \odot L_0 = L_a$.

Also consider

$$\begin{aligned}(L_a \odot (L_b \odot L_c) \odot L_a)(x) &= (a*x)*((b*x)*((c*x)*(a*x))) \\ &= (L_a \odot (L_b \odot (L_c \odot L_a)))(x).\end{aligned}$$

So,

$$(L_a \odot (L_b \odot L_c) \odot L_a) = (L_a \odot (L_b \odot (L_c \odot L_a))).$$

And clearly,

$$(L_a \odot L_a)(x) = L_a(x) * L_a(x) = (a*x)*(a*x) = 0 = L_0(x). \qquad \square$$

Corollary 3.1 *Let X be a prime obic algebra. Then $(L(X); \odot, L_0)$ is an obic algebra.*

Corollary 3.2 *$(L(X); \odot, L_0)$ is prime if and only if X is prime.*

We therefore know that

Proposition 3.1 *Let X be an obic algebra. Then the translation $L_0 : X \to X$ commutes with any endomorphism on X.*

Definition 3.3 *An obic algebra X is said to have the distributive property if $0*(x*y) = (0*x)*(0*y)$ for all $x, y \in X$.*

Proposition 3.2 *Let X be an obic algebra with distributive property. Then the translation $L_0 : X \to X$ is the only homomorphism in the collection $L(X)$.*

Proof Clearly, L_0 is a homomorphism. Let $x \in X$ such that $x \neq 0$. Let Suppose L_x is a homomorphism on X. Consider $x = (x*0) = L_x(0) = L_x(0*0) = L_x(0)*L_x(0) = 0$; which is a contradiction. $\square$

§4. Krib Maps in Obic Algebras

Definition 4.1 *Let X be an obic algebra. A self map $\alpha : X \to X$ is called a right krib map if $\alpha(x*y) = x*\alpha(y)$ for all $x, y \in X$.*

If $\alpha(x*y) = \alpha(x)*y$ for all $x, y \in X$, then α is called a left krib map. α is called a krib map if it is both a right and a left krib map.

Example 4.1 Consider the obic algebra X in Example 1.3. Define $\alpha : X \to X$ by $\alpha(1) = 0, \alpha(0) = 1$. Then α is a right krib map of X.

Denote by $D(X)$ the collection of right krib maps on an obic algebra X. We know the next result by definition.

Proposition 4.1 *Let X be an obic algebra. Then $D(X)$ is a monoid.*

Definition 4.2 *Let X be an obic algebra. A map $\alpha : X \to X$ is called regular if $\alpha(0) = 0$. If $\alpha(0) \neq 0$, then α is called irregular.*

The following results can be verified immediately.

Lemma 4.1 *Let α be an irregular right krib map of an obic algebra X. Then the following hold for all $x \in X$:*

(1) $x * \alpha(x) \neq 0$;
(2) $\alpha(x) = x * \alpha(0)$.

Proposition 4.2 *Every right krib map of a prime obic algebra is regular.*

Corollary 4.1 *Let α be a krib map on a prime obic algebra X. Then α is regular.*

Proposition 4.3 *A right krib map α of an obic algebra X is regular if and only if $x * \alpha(x) = 0$ for all $x \in X$.*

Proof Suppose α is regular. Then $0 = \alpha(0) = x\alpha(x)$. Conversely, suppose $x * \alpha(x) = 0$. Then $\alpha(0) = x * \alpha(x) = 0$. □

Proposition 4.4 *A left krib map α of an obic algebra X is regular if and only if $\alpha(x) * x = 0$ for all $x \in X$.*

Theorem 4.1 *Let α be a krib map of an obic algebra X. Then the following are equivalent:*

(1) $x * \alpha(x) = 0$;
(2) α *is regular;*
(3) $\alpha(x) * x = 0$.

Proof The proof is straightforward by definition. □

§5. Monics of Obic Algebras

Let X be an obic algebra. Define '$\wedge$' by $x \wedge y = y * (y * x)$ for all $x, y \in X$.

Definition 5.1 *Let X be an obic algebra. A function $\theta : X \to X$ is called a left (resp. right)monic if $\theta(x * y) = (\theta(x) * y) \wedge (x * \theta(y))$ (resp. $\theta(x * y) = (x * \theta(y)) \wedge (\theta(x) * y)$) for all $x, y \in X$.*

If $\theta : X \to X$ is both a left and a right monic, then θ is called a monic.

Example 5.1 Let X be the obic algebra given by Table 2.

$*$	0	1
0	0	1
1	1	0

Table 2

The map $\theta : X \to X$ such that $\theta(1) = 0, \theta(0) = 1$ is a left monic.

Definition 5.2 *Let X be an obic algebra. A map $\alpha : X \to X$ is called regular if $\alpha(0) = 0$.*

Definition 5.3 *Let X be an obic algebra. A self map θ on X is called self preserving if $\theta(x) * x = x$ for all $x \in X$.*

Definition 5.4 *Let X be an obic algebra. A self map θ on X is called anti-self preserving if $x * \theta(x) = x$ for all $x \in X$.*

Definition 5.5 *Let X be an obic algebra. A self map θ on X is called preserving if it is both self-preserving and ant-self-preserving.*

Proposition 5.1 *Let θ be a regular left monic on an obic algebra X. Then,*

$$(x * \theta(x)) * [(x * \theta(x)) * (\theta(x) * x)] = (y * \theta(y)) * [(y * \theta(y)) * (\theta(y) * y)]$$

for all $x, y \in X$.

Proof Now, $0 = \theta(0) = \theta(x * x) = (x * \theta(x)) * [(x * \theta(x)) * (\theta(x) * x)]$. Similar argument gives also $(y * \theta(y)) * [(y * \theta(y)) * (\theta(y) * y)] = 0$. Hence, the conclusion follows. □

Proposition 5.2 *Let X be a regular left monic on an associative obic algebra X. Then $0 * [\theta(x) * x] = 0 * [\theta(y) * y]$ for all $x, y \in X$.*

Proof By proposition 5.1,

$$\begin{aligned} 0 &= [x * \theta(x)] * [(x * \theta(x)) * (\theta(x) * x)] \\ &= [(x * \theta(x)) * (x * \theta(x))] * [\theta(x) * x] \\ &= 0 * [\theta(x) * x]. \end{aligned}$$

Similarly, we have $0 * [\theta(y) * y] = 0$. The conclusion follows. □

Proposition 5.3 *Let θ be a self preserving left monic on an obic algebra X. Then $[x * \theta(x)] * [(x * \theta(x)) * x)] = \theta(0)$ for all $x \in X$.*

Proof Now,

$$\begin{aligned} \theta(0) &= \theta(x * x) \\ &= [\theta(x) * x] \wedge [x * \theta(x)] \\ &= [x * \theta(x))] * [(x * \theta(x)) * x]. \end{aligned}$$ □

Corollary 5.1 *Let θ be a regular self preserving left monic on an obic algebra X. Then $[x * \theta(x))] * [(x * \theta(x)) * x] = 0$ for all $x \in X$.*

The following propositions can be immediately verified.

Proposition 5.4 *Let θ be a regular self preserving left monic on an obic algebra X. Then* $[x * \theta(x))] * [(x * \theta(x)) * x] = [y * \theta(y))] * [(y * \theta(y)) * y]$ *for all $x, y \in X$.*

Proposition 5.5 *Let θ be a self preserving left monic on an associative obic algebra X. Then* $0 * x = \theta(0)$ *for all $x \in X$.*

Proposition 5.6 *Let θ be a regular self preserving left monic on an associative obic algebra X. Then* $0 * [\theta(x) * x] = 0 * [\theta(y) * y]$ *for all $x, y \in X$.*

Proposition 5.7 *Let θ be a regular self preserving left monic on an associative obic algebra X. Then* $0 * x = 0$ *for all $x \in X$.*

Theorem 5.1 *Let X be an associative obic algebra with a self preserving left monic θ. Then X is prime if and only if θ is regular.*

Proof Suppose X is prime. Then,

$$\begin{aligned} 0 = 0 * x &= [(x * \theta(x)) * (x * \theta(x))] * x \\ &= (x * \theta(x)) * [(x * \theta(x)) * x] \\ &= x \wedge [x * \theta(x)] \\ &= \theta(x * x) = \theta(0). \end{aligned}$$

Conversely, suppose θ is regular. Then,

$$\begin{aligned} 0 = \theta(0) &= \theta(x * x) = [(\theta(x) * x)] \wedge [x * \theta(x)] \\ &= x \wedge [x * \theta(x)] = 0 * x. \qquad \square \end{aligned}$$

The two conclusions following can be easily verified by definition.

Proposition 5.8 *Let θ be an anti-self preserving left monic on an obic algebra X. Then* $x * [x * \theta(x)] = \theta(0)$ *for all $x \in X$.*

Proposition 5.9 *Let θ be a regular anti-self preserving left monic on an obic algebra X. Then* $y * [y * \theta(y)] = x * [x * \theta(x)]$ *for all $x, y \in X$.*

Theorem 5.2 *Let θ be an anti-self preserving left monic on an associative obic algebra X. Then $0 * [\theta(x) * x] = \theta(0)$. Moreover, if θ is regular, then $0 * [\theta(x) * x] = 0$ for all $x \in X$.*

Proof Notice that

$$\begin{aligned} \theta(0) = \theta(x * x) &= [\theta(x) * x] \wedge [x * \theta(x)] \\ &= [\theta(x) * x] \wedge x \\ &= 0 * [\theta(x) * x]. \end{aligned}$$

The second part of the theorem is obvious. $\square$

Proposition 5.10 *Let X be an obic algebra with a left monic θ. Then $\theta(x) = x * [x * \theta(x)]$ for all $x \in X$.*

Proof Notice that

$$\begin{aligned} \theta(x) &= \theta(x * 0) = \theta(x) \\ &= [\theta(x) * 0] \wedge [x * \theta(0)] \\ &= x * [x * \theta(x)]. \end{aligned}$$

This completes the proof. □

Corollary 5.2 *Let θ be an anti-self preserving regular left monic on an obic algebra X. Then $\theta(x) = 0$ for all $x \in X$.*

Corollary 5.3 *Let θ be a regular left monic on an associative obic algebra X. Then $\theta(x) = 0 * \theta(x)$ for all $x \in X$.*

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Prime BI-Ideals of Po-Γ-Groupoids

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Abstract: In this paper, we have studied the notion of prime bi-ideals and semi prime bi-ideals of Po-Γ-groupoids and explored the various properties of prime bi-ideals in Po-Γ-groupoids. Also we obtained the condition for Po-Γ-groupoids to be regular.

Key Words: Partially ordered Γ-groupoid (po-Γ-groupoid), left (right, two-sided) ideal, quasi-ideal, bi-ideal, prime (semi prime) bi-ideal, regular partially ordered -Γ-groupoid.

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§1. Introduction

In 1983, A.P.J. van der Walt [5] introduced the interesting concepts of prime and semi prime bi-ideals for an associative ring with unity. In 1995, using the concepts defined by A. P. J. van der Walt, the structure of a ring containing prime and semi prime bi-ideals were studied by H. J. le Roux [2]. In 2001, Kehayopulu and Tsingelis [1] studied prime ideals of groupoids. Following [1], in 2005, S.K. Lee developed prime left (right) ideals of groupoids [3] and obtained some results on prime bi-ideals of groupoids [4]. In this paper we have studied the notion of prime bi-ideals and semi prime bi-ideals of Po-Γ-groupoids. Let M be a non empty set. M is called Γ-groupoid if for all $a, b \in M$ and $\gamma \in \Gamma$, $a\gamma b \in M$.

A set $(G, \Gamma, \leq)$ is called a partial order-Γ-groupoid(or simply Po-Γ-groupoid) if

(i) $(G, \leq)$ is a partial ordered set;

(ii) (G, Γ) is a Γ-groupoid such that $a \leq b \Rightarrow a\gamma x \leq b\gamma x$ and $x\gamma_1 a \leq x\gamma_1 b$ for all $a, b, x \in G; \gamma, \gamma_1 \in \Gamma$.

Throughout this paper G denotes a Po-Γ-groupiod.

A non empty subset A of G is called right(resp.left) ideal of G if

(i) $A\Gamma G \subseteq A$ (resp.$G\Gamma A \subseteq A$);

(ii) $a \in A$, $b \leq a$ for $b \in G$ implies $b \in A$.

A non empty subset A is called an ideal of G if it is a right and left ideal of G.

For non-empty subsets A and B of a po-Γ-groupoid G, the product $A\Gamma B$ of A and B and the subset $(A]$ of G are defined by $A\Gamma B = \{a\gamma b \in S : a \in A, b \in B, \gamma \in \Gamma\}$; $(A]=\{x \in G : \exists a \in A(x \leq a)\}$.

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A non empty subset Q of G is called a quasi ideal if

(i) $(Q\Gamma G] \cap (G\Gamma Q] \subseteq Q$;

(ii) $a \leq q; q \in Q$ implies $a \in Q$.

A non empty subset B of G is called a bi-ideal if

(i) $(B\Gamma G\Gamma B] \subseteq B$;

(ii) $a \leq b; b \in B$ implies $a \in B$.

Every quasi ideal is a bi-ideal. But the converse need not be true. A bi-ideal B of G is prime, for $x, y \in G, (x\Gamma G\Gamma y] \subseteq B$ implies $x \in B$ or $y \in B$. A bi-ideal B of G is semi-prime, for $x \in G, (x\Gamma G\Gamma x] \subseteq B$ implies $x \in B$. A non-empty subset I of G is prime if I is an ideal of G such that for any ideals A, B of G , $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$. It is clear that $(x)_l = (x \cup G\Gamma x]$(resp.$(x)_r = (x \cup x\Gamma G]$) is the principle left(resp.right) ideal generated by x.

§2. Main Results

Theorem 2.1 *A bi-ideal B of G is prime if and only if for a right ideal R and a left ideal L of G $(R\Gamma L] \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$.*

Proof Suppose that $(R\Gamma L] \subseteq B$ for a right ideal R and a left ideal L of G and $R \nsubseteq B$. Then there exists $x \in R \backslash B$ such that $(x\Gamma G\Gamma y] \subseteq (R\Gamma G\Gamma L] \subseteq (R\Gamma L] \subseteq B$ for any $y \in L$ which implies $y \in B$. So $L \subseteq B$.

Conversely, let $(x\Gamma G\Gamma y] \subseteq B$ for $x, y \in G$. Then $(x\Gamma G]\Gamma(G\Gamma y] \subseteq (x\Gamma G\Gamma G\Gamma y] \subseteq (x\Gamma G\Gamma y] \subseteq B$. By hypothesis, we have $(x\Gamma G] \subseteq B$ or $G\Gamma y] \subseteq B$. If $(x\Gamma G] \subseteq B$, then $x\Gamma x \in (x\Gamma G]\Gamma G \subseteq B$. Now, $(x)_r(x)_l = (x \cup x\Gamma G]\Gamma(x \cup G\Gamma x] = (x\Gamma x \cup x\Gamma G\Gamma x \cup x\Gamma G\Gamma x \cup x\Gamma G\Gamma G\Gamma x] \subseteq (x\Gamma x \cup x\Gamma G] \subseteq B$ which implies $(x)_r \subseteq B$ or $(x)_l \subseteq B$. Therefore $x \in B$. If $(G\Gamma y] \subseteq B$, then by the similar method $y \in B$. □

Theorem 2.2 *If a bi-ideal B of G is prime, then B is a left or right ideal of G.*

Proof Let B be a prime bi-ideal of G.Then $(B\Gamma G] \subseteq B$ or $(G\Gamma B] \subseteq B$ as $(B\Gamma G]\Gamma(G\Gamma B] \subseteq (B\Gamma G\Gamma B] \subseteq B$ and by Theorem 2.1. So B is a a left ideal or right ideal of G. □

Theorem 2.3 *Let G be a po-Γ-groupoid. Then the following statements are hold:*

(i) *Any left/right/both sided ideal of G is a bi-ideal of G;*

(ii) *Intersection of right and left ideals of G is a bi-ideal of G;*

(iii) *Arbitrary intersection of bi-ideals of G is also a bi-ideal of G;*

(iv) *If B is a bi-ideal of G, then $B\Gamma r$ and $r\Gamma B$ are bi-ideals of G, for any $r \in G$.*

Proof This result can be immediately verified by definition. □

Notation 1 For a bi-ideal of B of G, we define $L_B = \{x \in B : G\Gamma x \subseteq B\}$, $R_B = \{x \in B : x\Gamma G \subseteq B\}$, $I_L = \{y \in L_B : y\Gamma G \subseteq L_B\}$ and $I_R = \{y \in R_B : G\Gamma y \subseteq R_B\}$.

Theorem 2.4 *Let B be bi-ideal of G . Then L_B is a left ideal of G contained in B if L_B is non empty.*

Proof Let $x \in L_B$. Let $g \in G$ and $\gamma \in \Gamma$. Then $g\gamma x \in G\Gamma x \subseteq B$. Now $G\Gamma g\gamma x \subseteq G\Gamma G\Gamma x \subseteq G\Gamma x \subseteq B$ which implies $g\gamma x \in L_B$. $G\Gamma L_B \subseteq L_B$. hence L_B is a left ideal. □

Theorem 2.5 *Let B be bi-ideal of G. Then I_L is the largest ideal of G contained in B if I_L is non empty. Furthermore, I_L coincides with I_R.*

Proof Let $x \in I_L$.Then $x\Gamma G \subseteq L_B$. For any $g \in G$ and $\gamma \in \Gamma$, we have $x\gamma g \in x\Gamma G \subseteq L_B$ and $x\gamma g\Gamma G \subseteq x\Gamma G\Gamma G \subseteq x\Gamma G \subseteq L_B$, So I_L is a right ideal of G.

Since $I_L \subseteq L_B \subseteq B$, we have $x \in L_B$ which implies $x\gamma g \in I_L$ and $G\Gamma x \subseteq B$.

Now, $G\Gamma g\gamma x \subseteq G\Gamma G\Gamma x \subseteq G\Gamma x \subseteq B$. So $g\gamma x \in L_B$. By Theorem 2.4 and $x \in I_L$, we have $x\Gamma G \subseteq L_B$. Then $g\gamma x\Gamma G \subseteq G\Gamma L_B \subseteq L_B$, and we have $g\gamma x \in I_L$. Therefore I_L is a left ideal.

Let A be an ideal of G such that $A \subseteq B$. If $x \in A$, then $x \in B$ and $G\Gamma \subseteq x \subseteq A \subseteq B$ which implies $x \in L_B$ and $A \subseteq L_B$.

Let $x \in A$. Then $x\Gamma G \subseteq A \subseteq L_B$. Hence $x \in I_L$ and $A \subseteq I_L$ which implies I_L is the largest ideal of G contained in B. Similarly I_R is the largest ideal of G contained in B. □

Notation 2 We denote I_B as $I_B :\equiv I_R = I_L$ by Theorem 2.5.

Theorem 2.6 *If B is a prime bi-ideal of G, then I_B is a prime ideal of G contained in B.*

Proof Let B be a prime bi-ideal of G. Then by Theorem 2.5, I_B is an ideal of G.

Suppose $X\Gamma Y \subseteq I_B$ for any ideals X, Y of G. Since $I_B \subseteq L_B \subseteq B$, we have $X\Gamma Y \subseteq B$. By Theorem 2.1, $X \subseteq B$ or $Y \subseteq B$. But I_B is the largest ideal contained in B, so $X \subseteq I_B$ or $Y \subseteq I_B$ which implies I_B is a prime ideal of G. □

Corollary 2.7 *If B be a semi-prime bi-ideal of G, then I_B is a semi-prime ideal of G if I_B is non empty.*

Theorem 2.8 *If a bi-ideals B of G is semi-prime, then*

(i) *for any left ideal L of G, $L\Gamma L \subseteq B$ implies $L \subseteq B$;*

(ii) *for any right ideal R of G, $R\Gamma R \subseteq B$ implies $R \subseteq B$.*

Proof Suppose $L\Gamma L \subseteq B$ for a left ideal L of G and $L \nsubseteq B$. Then there exists $x \in L\backslash B$, $x\Gamma G\Gamma x \subseteq L\Gamma G\Gamma L \subseteq L\Gamma L \subseteq B$. Since B is a semi-prime we have $x \in B$, a contradiction.

The second assertion can be proved similarly. □

Theorem 2.9 *If a bi-ideal B of G is semi-prime, then B is a quasi-ideal of G.*

Proof Let $y \in (B\Gamma G] \cap (G\Gamma B]$. Then $(y\Gamma G\Gamma y] \subseteq ((B\Gamma G]\Gamma G\Gamma(G\Gamma B]] \subseteq (B\Gamma G\Gamma B] \subseteq B$. Since B is a semi prime, we have $y \in B$. Hence B is a quasi-ideal of G. □

Remark 2.10 For a Po-Γ-groupoid G,

(a) The set of all prime ideal of G is denoted by $spec(G)$;

(*b*) $Bspec(G)$ denotes the set of prime bi-ideals of G;

(*c*) $Sspec(G)$ denotes the set of all semi prime bi-ideals of G.

Then we know conclusions following easily by definition.

Theorem 2.11 *If G is finite, then $\bigcap spec(G) = \bigcap Bspec(G)$.*

Theorem 2.12 *A bi-ideal B of G is semi prime if and only if for a right ideal (left ideal) A of G $(A\Gamma A] \subseteq B$ implies $A \subseteq B$.*

Theorem 2.13 *The intersection of any family of prime bi-ideals of G is a semi prime bi-ideal of G.*

Theorem 2.14 *If G is finite, then $\bigcap spec(G) = \bigcap Sspec(G)$.*

We note that G is regular if for any $x \in G$, there exist $a \in G$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $x \leq x\gamma_1 a\gamma_2 x$.

The following results shows the necessary and sufficient condition for a Po-Γ-groupiod to be regular.

Theorem 2.15 *Let G be Po-gamma-groupiod. Then G is regular if and only if every bi-ideal of G is semi-prime.*

Proof Let G be regular and B a bi-ideal of G. Suppose that $x\Gamma G\Gamma x \subseteq B$ for $x \in G$. Then there exist $a \in G$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $x \leq x\gamma_1 a\gamma_2 x \in x\Gamma a\Gamma x \in x\Gamma G\Gamma x \subseteq B$ which implies $x \in B$. Hence B is semi prime.

Conversely , assume that every bi-ideal of G is semi-prime. Let $B = (a\Gamma G\Gamma a]$ for $a \in G$. Then $B\Gamma G\Gamma B = (a\Gamma G\Gamma a]\Gamma G\Gamma(a\Gamma G\Gamma a] \subseteq (a\Gamma G\Gamma a] = B$, which implies B is a bi-ideal of G and by assumption $(a\Gamma G\Gamma a]$ is semi-prime. Since $a\Gamma G\Gamma a \subseteq (a\Gamma G\Gamma a] = B$, we get $a \in (a\Gamma G\Gamma a] = B$. Then there exist $x \in G$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $a \leq a\gamma_1 x\gamma_2 a$. □

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4-Total Prime Cordiality of Certain Subdivided Graphs

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Abstract: Let G be a (p, q) graph. Let $f : V(G) \to \{1, 2, \cdots, k\}$ be a map where $k \in \mathbb{N}$ and $k > 1$. For each edge uv, assign the label $\gcd(f(u), f(v))$. f is called k-total prime cordial labeling of G if $|t_f(i) - t_f(j)| \leq 1$, $i, j \in \{1, 2, \cdots, k\}$ where $t_f(x)$ denotes the total number of vertices and the edges labelled with x. A graph with a k-total prime cordial labeling is called k-total prime cordial graph. In this paper we investigate the 4-total prime cordial labeling for some subdivided graphs.

Key Words: Corona, ladder, triangular snake, k-total prime cordial labeling, Smarandache k-total prime cordial labeling.

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§1. Introduction

In this paper we consider simple, finite and undirected graphs only. The notion of k- total prime cordial labelling has been introduced in [4]. In [4–9], they investigate the k-total prime cordial labeling of some graphs and investigate the 4-total prime cordial labeling behaviour of path, cycle, star, bistar, ladder, triangular snake, friendship graph, comb, double comb, double triangular snake, flower graph, gear graph, Jelly fish, book, irregular triangular snake, prism, helm, dumbbell graph, sunflower graph, dragon, mobius ladder and subdivision of some graphs. In this paper we examine the 4-total prime cordial labeling of subdivision of some graphs like star, bistar, comb, double comb, ladder, triangular snake and double triangular snake. Terms are not defined here follows from [1], [3].

§2. Preliminary Results

Definition 2.1 *Let G_1, G_2 respectively be (p_1, q_1), (p_2, q_2) graphs. The corona of G_1 with G_2, $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2.*

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Definition 2.2 *If $e = uv$ is an edge of G then e is said to be subdivided when it is replaced by the edges uw and wv. The graph obtained by subdividing each edge of a graph G is called the subdivision graph of G and is denoted by $S(G)$.*

§3. k-Total Prime Cordial Labeling

Definition 3.1 *Let G be a (p, q) graph. Let $f : V(G) \to \{1, 2, \cdots, k\}$ be a function where $k \in \mathbb{N}$ and $k > 1$. For each edge uv, assign the label* $\gcd(f(u), f(v))$. *f is called k-total prime cordial labeling of G if $|t_f(i) - t_f(j)| \leq 1$, $i, j \in \{1, 2, \cdots, k\}$ where $t_f(x)$ denotes the total number of vertices and the edges labelled with x. A graph with a k-total prime cordial labeling is called k-total prime cordial graph. Generally, if there are integers $i, j \in \{1, 2, \cdots, k\}$ such that $|t_f(i) - t_f(j)| > 1$, f is called a Smarandache k-total prime cordial labeling and G a Smarandache k-total prime cordial labeling graph.*

Theorem 3.2 *The subdivision of comb $S(P_n \odot K_1)$ is 4-total prime cordial.*

Proof Let P_n be the path $u_1u_2\cdots u_n$. Let x_i be the vertex which subdivide the edge u_iu_{i+1}. Let v_i be the vertex adjacent to u_i. Let w_i be the pendent vertices v_i. Clearly $|V(S(P_n \odot K_1))| + |E(S(P_n \odot K_1))| = 8n - 3$.

Case 1. $n \equiv 0 \pmod 4$.

Let $n = 4t$, $t \in \mathbb{N}$. Assign the label 4 to the vertices $u_1, u_2, \cdots, u_t$ and assign the label 3 to the vertices $u_{t+1}, u_{t+2}, \cdots, u_{2t}$. Next we assign the label 2 to the vertices $u_{2t+1}, u_{2t+2}, \cdots, u_{3t}$ and assign 1 to the vertices $u_{3t+1}, u_{3t+2}, \cdots, u_n$. Now we consider the vertices x_i $(1 \leq i \leq n-1)$. Assign the label 4 to the vertices $x_1, x_2, \cdots, x_t$ and assign the label 3 to the vertices $x_{t+1}, x_{t+2}, \ldots, x_{2t}$. Next we assign the label 2 to the vertices $x_{2t+1}, x_{2t+2}, \cdots, x_{3t}$. Finally we assign 1 to the vertices $x_{3t+1}, x_{3t+2}, \cdots, x_{n-1}$. Now we move to the vertices v_i, w_i $(1 \leq i \leq n)$. Assign the label 4 to the vertices $v_1, v_2, \ldots, v_t$ and $w_1, w_2, \cdots, w_t$. Then assign the label 3 to the vertices $v_{t+1}, v_{t+2}, \cdots, v_{2t}$ and $w_{t+1}, w_{t+2}, \cdots, w_{2t}$. Now we assign the label 2 to the vertices $v_{2t+1}, v_{2t+2}, \cdots, v_{3t}$ and $w_{2t+1}, w_{2t+1}, \cdots, w_{3t}$. Finally we assign the label 1 to the vertices $v_{3t+1}, v_{3t+2}, \cdots, v_n$ and $w_{3t+1}, w_{3t+2}, \cdots, w_n$.

Case 2. $n \equiv 1 \pmod 4$.

Let $n = 4t + 1$, $t \in \mathbb{N}$. As in case 1, assign the label to the vertices u_i $(1 \leq i \leq n-2)$, x_i $(1 \leq i \leq n-2)$, v_i, w_i $(1 \leq i \leq n-1)$. Now we assign the labels 3, 4, 2, 3, 4 respectively to the vertices u_{n-1}, u_n, x_{n-1}, v_n and w_n.

Case 3. $n \equiv 2 \pmod 4$.

Let $n = 4t + 2$, $t \in \mathbb{N}$. Assign the label to the vertices u_i, v_i, w_i $(1 \leq i \leq n-2)$ and x_i $(1 \leq i \leq n-3)$ by case 1. Next we assign the labels 4, 3, 4, 3, 2, 2, 4, 3 to the vertices u_{n-1}, u_n, x_{n-2}, x_{n-1}, v_{n-1}, v_n, w_{n-1} and w_n respectively.

Case 4. $n \equiv 3 \pmod 4$.

Let $n = 4t+3$, $t \in \mathbb{N}$. As in Case 3, assign the label to the vertices u_i, v_i, w_i $(1 \le i \le n-2)$ and x_i $(1 \le i \le n-3)$. Next we assign the labels 2, 3, 4, 3, 3, 3, 2,1 respectively to the vertices u_{n-1}, u_n, x_{n-2}, x_{n-1}, v_{n-1}, v_n, w_{n-1} and w_n. □

Theorem 3.3 *The subdivision of double comb $S(P_n \odot 2K_1)$ is 4-total prime cordial.*

Proof Let P_n be the path $u_1u_2\cdots u_n$. Let z_i be the vertex which subdivide the edge u_iu_{i+1}. Let v_i, w_i be the vertices adjacent to u_i. Let x_i, y_i be the pendent vertices adjacent to u_i,v_i respectively. Obviously $|V(S(P_n \odot 2K_1))| + |E(S(P_n \odot 2K_1))| = 12n-3$.

Case 1. $n \equiv 0 \pmod 4$.

Let $n = 4t$, $t \in \mathbb{N}$. Assign the label 4 to the vertices $u_1, u_2, \cdots, u_t$ and assign the label 3 to the vertices $u_{t+1}, u_{t+2}, \cdots, u_{2t}$. Now we assign the label 2 to the vertices $u_{2t+1}, u_{2t+2}, \cdots, u_{3t}$. Next we assign 1 to the vertices $u_{3t+1}, u_{3t+2}, \cdots, u_{n-1}$. Finally we assign the label 3 to the vertex u_n. Now we consider the vertices z_i $(1 \le i \le n-1)$. Assign the label 4 to the vertices $z_1, z_2, \cdots, z_{t-1}$ and assign the label 3 to the vertices $z_{t+1}, z_{t+2}, \cdots, z_{2t-1}$. Next we assign the label 2 to the vertices $z_{2t+1}, z_{2t+2}, \cdots, z_{3t-1}$. Now we assign the label 1 to the vertices $z_t, z_{2t}, z_{3t}, z_{3t+1}, \cdots, z_{n-1}$. Assign the label to the vertices v_i, w_i, x_i, y_i $(1 \le i \le n-1)$. Finally we assign 2, 4, 4, 3 respectively to the vertices x_n, v_n, w_n and y_n.

Case 2. $n \equiv 1 \pmod 4$.

Let $n = 4t+1$, $t \in \mathbb{N}$. As in case 1, assign the label to the vertices u_i, v_i, x_i, w_i, y_i $(1 \le i \le n-1)$ and z_i $(1 \le i \le n-2)$. Now we assign the labels 2, 4, 4, 2, 3, 3 respectively to the vertices z_{n-1}, x_n, v_n, u_n, w_n and y_n.

Case 3. $n \equiv 2 \pmod 4$.

Let $n = 4t+2$, $t \in \mathbb{N}$. Assign the label to the vertices u_i, v_i, x_i, w_i, y_i $(1 \le i \le n-1)$ and z_i $(1 \le i \le n-2)$ by case 2. Next we assign the label 1 to z_{n-1} and assign the labels 4, 4, 2, 3, 3 to the vertices x_n, v_n, u_n, w_n and y_n respectively.

Case 4. $n \equiv 3 \pmod 4$.

Let $n = 4t+3$, $t \in \mathbb{N}$. As in Case 3, assign the label to the vertices u_i, v_i, x_i, w_i, y_i $(1 \le i \le n-1)$ and z_i $(1 \le i \le n-2)$. Next we assign the labels 2, 4, 4, 2, 3, 3 respectively to the vertices z_{n-1}, x_n, v_n, u_n, w_n and y_n. □

Theorem 3.4 *The subdivision of star $S(K_{1,n})$ is 4-total prime cordial.*

Proof Let u be the vertex of degree n and $u_1, u_2, \cdots, u_n$ be the vertices of degree 2. Let $v_1, v_2, \cdots, v_n$ be the pendent vertices.

Case 1. $n \equiv 0 \pmod 4$.

Let $n = 4t$, $t \in \mathbb{N}$. Assign the label 4 to the vertex u. Next we now move to the vertices $u_1, u_2, \cdots, u_n$. Assign the label 4 to the vertices $u_1, u_2, \cdots, u_t$ and assign the label 2 to the vertices $u_{t+1}, u_{t+2}, \cdots, u_{2t}$. Next we assign the label 3 to the vertices $u_{2t+1}, u_{2t+2}, \cdots, u_{3t}$. Finally assign the label 1 to the non-labelled vertices of u_n. Now we consider the pendent vertices $v_1, v_2, \cdots, v_n$. Assign the label 4 to the vertices $v_1, v_2, \cdots, v_t$ and assign the label 2 to the vertices $v_{t+1}, v_{t+2}, \ldots, v_{2t}$. Finally assign 3 to the non-labelled vertices of v_n.

Case 2. $n \equiv 1 \pmod 4$.

Let $n = 4t + 1$, $t \in \mathbb{N}$. In this case, assign the label to the vertices u, u_i $(1 \leq i \leq n)$ and v_i $(1 \leq i \leq n-2)$ by in case 1. Next assign the labels 2 and 4 to the vertices v_{n-1} and v_n respectively.

Case 3. $n \equiv 2 \pmod 4$.

Let $n = 4t + 2$, $t \in \mathbb{N}$. As in Case 1, assign the label to the vertices u, u_i $(1 \leq i \leq n-1)$ and v_i $(1 \leq i \leq n-2)$. Next assign the labels 2, 1 and 4 to the vertices respectively u_n, v_{n-1} and v_n.

Case 4. $n \equiv 3 \pmod 4$.

Let $n = 4t + 3$, $t \in \mathbb{N}$. Assign the label to the vertices u, u_i $(1 \leq i \leq n-2)$ and v_i $(1 \leq i \leq n-2)$ as in case 1. Finally assign the labels 3, 2, 4 and 4 to the vertices u_{n-1}, u_n, v_{n-1} and v_n respectively. □

Theorem 3.5 *The subdivision of bistar $S(B_{n,n})$ is 4-total prime cordial.*

Proof Let u, v be the vertices of degree n and w be the vertex of degree 2 adjacent to both u and v. Let u_i be the vertex of degree 2 adjacent to u and v_i be the vertex of degree 2 adjacent to v. Let x_i and y_i $(1 \leq i \leq n)$ be the pendent vertex adjacent to u_i and v_i respectively.

Case 1. $n \equiv 0 \pmod 4$.

Let $n = 4t$, $t \in \mathbb{N}$. Assign the labels 4, 2 and 3 to the vertex u, w and v respectively. Next we move to the vertices $u_1, u_2, \cdots, u_n$. Assign the label 4 to the vertices $u_1, u_2, \cdots, u_{2t}$ and assign the label 2 to the vertices $u_{2t+1}, u_{2t+2}, \cdots, u_{4t}$. Now we consider the pendant vertices of u_n. Assign the label 4 to the vertices $x_1, x_2, \cdots, x_{2t}$ and assign the label 2 to the vertices $x_{2t+1}, x_{2t+2}, \cdots, x_{4t}$. Now we consider the vertices $v_1, v_2, \cdots, v_n$. Assign the label 3 to the vertices $v_1, v_2, \cdots, v_{2t}$ and assign the label 1 to the vertices $v_{2t+1}, v_{2t+2}, \cdots, v_{4t}$. Finally we move to the pendant vertices of v_n. Assign the label 3 to the vertices $y_1, y_2, \cdots, y_{2t}$ and assign the label 1 to the vertices $y_{2t+1}, y_{2t+2}, \cdots, y_{4t}$.

Case 2. $n \equiv 1 \pmod 4$.

Let $n = 4t+1$, $t \in \mathbb{N}$. In this case assign the label to the vertices u, v, w, u_i $(1 \leq i \leq n-1)$, v_i $(1 \leq i \leq n-1)$, x_i $(1 \leq i \leq n-1)$ and y_i $(1 \leq i \leq n-1)$ as in case 1. Next assign the labels 4, 2, 3 and 1 respectively to the vertices u_n, x_n, v_n and y_n.

Case 3. $n \equiv 2, 3 \pmod 4$.

Let $n = 4t + 1$ and $n = 4t + 2$ $t \in \mathbb{N}$. The proof is similar to that of Case 2. □

Theorem 3.6 *The subdivision of triangular snake $S(T_n)$ is 4-total prime cordial.*

Proof Let P_n be the path $u_1 u_2 \cdots u_n$. Let w_i be the vertex adjacent to u_i and u_{i+1}. Let

v_i be the vertices which subdivide the edge u_iu_{i+1} and x_i,y_i be the vertex which subdivided u_iw_i and $u_{i+1}w_i$ respectively. It is easy to verify that $|V(S(T_n))| + |E(S(T_n))| = 11n - 10$.

Case 1. $n \equiv 0 \pmod 4$.

Let $n = 4t$, $t \in \mathbb{N}$. Assign the label 4 to the vertices $u_1, u_2, \cdots, u_t$ and assign the label 2 to the vertices $u_{t+1}, u_{t+2}, \cdots, u_{2t}$. Next we assign the label 3 to the vertices $u_{2t+1}, u_{2t+2}, \cdots, u_{3t}$ then we assign the label 1 to the vertices $u_{3t+1}, u_{3t+2}, \cdots, u_{n-1}$. Finally, we assign 3 to the vertex u_n. Assign the label 4 to the vertices $v_1, v_2, \cdots, v_t$ and assign the label 2 to the vertices $v_{t+1}, v_{t+2}, \cdots, v_{2t-1}$ and assign the label 3 to the vertices $v_{2t}, v_{2t+1}, \cdots, v_{3t-1}$ then we assign the label 1 to the vertices $v_{3t}, v_{3t+1}, \cdots, v_{n-1}$. Assign the label to the vertices x_i $(1 \le i \le n-1)$ as in v_i $(1 \le i \le n-1)$. Now relabel the vertex x_{2t} by 2. Assign the label 4 to the vertices $y_1, y_2, \cdots, y_{t-1}$ and assign the label 2 to the vertices $y_t, y_{t+1}, \cdots, y_{2t-1}$ and assign the label 3 to the vertices $y_{2t}, y_{2t+1}, \cdots, y_{3t-1}$ then we assign the label 1 to the vertices $y_{3t}, y_{3t+1}, \cdots, y_{n-2}$. Finally we assign the label 2 to the vertices y_{n-1}. Now we consider the vertices w_i $(1 \le i \le n-1)$. Assign the label 4 to the vertices $w_1, w_2, \cdots, w_t$ and assign the label 2 to the vertices $w_{t+1}, w_{t+2}, \cdots, w_{2t-1}$ and assign the label 3 to the vertices $w_{2t}, w_{2t+1}, \cdots, w_{3t-1}$ then we assign the label 1 to the vertices $w_{3t}, w_{3t+1}, \cdots, w_{n-2}$. Finally we assign 4 to the vertex w_{n-1}.

Case 2. $n \equiv 1 \pmod 4$.

Let $n = 4t + 1$, $t \in \mathbb{N}$. As in Case 1, assign the label to the vertices u_i $(1 \le i \le n-1)$, v_i, x_i, y_i, w_i $(1 \le i \le n-2)$. Next we assign the labels 3, 3, 2, 4, 4 to the vertices v_{n-1}, u_n, x_{n-1}, y_{n-1} and w_{n-1} respectively.

Case 3. $n \equiv 2 \pmod 4$.

Let $n = 4t + 2$, $t \in \mathbb{N}$. Assign the label to the vertices u_i $(1 \le i \le n-3)$, v_i, x_i, w_i $(1 \le i \le n-3)$ and y_i $(1 \le i \le n-4)$ as in Case 2. Now we assign the labels 4, 3, 3 respectively to the vertices u_{n-2}, u_{n-1} and u_n. Next we assign the labels to the vertices 4, 3, 2, 2, 2, 1 to the vertices v_{n-2}, v_{n-1}, x_{n-2}, x_{n-1}, w_{n-2}, and w_{n-1} respectively. Finally we assign the labels 4, 2, 3 respectively to the vertices y_{n-3}, y_{n-2} and y_{n-1}.

Case 4. $n \equiv 3 \pmod 4$.

Let $n = 4t + 3$, $t \in \mathbb{N}$. Assign the label to the vertices u_i, x_i, y_i, w_i $(1 \le i \le n-3)$ and v_i $(1 \le i \le n-4)$ as in Case 3. Now we assign the labels 4, 3, 3, 3, 1, 1, 3, 4, 3 respectively to the vertices u_{n-2}, u_{n-1}, u_n, x_{n-2}, x_{n-1}, y_{n-2}, y_{n-1}, w_{n-2} and w_{n-2}. Finally we assign the labels 2, 4, 3 to the vertices v_{n-3}, v_{n-2} and v_{n-1} respectively. □

Theorem 3.7 *The subdivision of ladder $S(L_n)$ is 4-total prime cordial.*

Proof Let $V(L_n) = \{u_i, v_i : 1 \le i \le n\}$ and $E(L_n) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \le i \le n-1\} \cup \{u_iv_i : 1 \le i \le n\}$. Let y_i, w_i and x_i be the vertices which subdivide the edges u_iu_{i+1}, u_iv_i and v_iv_{i+1} respectively. Clearly $|V(S(L_n))| + |E(S(L_n))| = 11n - 6$.

Case 1. $n \equiv 0 \pmod 4$.

Let $n = 4t$, $t \in \mathbb{N}$. Assign the label 4 to the vertices $u_1, u_2, \cdots, u_t$ and $v_1, v_2, \cdots, v_t$.

Assign the label 2 to the vertices $u_{t+1}, u_{t+2}, \cdots, u_{2t}$ and $v_{t+1}, v_{t+2}, \cdots, v_{2t}$. Next we assign the label 3 to the vertices u_{2t+1}, $u_{2t+2}, \ldots$, u_{3t} and $v_{2t+1}, v_{2t+2}, \cdots, v_{3t}$ then we assign the label 1 to the vertices $u_{3t+1}, u_{3t+2}, \cdots, u_{n-1}$ and $v_{3t+1}, v_{3t+2}, \cdots, v_{n-1}$. Finally, we assign the labels 4 and 3 to the vertices u_n and v_n respectively. Next we consider the vertices x_i $(1 \leq i \leq n)$. Assign the label 4 to the vertices $x_1, x_2, \cdots, x_t$ and assign the label 2 to the vertices $x_{t+1}, x_{t+2}, \cdots, x_{2t}$. Now we assign the label 3 to the vertices $x_{2t+1}, x_{2t+2}, \cdots, x_{3t}$. Finally we assign the label 1 to the vertices $x_{3t+1}, x_{3t+2}, \cdots, x_n$. Now we consider the vertices y_i, w_i $(1 \leq i \leq n-1)$. Assign the label 4 to the vertices $y_1, y_2, \cdots, y_t$ and $w_1, w_2, \cdots, w_t$. Assign the label 2 to the vertices $y_{t+1}, y_{t+2}, \cdots, y_{2t-1}$ and $w_{t+1}, w_{t+2}, \cdots, w_{2t-1}$. Next we assign the label 3 to the vertices $y_{2t}, y_{2t+1}, \cdots, y_{3t-1}$ and $w_{2t}, w_{2t+1}, \cdots, w_{3t-1}$ then we assign the label 1 to the vertices $y_{3t}, y_{3t+1}, \cdots, y_{n-2}$ and w_{3t}, $w_{3t+1}, \cdots$, w_{n-2}, w_{n-1}. Finally, we assign the labels 2 vertex y_{n-1}.

Case 2. $n \equiv 1 \pmod 4$.

Let $n = 4t+1$, $t \in \mathbb{N}$. As in Case 1, assign the label to the vertices u_i $(1 \leq i \leq n-1)$, v_i $(1 \leq i \leq n-1)$, x_i $(1 \leq i \leq n)$, y_i $(1 \leq i \leq n-2)$ and w_i $(1 \leq i \leq n-2)$. Finally we assign the labels 2, 2, 4, 3 respectively to the vertices u_n, v_n, y_{n-1} and w_{n-1}.

Case 3. $n \equiv 2 \pmod 4$.

Let $n = 4t+2$, $t \in \mathbb{N}$. As in Case 2, assign the label to the vertices u_i $(1 \leq i \leq n-1)$, v_i $(1 \leq i \leq n-1)$, x_i $(1 \leq i \leq n-1)$, y_i $(1 \leq i \leq n-2)$ and w_i $(1 \leq i \leq n-2)$. Finally we assign the labels 4, 3, 2, 4, 3 to the vertices u_n, v_n, x_n, y_{n-1} and w_{n-1} respectively.

Case 4. $n \equiv 3 \pmod 4$.

Let $n = 4t+3$, $t \in \mathbb{N}$. As in Case 3, assign the label to the vertices u_i $(1 \leq i \leq n-1)$, v_i $(1 \leq i \leq n-1)$, x_i $(1 \leq i \leq n-1)$, y_i $(1 \leq i \leq n-2)$ and w_i $(1 \leq i \leq n-2)$. Finally we assign the labels 3, 4, 2, 3, 4 respectively to the vertices u_n, v_n, x_n, y_{n-1} and w_{n-1}. □

Theorem 3.8 *The subdivision of double triangular snake $S(DT_n)$ is 4-total prime cordial.*

Proof Let P_n be the path $u_1u_2\cdots u_n$. Let v_i, w_i be the vertex adjacent to u_iu_{i+1}. Let x_i, y_i, z_i, s_i and r_i be the vertex which subdivide the edges u_iu_{i+1}, u_iv_i, v_iu_{i+1}, u_iw_i and w_iu_{i+1} respectively. Clearly $|V(S(DT_n))| + |E(S(DT_n))| = 18n - 17$.

Case 1. $n \equiv 0 \pmod 4$, $n \geq 8$.

Let $n = 4t$, $t \in \mathbb{N}$. Assign the label 4 to the vertices $u_1, u_2, \cdots, u_t$ and assign the label 2 to the vertices $u_{t+1}, u_{t+2}, \cdots, u_{2t}$. Next we assign the label 3 to the vertices $u_{2t+1}, u_{2t+2}, \cdots, u_{3t}$ then we assign the label 1 to the vertices $u_{3t+1}, u_{3t+2}, \cdots, u_{n-1}$. Finally, we assign the label 3 to the vertex u_n. Now we consider the vertices v_i,w_i $(1 \leq i \leq n-1)$. Assign the label 4 to the vertices $v_1, v_2, \cdots, v_t$ and $w_1, w_2, \cdots, w_t$. Assign the label 2 to the vertices $v_{t+1}, v_{t+2}, \cdots, v_{2t-1}$ and $w_{t+1}, w_{t+2}, \cdots, w_{2t-1}$. Next we assign the label 3 to the vertices $v_{2t}, v_{2t+1}, \cdots, v_{3t-1}$ and $w_{2t}, w_{2t+1}, \cdots, w_{3t-1}$ then we assign the label 1 to the vertices $v_{3t}, y_{3t+1}, \cdots, v_{n-3}$ and $w_{3t}, w_{3t+1}, \cdots, w_{n-3}$. Finally we assign the labels 2, 4, 2, 4 respectively to the vertices v_{n-2}, v_{n-1}, w_{n-2} and w_{n-1}. Next we move to the vertices x_i $(1 \leq i \leq n-1)$. Assign the label 4 to the vertices $x_1, x_2, \cdots, x_t$ and assign the label 2 to the vertices $x_{t+1}, u_{t+2}, \cdots, x_{2t-1}$. Next we assign the label 3 to the vertices $x_{2t}, x_{2t+1}, \cdots, x_{3t-1}$ then we assign the label 1 to the vertices $x_{3t}, x_{3t+1}, \cdots, x_{n-2}$. Finally, we assign the label 3 to the vertex x_{n-1}. Now we consider the vertices y_i,s_i $(1 \leq i \leq n-1)$. Assign the label 4 to the vertices $y_1, y_2, \cdots, y_t$ and $s_1, s_2, \cdots, s_t$. Assign the label 2 to the vertices $y_{t+1}, y_{t+2}, \cdots, y_{2t}$ and $s_{t+1}, s_{t+2}, \cdots, s_{2t}$. Next we assign the label 3 to the vertices $y_{2t+1}, y_{2t+2}, \cdots, y_{3t-1}$ and $s_{2t+1}, s_{2t+2}, \cdots, s_{3t-1}$. Finally we assign the label 1 to the vertices $y_{3t}, y_{3t+1}, \cdots, y_{n-1}$ and $s_{3t}, s_{3t+1}, \cdots, s_{n-1}$. Next we move to the vertices z_i, r_i $(1 \leq i \leq n-1)$. Assign the label 4 to the vertices $z_1, z_2, \cdots, z_{t-1}$ and $r_1, r_2, \cdots, r_{t-1}$. Assign the label 2 to the vertices $z_t, z_{t+1}, \cdots, z_{2t-1}$ and $r_t, r_{t+1}, \cdots, r_{2t-1}$. Next we assign the label 3 to the vertices $z_{2t}, z_{2t+1}, \cdots, z_{3t-1}$ and $r_{2t}, r_{2t+1}, \cdots, r_{3t-1}$. Finally we assign the label 1 to the vertices $z_{3t}, z_{3t+1}, \cdots, z_{n-1}$ and $r_{3t}, r_{3t+1}, \cdots, r_{n-1}$. Clearly $t_f(1) = t_f(2) = t_f(3) = 18t - 4$ and $t_f(4) = 18t - 5$.

Case 2. $n \equiv 1 \pmod 4$, $n \geq 9$.

Let $n = 4t + 1$, $t \in \mathbb{N}$. As in Case 1, assign the label to the vertices u_i $(1 \leq i \leq n-1)$, v_i, w_i, x_i, y_i, z_i, s_i, r_i $(1 \leq i \leq n-2)$. Finally we assign the labels 4, 4, 2, 3, 1, 4, 3, 2 respectively to the vertices u_n, v_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}, z_{n-1}, s_{n-1} and r_{n-1}. Obviously $t_f(1) = 18t + 1$ and $t_f(2) = t_f(3) = t_f(4) = 18t$.

Case 3. $n \equiv 2 \pmod 4$, $n \geq 10$.

Let $n = 4t + 2$, $t \in \mathbb{N}$. Assign the label to the vertices u_i $(1 \leq i \leq n-2)$, v_i, w_i, x_i, y_i, z_i $(1 \leq i \leq n-3)$, s_i, r_i $(1 \leq i \leq n-2)$ by in case 1. Finally we assign the labels 2, 4, 3, 4, 3, 3, 1, 2, 3, 4, 2, 4, 1, 4 to the vertices u_{n-1}, u_n, v_{n-2}, v_{n-1}, w_{n-2}, w_{n-1}, x_{n-2}, x_{n-1}, y_{n-2}, y_{n-1}, z_{n-2}, z_{n-1}, s_{n-1} and r_{n-1} respectively. It is easy to verify that $t_f(1) = t_f(2) = t_f(3) = 18t + 5$ and $t_f(4) = 18t + 4$.

Case 4. $n \equiv 3 \pmod 4$, $n \geq 11$.

Let $n = 4t + 3$, $t \in \mathbb{N}$. As in Case 3, assign the label to the vertices u_i $(1 \leq i \leq n-1)$, v_i, w_i, x_i, y_i, z_i, s_i, r_i $(1 \leq i \leq n-2)$. Finally we assign the labels 3, 3, 2, 2, 4, 3, 4, 1 respectively to the vertices u_n, v_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}, z_{n-1}, s_{n-1} and r_{n-1}. Clearly $t_f(1) = t_f(2) = t_f(4) = 18t + 9$ and $t_f(3) = 18t + 10$.

Case 5. $t = 2, 3, 4, 5, 6, 7$.

A 4-total prime cordial labeling is given in Table 1.

n	2	3	4	5	6	7
u_1	3	4	4	4	4	4
u_2	4	2	2	4	4	4
u_3		3	3	2	2	2
u_4			3	3	3	2
u_5				1	1	3

u_6					2	1
u_7						1
v_1	4	4	4	4	4	4
v_2		3	3	2	2	4
v_3			4	3	3	2
v_4				3	3	3
v_5					4	3
v_6						1
w_1	2	4	4	4	4	4
w_2		1	2	2	2	4
w_3			1	3	3	2
w_4				1	1	3
w_5					3	3
w_6						1
x_1	1	2	4	4	4	4
x_2		3	2	2	2	2
x_3			3	3	3	2
x_4				3	3	3
x_5					4	3
x_6						1
y_1	3	4	4	4	4	4
y_2		3	3	2	2	4
y_3			1	3	3	2
y_4				1	1	3
y_5					3	3
y_6						1
z_1	4	2	2	4	4	4
z_2		3	3	2	2	2
z_3			4	3	3	2
z_4				1	1	3
z_5					4	1
z_6						1
s_1	3	4	4	4	4	4
s_2		1	2	2	2	4

s_3			1	3	3	2
s_4				1	1	3
s_5					3	3
s_6						1
r_1	2	2	2	4	4	4
r_2		1	3	2	2	2
r_3			1	3	3	2
r_4				1	1	3
r_5					2	3
r_6						1

Table 1

This completes the proof. □

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The Signed Product Cordial for Corona of Paths and Fourth Power of Paths

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Abstract: A graph $G = (V, E)$ is called signed product cordial if it is possible to label the vertex by the function $f : V \rightarrow \{-1, 1\}$ and label the edges by $f^* : E \rightarrow \{-1, 1\}$, where $f^*(uv) = f(u) \cdot f(v)$, $u, v \in V$ so that $|v_{-1} - v_1| \leqslant 1$ and $|e_{-1} - e_1| \leqslant 1$. In [3] J.Devaraj and P.Delphy, they have defined signed graphs, and they have started by labeling edges and then induced the labeling of vertices. In this paper, we contribute some new results on signed product cordial labeling and present necessary and sufficient conditions for signed product cordial for corona of paths and fourth power of paths.

Key Words: Second power, fourth power, corona graph, signed product cordial graph, Smarandachely signed product cordial labeling.

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§1. Introduction

The labeling of graphs is perceived to be a primarily theoretical subject in the field of graph theory and discrete mathematics, it serves as models in a wide range of application like astronomy, coding theory, circuit design and communication networks addressing. The concept of graph labeling was introduced during the sixties' of the last century by Rosa [12]. Many researches have been working with different types of labeling graphs [1,4,5]. In 1954 Harray introduced S-cordiality [10]. An excellent reference for this purpose is the survey written by Gallian [6]. All graphs considered in this theme are finite, simple and undirected. The original concept of cordial graphs is due to Chait[2]. He showed that each tree is cordial; an Euerlian graph is not cordial if its size is congruent to $2(mod\ 4)$. Let $G = (V, E)$ be a graph and let $f : V \rightarrow \{-1, 1\}$ be a labeling of its vertices, and let the induced edge labeling $f^* : E \rightarrow \{-1, 1\}$ be given by $f^*(e) = (f(u) \cdot f(v))$, where $e = uv$ and $u, v \in V$.

Let v_{-1} and v_1 be the numbers of vertices that are labeled by -1 and 1, respectively, and let e_{-1} and e_1 be the corresponding numbers of edges. Such a labeling is called *signed product*

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cordial if both $|v_{-1} - v_1| \leqslant 1$ and $|e_{-1} - e_1| \leqslant 1$ hold. Otherwise, it is called *Smarandachely signed product cordial* if $|v_{-1} - v_1| > 1$ or $|e_{-1} - e_1| > 1$. The corona $G_1 \bigodot G_2$ of two graphs G_1 (with n_1 vertices, m_1 edges) and G_2 (with n_2 vertices, m_2 edges) is defined as the graph obtained by taking one copy of G_1 and copies of G_2, and then joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2. It is easy to see that the corona $G_1 \bigodot G_2$ that has $n_1 + n_1n_2$ vertices and $m_1 + n_1m_2 + n_1n_2$ edges. The fourth power of a paths P_n, denoted by P_n^4, is $P_n \bigcup J$, where J is the set of all edges of the form edges v_iv_j such that $2 \leq d(v_iv_j) \leq 4$ and $i < j$ where $d(v_iv_j)$ is the shortest path from v_i to v_j.

§2. Terminologies and Notations

A path with m vertices and $m-1$ edges, denoted by P_m, and its fourth power P_n^4 has n vertices and $4n - 10$ edges. We let L_{4r} denote the labeling $(-1)_2 11$ $(-1)_2 11 \cdots (-1)_2 11$ (repeated r-times), Let L'_{4r} denote the labeling $(-1)11(-1)$ $(-1)11(-1) \cdots (-1)11(-1)$ (repeated r-times).

The labeling $11(-1)_2$ $11(-1)_2 \cdots 11(-1)_2$ (repeated r-times) and labeling $1(-1)_2 11(-1)_2 1$ $\cdots$ $1(-1)_2 1$ (repeated r-times) are written S_{4r} and S'_{4r}. Let M_r denote the labeling $(-1)1$ $(-1)1 \cdots (-1)1$, zero-one repeated $r$times if r is even and $(-1)1$ $(-1)1 \cdots (-1)1(-1)$ if r is odd; for example, $M_6 = (-1)1(-1)1(-1)1$ and $M_5 = (-1)1(-1)1(-1)$. We let M'_r denote the labeling $1(-1)1(-1) \cdots 1(-1)$. Sometimes, we modify the labeling M_r or M'_r by adding symbols at one end or the other (or both). Also, L_{4r} (or L'_{4r}) with extra labeling from right or left (or both sides). If L is a labeling for a path p_m and M is a labeling for fourth power of path P_n, then we use the notation $[L; M]$ to represent the labeling of the corona $P_m \bigodot P_n^4$. Additional notation that we use is the following: for a given labeling of the corona $P_m \bigodot P_n^4$, we let v_i and e_i (for $i = -1, 1$) be the numbers of vertices and edges, respectively, that are labeled by i of the corona $P_m \bigodot P_n^4$, and let x_i and a_i be the corresponding quantities for p_m, and we let y_i and b_i be those for P_n^4, which are connected with vertices labeled (-1) of P_m. Similarly, let y'_i and b'_i for P_n^4 which are connected with vertices labeled 1 of P_m. It is easy to verify that $v_{-1} = x_{-1} + x_{-1}y_{-1} + x_1y'_{-1}$, $v_1 = x_1 + x_{-1}y_1 + x_1y'_1$, $e_{-1} = a_{-1} + x_{-1}b_{-1} + x_1b'_{-1} + x_{-1}y_{-1} + x_1y'_1$ and $e_1 = a_1 + x_{-1}b_1 + x_1b'_1 + x_{-1}(x_{-1}y_1) + x_1y'_{-1}$. Thus, $v_{-1} - v_1 = (x_{-1} - x_1) + x_{-1}(y_{-1} - y_1) + x_1(y'_{-1} - y'_1)$ and $e_{-1} - e_1 = (a_{-1} - a_1) + x_{-1}(b_{-1} - b_1) + x_1(b'_{-1} - b'_1) + x_{-1}(y_{-1} - y_1) - x_1(y'_{-1} - y'_1)$. When it comes to the proof, we only need to show that, for each specified combination of labeling, $|v_{-1} - v_1| \leq 1$ and $|e_{-1} - e_1| \leq 1$.

§3. Main Results

In this section, we show that the corona $P_m \bigodot P_n^4$ is signed product cordial for all m, $n \geq 7$ and also for $n = 3$ with $m \geq 1$. This target will be achieved after the following series of lemmas.

Lemma 3.1 *The corona $P_m \bigodot P_3^4$ is signed product cordial if and only if $m \neq 1$.*

Proof Suppose that $n = 3$. The following cases will be examined.

Case 1. Obviously, $P_1 \bigodot P_3^4$ isomorphic to the complete graph K_4. Since K_4 is not cordial, $P_1 \bigodot P_3^4$ is not is signed product cordial.

Case 2. Suppose that $m = 2$. Then we label the vertices of $P_2 \bigodot P_3^4$ by $[-11; -11-1, 1-11]$; hence $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. So, $P_2 \bigodot P_3^4$ is signed product cordial.

Case 3. Suppose that $m = 3$. Then we label the vertices of $P_3 \bigodot P_3^4$ by $[-1-1-1; -111, 111, -11-1]$; hence $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. So, $P_3 \bigodot P_3^4$ is signed product cordial.

Case 4. $m \equiv 0(mod 4)$.

Suppose that $m = 4r, r \geq 1$. We choose the labeling $[L_{4r}; -11-1, -11-1,\ 1-11, 1-11, \cdots, (r-times)]$ for $P_{4r} \bigodot P_3^4$. Therefore $x_{-1} = x_1 = 2r$, $a_{-1} = 2r-1$, $a_1 = 2r$, $y_{-1} = 2$, $y_1 = 1$, $y'_{-1} = 1$, $y'_1 = 2$, $b_{-1} = 2$, $b'_{-1} = 2$, $b_1 = 1$ and $b'_1 = 1$. Hence $v_{-1} - v_1 = (x_{-1} - x_1) + x_{-1}.(y_{-1} - y_1) + x_1.(y'_{-1} - y'_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + x_{-1}.(b_{-1} - b_1) + x_1.(b'_{-1} - b'_1) + x_{-1}.(y_{-1} - y_1) - x_1.(y'_{-1} - y'_1) = -1$. Thus $P_{4r} \bigodot P_3^4$ is signed product cordial.

Case 5. $m \equiv 1(mod 4)$.

Suppose that $m = 4r+1, r \geq 1$. We choose the labeling $[L_{4r}1; -11-1, -11-1, 1-11, 1-11, \cdots, (r-times), -11-1]$ for $P_{4r+1} \bigodot P_3^4$. Therefore $x_{-1} = 2r$, $x_1 = 2r+1$, $a_{-1} = 2r-1$, $a_1 = 2r+1$, and for the first $4r$- vertices $y_{-1} = 2$, $y_1 = 1$, $y'_{-1} = 1$, $y'_1 = 2$, $b_{-1} = b'_{-1} = 2$ and $b_1 = b'_1 = 1$, and for the cycle c_3 which is connected to last vertex in P_{4r+1}, we have $y''_{-1} = 2$, $y''_1 = 1$, $b''_{-1} = 2$ and $b''_{(1)} = 1$, where y''_i and b''_i are the numbers of vertices and edges labeled by i in P_3^4 that is connected to the last vertex of P_{4r+1}. It is easily to verify that $v_{-1} = x_{-1} + x_{-1}.y_{-1} + (x_1 - 1).y'_{-1} + y''_{-1} = 8r+2$, $v_1 = x_1 + x_{-1}.y_1 + (x_1 - 1).y'_1 + y''_1 = 8r+2$, $e_{-1} = a_{-1} + x_{-1}.b_{-1} + (x_1 - 1).b'_{-1} + x_{-1}.y_{-1} + (x_1 - 1).y'_1 + b''_{-1} + 1 = 14r+3$ and $e_1 = a_1 + x_{-1}.b_1 + (x_1 - 1).b'_1 + x_{-1}.y_1 + (x_1 - 1).y'_{-1} + b''_{(1)} + 2 = 14r+3$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Thus $P_{4r+1} \bigodot P_3^4$ is cordial.

Case 6. $m \equiv 2(mod 4)$.

Suppose that $m = 4r+2, r \geq 1$. We choose the labeling $[L_{4r}1(-1); (-1)1(-1), (-1)1(-1), 1(-1)1, 1(-1)1, ..., (r-times), 1(-1)1, (-1)1(-1)]$ for $P_{4r+2} \bigodot P_3^4$. Therefore $x_{-1} = x_1 = 2r+1$, $a_{-1} = 2r$, $a_1 = 2r+1$, $y_{-1} = 2$, $y_1 = 1$, $y'_{-1} = 1$, $y'_1 = 2$, $b_{-1} = 2$, $b'_{-1} = 2$, $b_1 = 1$ and $b'_1 = 1$. Hence $v_{-1} - v_1 = (x_{-1} - x_1) + x_{-1}.(y_{-1} - y_1) + x_1.(y'_{-1} - y'_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + x_{-1}.(b_{-1} - b_1) + x_1.(b'_{-1} - b'_1) + x_{-1}.(y_{-1} - y_1) - x_1.(y'_{-1} - y'_1) = -1$. Thus $P_{4r+2} \bigodot P_3^4$ is cordial.

Case 7. $m \equiv 3(mod 4)$.

Let $m = 4r+3, r \geq 1$. We choose the labeling $[L_{4r}1(-1)(-1); (-1)1(-1), (-1)1(-1), 1(-1)1, 1(-1)1, \cdots, (r-times), 1(-1)1, (-1)1(-1), 1(-1)1]$ for $P_{4r+3} \bigodot P_3^4$. Therefore $x_{-1} = 2r+2$, $x_1 = 2r+1$, $a_{-1} = 2r$, $a_1 = 2r+2$, and for the first $4r$- vertices $y_{-1} = 2$, $y_1 = 1$, $y'_{-1} = 1$, $y'_1 = 2$, $b_{-1} = b'_{-1} = 2$ and $b_1 = b'_1 = 1$, and for the cycle P_3^4 which is connected to last vertex of P_{4r+3}, we have $y''_0 = 1$, $y''_1 = 2$, $b''_0 = 2$ and $b''_1 = 1$, where y''_i and b''_i are the numbers of vertices and edges labeled by i in P_3^4 that is connected to the last vertex of P_{4r+3}. Similar to Case 2, we conclud that $v_{-1}-v_1=(x_{-1}-x_1)+x_{-1}.(y_{-1}-y_1)+(x_1-1).(y'_{-1}-y'_1)+(y''_{-1}-y''_1)=0$ and $e_{-1}-e_1=(a_{-1}-a_1)+x_{-1}.(b_{-1}-b_1)+(x_1-1).(b'_{-1}-b'_1)+x_{-1}.(y_{-1}-y_1)-(x_1-1).(y'_{-1}-y'_1)+(c_{-1}-c_1)-1=0$. Hence $P_{4r+3}\bigodot P_3^4$ is cordial. Thus the lemma is proved. □

Lemma 3.2 *If $n\equiv 0(mod4)$, then $P_m\bigodot P_n^4$ is signed product cordial for all $m\geq 1$.*

Proof Suppose that $n=4s$, where $s\geq 2$. The following cases will be examined.

Case 1. Suppose that $m=1$. Then we label the vertices of $P_1\bigodot P_{4s}^4$ by $[-1;-1L_{4s-4}-11_2]$. Therefore $x_{-1}=1, x_1=0, a_{-1}=a_1=0, y_{-1}=y_1=2s, b_{-1}=b_1=8s-5$. It follows that $v_{-1}-v_1=1$ and $e_{-1}-e_1=0$. As an example, Figure 1 illustrates $P_1\bigodot P_8^4$. Hence, $P_1\bigodot P_{4s}^4$ is signed product cordial.

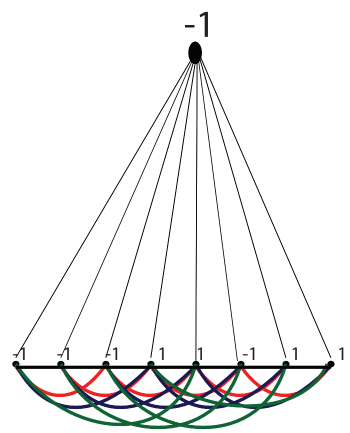

Figure 1

Case 2. Suppose that $m=2$. Then we label the vertices of $P_2\bigodot P_{4s}^4$ by $[-11;-1L_{4s-4}-11_2, 1_2L'_{4s-4}-1_2]$. Therefore $x_{-1}=x_1=1, a_{-1}=1, a_1=0, y_{-1}=y_1=2s, b_{-1}=b_1=8s-5, y'_{-1}=y'_1=2s$ and $b'_{-1}=b'_1=8s-5$. It follows that $v_{-1}-v_1=0$ and $e_{-1}-e_1=1$. As an example, Figure 2 illustrates $P_2\bigodot P_8^4$. Hence, $P_2\bigodot P_{4s}^4$ is signed product cordial.

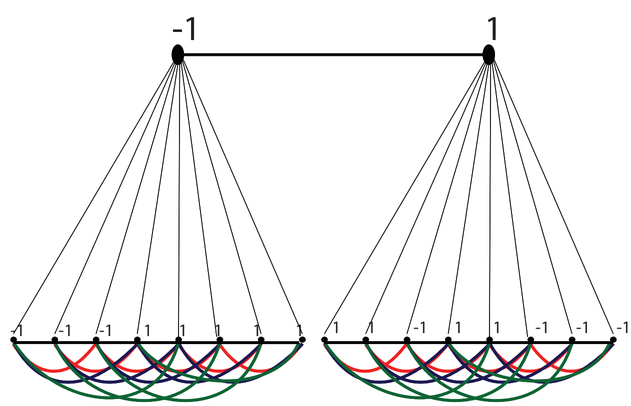

Figure 2

Case 3. Suppose that $m = 3$. Then we label the vertices of $P_3 \bigodot P^4_{4s}$ by $[-1_2 1; -1L_{4s-4} - 11_2, -1L_{4s-4} - 11_2, 1_2 L'_{4s-4} - 1_2]$. Therefore $x_{-1} = 2, x_1 = 2, a_{-1} = a_1 = 1, y_{-1} = y_1 = 2s, b_{-1} = b_1 = 8s - 5, y'_{-1} = y'_1 = 2s$ and $b'_{-1} = b'_1 = 8s - 5$. It follows that $v_0 - 1 - v_1 = 1$ and $e_{-1} - e_1 = 0$. As an example, Figure 3 illustrates $P_3 \bigodot P^4_8$. Hence, $P_3 \bigodot P^4_{4s}$ is signed product cordial.

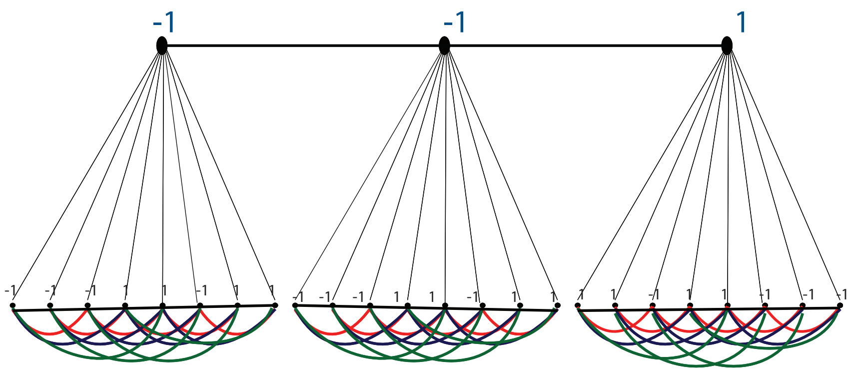

Figure 3

Case 4. $m = 0(mod\ 4)$.

Suppose that $m = 4r$, where $r \geq 2$. Then we label the vertices of $P_{4r} \bigodot P^4_{4s}$ by $[L_{4r}; -1L_{4s-4} - 11_2, -1L_{4s-4} - 11_2, 1_2 L'_{4s-4} - 1_2,$
$1_2 L'_{4s-4} - 1_2, ..., (r - time)]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = 2r - 1, a_1 = 2r, y_{-1} = y_1 = 2s, b_{-1} = b_1 = 8s - 5, y'_{-1} = y'_1 = 2s$ and $b'_{-1} = b'_1 = 8s - 5$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. As an example, Figure 4 illustrates $P_4 \bigodot P^4_8$. Hence, $P_{4r} \bigodot P^4_{4s}$ is signed product cordial.

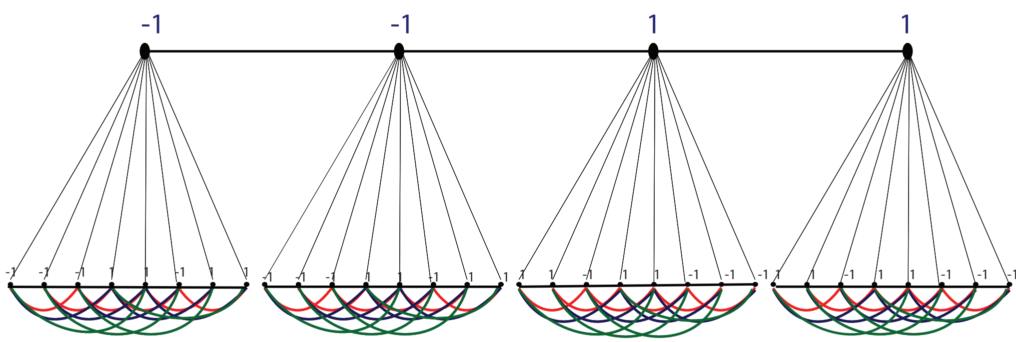

Figure 4

Case 5. $m = 1(mod\ 4)$.

Suppose that $m = 4r + 1$, where $r \geq 1$. Then we label the vertices of $P_{4r+1} \bigodot P^4_{4s}$ by $[L_{4r} - 1; -1L_{4s-4} - 11_2, -1L_{4s-4} - 11_2, 1_2 L'_{4s-4} - 1_2, 1_2 L'_{4s-4} - 1_2, \cdots, (r - time), -1L_{4s-4} - 11_2]$. Therefore $x_{-1} = 2r + 1, x_1 = 2r, a_{-1} = a_1 = 2r, y_{-1} = y_1 = 2s, b_{-1} = b_1 = 8s - 5, y'_{-1} = y'_1 = 2s$

and $b'_{-1} = b'_1 = 8s-5$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r+1} \bigodot P^4_{4s}$ is signed product cordial.

Case 6. $m = 2(mod\ 4)$.

Suppose that $m = 4r+2$, where $r \geq 1$. Then we label the vertices of $P_{4r+2} \bigodot P^4_{4s}$ by $[L_{4r}-11; -1L_{4s-4}-11_2, -1L_{4s-4}-11_2, 1_2L'_{4s-4}-1_2, 1_2L'_{4s-4}-1_2, \cdots, (r-time), -1L_{4s-4}-11_2, 1_2L'_{4s-4}-1_2]$. Therefore $x_{-1} = x_1 = 2r+1, a_{-1} = 2r+1, a_1 = 2r, y_{-1} = y_1 = 2s, b_{-1} = b_1 = 8s-5, y'_{-1} = y'_1 = 2s$ and $b'_{-1} = b'_1 = 8s-5$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. Hence, $P_{4r+2} \bigodot P^4_{4s}$ is signed product cordial.

Case 7 $m = 3(mod\ 4)$.

Suppose that $m = 4r+3$, where $r \geq 1$. Then we label the vertices of $P_{4r+3} \bigodot P^4_{4s}$ by $[L_{4r}-1_21; -1L_{4s-4}-11_2, -1L_{4s-4}-11_2, 1_2L'_{4s-4}-1_2,$
$1_2L'_{4s-4}v_2, \cdots, (r-time), -1L_{4s-4}-11_2, -1L_{4s-4}-11_2, 1_2L'_{4s-4}-1_2]$. Therefore $x_{-1} = 2r+2, x_1 = 2r+1, a_{-1} = a_1 = 2r+1, y_{-1} = y_1 = 2s, b_{-1} = b_1 = 8s-5, y'_{-1} = y'_1 = 2s$ and $b'_{-1} = b'_1 = 8s-5$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r+3} \bigodot P^4_{4s}$ is signed product cordial. □

Lemma 3.3 *If $n \equiv 1(mod4)$, then $P_m \bigodot P^4_n$ is cordial for all $m \geq 1$.*

Proof Suppose that $n = 4s+1$, where $s \geq 2$. The following cases will be examined.

Case 1. Suppose that $m = 1$. Then we label the vertices of $P_1 \bigodot P^4_{4s+1}$ by $[-1; 1_2L'_{4s-4}-11-1]$. Therefore $x_{-1} = 1, x_1 = 0, a_{-1} = a_1 = 0, y_{-1} = 2s, y_1 = 2s+1, b_{-1} = b_1 = 8s-3$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. Hence, $P_1 \bigodot P^4_{4s+1}$ is signed product cordial.

Case 2. Suppose that $m = 2$. Then we label the vertices of $P_2 \bigodot P^4_{4s+1}$ by $[-11; -1_2L_{4s-4}1-11, 1_2L'_{4s-4}-11-1]$. Therefore $x_{-1} = x_1 = 1, a_{-1} = 1, a_1 = 0, y_{-1} = 2s+1, y_1 = 2s, b_{-1} = b_1 = 8s-3, y'_{-1} = 2s+1, y'_1 = 2s$ and $b'_{-1} = b'_1 = 8s-3$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = -1$. Hence, $P_2 \bigodot P^4_{4s+1}$ is signed product cordial.

Case 3. Let $m = 3$. Then we label the vertices of $P_3 \bigodot P^4_{4s+1}$ by $[-11-1; -1_2L_{4s-4}1-11, 1_2L'_{4s-4}-11-1, 1_2L'_{4s-4}-11-1]$. Therefore $x_{-1} = 2, x_1 = 1, a_{-1} = 2, a_1 = 0, y_{-1} = 2s+1, y_1 = 2s, b_{-1} = b_1 = 8s-3, y'_{-1} = 2s, y'_1 = 2s+1, b'_{-1} = b'_1 = 8s-3, y''_{-1} = 2s, y''_1 = 2s+1$ and$b''_{-1} = b''_1 = 8s-3$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. Hence, $P_3 \bigodot P^4_{4s+1}$ is signed product cordial.

Case 4. $m = 0(mod\ 4)$.

Suppose that $m = 4r$, where $r \geq 1$. Then we label the vertices of $P_{4r} \bigodot P^4_{4s+1}$ by $[M_{4r}; -1_2L_{4s-4}1-11, 1_2L'_{4s-4}-11-1, -1_2L_{4s-4}1-11,$
$1_2L'_{4s-4}-11-1, \cdots, (r-time)]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = 4r-1, a_1 = 0, y_{-1} = 2s+1, y_1 = 2s, b_{-1} = b_1 = 8s-3, y'_{-1} = 2s, y'_1 = 2s+1$ and $b'_{-1} = b_1 = 8s-3$. It follows that $v_{-1} - v_1 = 0$ and $e_0 - e_1 = -1$. Hence, $P_{4r} \bigodot P^4_{4s+1}$ is signed product cordial.

Case 5. $m = 1(mod\ 4)$.

Suppose that $m = 4r+1$, where $r \geq 1$. Then we label the vertices of $P_{4r+1} \bigodot P^4_{4s+1}$ by

$[M_{4r}{-}1; -1_2L_{4s-4}1{-}11, 1_2L'_{4s-4}{-}11{-}1, -1_2L_{4s-4}1{-}11, 1_2L'_{4s-4}{-}11{-}1, \cdots, (r{-}time), 1_2L'_{4s-4}$ $-11{-}1]$. Therefore $x_{-1} = 2r+1, x_1 = 2r, a_{-1} = 4r, a_1 = 0, y_{-1} = 2s+1, y_1 = 2s, b_{-1} = b_1 = 8s-3, y'_{-1} = 2s, y'_1 = 2s+1, b'_{-1} = b'_1 = 8s-3, y''_{-1} = 2s, y''_1 = 2s+1$and $b''_{-1} = b''_1 = 8s-3$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. Hence, $P_{4r+1} \bigodot P^4_{4s+1}$ is signed product cordial.

Case 6. $m = 2(mod\ 4)$.

Let $m = 4r{+}2$, where $r \geq 1$. Label the vertices of $P_{4r+2} \bigodot P^4_{4s+1}$ by $[M_{4r+2}; -1_2L_{4s-4}1{-}11, 1_2L'_{4s-4}{-}11{-}1, -1_2L_{4s-4}1{-}11, 1_2L'_{4s-4}{-}11{-}1, -1_2L_{4s-4}1{-}11, 1_2L'_{4s-4}{-}11{-}1, \cdots, (r{-}time)]$. Therefore $x_{-1} = x_1 = 2r{+}1, a_{-1} = 4r{+}1, a_1 = 0, y_{-1} = 2s{+}1, y_1 = 2s, b_{-1} = b_1 = 8s{-}3, y'_{-1} = 2s, y'_1 = 2s+1$ and $b'_{-1} = b'_1 = 8s-3$. It follows that $v_{-1} - v_1 = 0$ and $e_0 - e_1 = -1$. Hence $P_{4r+2} \bigodot P^4_{4s+1}$ is signed product cordial.

Case 7. $m = 3(mod\ 4)$.

Suppose that $m = 4r+3$, where $r \geq 1$. We then label the vertices of $P_{4r+3} \bigodot P^4_{4s+1}$ by $[M_{4r+2}{-}1; -1_2L_{4s-4}1{-}11, 1_2L'_{4s-4}{-}11{-}1, -1_2L_{4s-4}1{-}11, 1_2L'_{4s-4}{-}11{-}1, -1_2L_{4s-4}1{-}11, 1_2L'_{4s-4}{-}11{-}1, \cdots, (r-time), 1_2L'_{4s-4}{-}11{-}1]$. Therefore $x_{-1} = 2r+2, x_1 = 2r+1, a_{-1} = 4r+1, a_1 = 0, y_{-1} = 2s+1, y_1 = 2s, b_{-1} = b_1 = 8s-3, y'_0 = 2s, y'_1 = 2s+1, b'_0 = b'_1 = 8s-3, y''_{-1} = 2s, y''_1 = 2s+1$ and $b''_{-1} = b''_1 = 8s-3$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. Hence, $P_{4r+3} \bigodot P^4_{4s+1}$ is signed- cordial. □

Lemma 3.4 *If $n \equiv 2(mod4)$, then $P_m \bigodot P^4_n$ is cordial for all $m \geq 1$.*

Proof Suppose that $n = 4s+2$, where $s \geq 2$. The following cases will be examined.

Case 1. Suppose that $m = 1$. Then we label the vertices of $P_1 \bigodot P^4_{4s+2}$ by $[-1; -11_3{-}1s_{4s-4}{-}1]$. Therefore $x_{-1} = 1, x_1 = 0, a_{-1} = a_1 = 0, y_{-1} = y_1 = 2s+1, b_{-1} = b_1 = 8s-1$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $P_1 \bigodot P^4_{4s+2}$ is signed product cordial.

Case 2. Suppose that $m = 2$. Then, label the vertices of $P_2 \bigodot P^4_{4s+2}$ by $[-11; -11_3{-}1s_{4s-4}{-}1, -1l_{4s-4}{-}11_3{-}1]$. Therefore $x_{-1} = x_1 = 1, a_{-1} = 1, a_1 = 0, y_{-1} = y_1 = 2s+1, b_{-1} = b_1 = 8s-1, y'_{-1} = y'_1 = 2s{+}1$ and $b'_{-1} = b'_1 = 8s-1$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = -1$. Hence, $P_2 \bigodot P^4_{4s+2}$ is signed product cordial.

Case 3. Let $m = 3$. Then we label the vertices of $P_3 \bigodot P^4_{4s+2}$ by $[-1{-}11; -11_3{-}1s_{4s-4}{-}1, -11_3{-}1s_{4s-4}{-}1, -1l_{4s-4}{-}11_3{-}1]$. Therefore $x_{-1} = 2, x_1 = 1, a_{-1} = a_1 = 1, y_{-1} = y_1 = 2s+1, b_{-1} = b_1 = 8s-1, y'_{-1} = y'_1 = 2s+1$ and $b'_{-1} = b'_1 = 8s-1$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $P_3 \bigodot P^4_{4s+2}$ is signed product cordial.

Case 4. $m = 0(mod\ 4)$.

Suppose that $m = 4r$, where $r \geq 1$. Then we label the vertices of $P_{4r} \bigodot P^4_{4s+2}$ by $[l_{4r}; -11_3{-}1s_{4s-4}{-}1, -11_3{-}1s_{4s-4}{-}1, -1L_{4s-4}{-}11_3{-}1, -1L_{4s-4}{-}11_3{-}1, \cdots, (r{-}time)]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = 2r{-}1, a_1 = 2r, y_{-1} = y_1 = 2s{+}1, b_{-1} = b_1 = 8s{-}1, y'_{-1} = y'_1 = 2s{+}1$ and $b'_{-1} = b_1 = 8s-1$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = -1$. Hence, $P_{4r} \bigodot P^4_{4s+2}$ is signed product cordial.

Case 5. $m = 1(mod\ 4)$.

Suppose that $m = 4r + 1$, where $r \geq 1$. Then we label the vertices of $P_{4r+1} \bigodot P^4_{4s+2}$ by $[l_{4r}{-}1; -11_3{-}1s_{4s-4}{-}1, -11_3{-}1s_{4s-4}{-}1,$
$-1L_{4s-4}{-}11_3{-}1, -1L_{4s-4}{-}11_3{-}1, \cdots, (r - time), -11_3{-}1s_{4s-4}{-}1]$. Therefore $x_{-1} = 2r + 1, x_1 = 2r, a_{-1} = a_1 = 2r, y_{-1} = y_1 = 2s + 1, b_{-1} = b_1 = 8s - 1, y'_{-1} = y'_1 = 2s + 1$ and $b'_{-1} = b'_1 = 8s - 1$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r+1} \bigodot P^4_{4s+2}$ is signed- cordial.

Case 6. $m = 2(mod\ 4)$.

Suppose that $m = 4r + 2$, where $r \geq 1$. Then we label the vertices of $P_{4r+2} \bigodot P^4_{4s+2}$ by $[l_{4r}{-}11; -11_3{-}1s_{4s-4}{-}1, -11_3{-}1s_{4s-4}{-}1,$
$-1l_{4s-4}{-}11_3{-}1, -1l_{4s-4}{-}11_3{-}1, \cdots, (r - time), -11_3{-}1s_{4s-4}{-}1, -1L_{4s-4}{-}11_3{-}1]$. Therefore $x_{-1} = x_1 = 2r + 1, a_{-1} = 2r + 1, a_1 = 2r, y_{-1} = y_1 = 2s + 1, b_{-1} = b_1 = 8s - 1, y'_{-1} = y'_1 = 2s + 1$ and $b'_{-1} = b'_1 = 8s - 1$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. Hence, $P_{4r+2} \bigodot P^4_{4s+2}$ is signed product cordial.

Case 7. $m = 3(mod\ 4)$.

Suppose that $m = 4r + 3$, where $r \geq 1$. Then we label the vertices of $P_{4r+3} \bigodot P^4_{4s+2}$ by $[L_{4r}{-}1_21; -11_3{-}1s_{4s-4}{-}1, -11_3{-}1s_{4s-4}{-}1, -1L_{4s-4}{-}11_3{-}1, -1L_{4s-4}{-}11_3{-}1, \cdots, (r - time), -11_3{-}1s_{4s-4}{-}1, -11_3{-}1s_{4s-4}{-}1, -1L_{4s-4}{-}11_3{-}1]$. Therefore $x_{-1} = 2r + 2, x_1 = 2r + 1, a_{-1} = a_1 = 2r + 1, y_{-1} = y_1 = 2s + 1, b_{-1} = b_1 = 8s - 1, y'_{-1} = y'_1 = 2s + 1$ and $b'_{-1} = b'_1 = 8s - 1$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r+3} \bigodot P^4_{4s+2}$ is signed product cordial. □

Lemma 3.5 *If $n \equiv 3(mod4)$, then $P_m \bigodot P^4_n$ is signed product cordial for all $m \geq 1$.*

Proof Suppose that $n = 4s + 3$, where $s \geq 2$. The following cases will be examined.

Case 1. Suppose that $m = 1$. Then we label the vertices of $P_1 \bigodot P^4_{4s+3}$ by $[-1; 1_2s_{4s}{-}1]$. Therefore $x_{-1} = 1, x_1 = 0, a_{-1} = a_1 = 0, y_{-1} = 2s + 1, y_1 = 2s + 2, b_{-1} = b_1 = 8s + 1$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. Hence, $P_1 \bigodot P^4_{4s+3}$ is signed product cordial.

Case 2. Suppose that $m = 2$. Then, label the vertices of $P_2 \bigodot P^4_{4s+3}$ by $[-11; -1_21L_{4s}, 1_2s_{4s}{-}1]$. Therefore $x_{-1} = x_1 = 1, a_{-1} = 1, a_1 = 0, y_{-1} = 2s + 2, y_1 = 2s + 1, b_{-1} = b_1 = 8s + 1, y'_{-1} = 2s + 1, y'_1 = 2s + 2$ and $b'_{-1} = b'_1 = 8s + 1$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. Hence, $P_2 \bigodot P^4_{4s+3}$ is signed product cordial.

Case 3. Suppose that $m = 3$. Then we label the vertices of $P_3 \bigodot P^4_{4s+3}$ by $[-11{-}1; -1_21L_{4s}, 1_2s_{4s}{-}1, 1_2s_{4s}{-}1]$. Therefore $x_{-1} = 2, x_1 = 1, a_{-1} = 2, a_1 = 0, y_{-1} = 2s + 2, y_1 = 2s + 1, b_{-1} = b_1 = 8s + 1, y'_{-1} = 2s + 1, y'_1 = 2s + 2, b'_{-1} = b'_1 = 8s + 1, y''_{-1} = 2s + 1, y''_1 = 2s + 2$ and$b''_{-1} = b''_1 = 8s + 1$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. Hence, $P_3 \bigodot P^4_{4s+3}$ is signed product cordial.

Case 4. $m = 0(mod\ 4)$.

Suppose that $m = 4r$, where $r \geq 1$. Then we label the vertices of $P_{4r} \bigodot P^4_{4s+3}$ by $[M_{4r}; -1_21L_{4s}, 1_2s_{4s}{-}1, -1_21L_{4s}, 1_2s_{4s}{-}1, \cdots, (r - time)]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} =$

$4r-1, a_1 = 4r, y_{-1} = 2s+2, y_1 = 2s+1, b_{-1} = b_1 = 8s+1, y'_{-1} = 2s+1, y'_1 = 2s+2$ and $b'_{-1} = b'_1 = 8s+1$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = -1$. Hence, $P_{4r} \bigodot P^4_{4s+3}$ is signed product cordial.

Case 5. $m = 1(mod\ 4)$.

Suppose that $m = 4r+1$, where $r \geq 1$. Then we label the vertices of $P_{4r+1} \bigodot P^4_{4s+3}$ by $[M_{4r}{-}1; -1_2 1L_{4s}, 1_2 s_{4s}{-}1, -1_2 1L_{4s}, 1_2 s_{4s}{-}1, \cdots, (r-time),\ 1_2 s_{4s}{-}1]$. Therefore $x_{-1} = 2r+1, x_1 = 2r, a_{-1} = 4r, a_1 = 0, y_{-1} = 2s+2, y_1 = 2s+1, b_{-1} = b_1 = 8s+1, y'_{-1} = 2s+1, y'_1 = 2s+2, b'_{-1} = b'_1 = 8s+1, y''_{-1} = 2s+1, y''_1 = 2s+2$and $b''_{-1} = b''_1 = 8s+1$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. Hence, $P_{4r+1} \bigodot P^4_{4s+3}$ is signed product cordial.

Case 6. $m = 2(mod\ 4)$.

Suppose that $m = 4r+2$, where $r \geq 1$. Then we label the vertices of $P_{4r+2} \bigodot P^4_{4s+3}$ by $[M_{4r+2}; -1_2 1L_{4s}, 1_2 s_{4s}{-}1, -1_2 1L_{4s}, 1_2 s_{4s}{-}1, -1_2 1L_{4s}, 1_2 s_{4s}{-}1, \cdots, (r-time)]$. Therefore $x_{-1} = x_1 = 2r+1, a_{-1} = 4r+1, a_1 = 0, y_{-1} = 2s+2, y_1 = 2s+1, b_{-1} = b_1 = 8s+1, y'_{-1} = 2s+1, y'_1 = 2s+2$ and $b'_{-1} = b'_1 = 8s+1$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = -1$. Hence, $P_{4r+2} \bigodot P^4_{4s+3}$ is signed product cordial.

Case 7. $m = 3(mod\ 4)$.

Suppose that $m = 4r+3$, where $r \geq 1$. Then we label the vertices of $P_{4r+3} \bigodot P^4_{4s+3}$ by $[M_{4r+2}{-}1; -1_2 1L_{4s}, 1_2 s_{4s}{-}1, -1_2 1L_{4s}, 1_2 s_{4s}{-}1,$
$-1_2 1L_{4s}, 1_2 s_{4s}{-}1..., (r-time)$
$1_2 s_{4s}{-}1]$. Therefore $x_{-1} = 2r+2, x_1 = 2r+1, a_{-1} = 4r+2, a_1 = 0, y_{-1} = 2s+2, y_1 = 2s+1, b_{-1} = b_1 = 8s+1, y'_{-1} = 2s+1, y'_1 = 2s+2, b'_{-1} = b'_1 = 8s+1, y''_{-1} = 2s+1, y''_1 = 2s+2$ and $b''_{-1} = b''_1 = 8s+1$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. Hence, $P_{4r+3} \bigodot P^4_{4s+3}$ is signed product cordial. □

As a consequence of all lemmas mentioned above we conclude that the signed product cordial of the corona between paths and fourth power of paths is cordial for all m, $n \geq 7$ and $n = 3$ for all $m \geq 1$.

Theorem 3.1 *The corona between paths and fourth power of paths is signed product cordial for all m, $n \geq 7$ and also for $n = 3$ with $m \geq 1$.*

Proof The proof follows directly from Lemma 3.1 to Lemma 3.5. □

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On the Eccentric Sequence of Composite Graphs

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Abstract: The eccentric sequence of a graph is defined as list of eccentricity of its vertices. Eccentric sequence of composite graphs under seven graph products: line graph, sum, cartesian product, disjunction, symmetric difference, lexicographic product and corona product is investigated. Also some family of non vertex transitive graphs that are self centered are determined as product of graphs. It is proved that for any positive integer d, there is an infinite family of non-vertex transitive self centered graphs with diameter d. The relation between total eccentricity of a tree and total eccentricity of its line graph is given.

Key Words: Eccentricity, eccentric sequence, graph product, self centered graph.

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§1. Introduction

We consider only simple connected graphs in this paper. Let $G = (V(G), E(G))$ be a graph and u, v be two vertices of G. The *distance* between u and v, $d_G(u, v)$ (simply $d(u, v)$) is the length of shortest path connecting u and v. For a vertex $v \in V(G)$, the *eccentricity* of v, $\varepsilon_G(v)$ is the maximum distance from v to other vertices in G. The maximum and the minimum eccentricity among all vertices of G are called *diameter* $\mathrm{diam}(G)$ and *radius* $\mathrm{rad}(G)$ of G respectively. The *center* of G, $C(G)$ is the set of vertices whose eccentricity is equal to $\mathrm{rad}(G)$. A graph G is called *self centered* if all of its vertices have a same eccentricity. Let $V(G) = \{v_1,\ v_2, \cdots, v_n\}$ be the vertex set of G. *Eccentric sequence* of G is the sequence $\varepsilon_1,\ \varepsilon_2, \cdots, \varepsilon_n$ where ε_i is the eccentricity of vertex v_i. The *minimum eccentric sequence* of G, $es(G)$ is the sequence $\{\varepsilon_1^{t_1},\ \varepsilon_2^{t_2}, \cdots,\ \varepsilon_k^{t_k}\}$ where $\varepsilon_1,\ \varepsilon_2 \cdots, \varepsilon_k$ are the different eccentricities of vertices and t_i denotes the number of vertices with eccentricity ε_i and more over $\varepsilon_{i+1} = \varepsilon_i + 1$ for $1 \leqslant i \leqslant k-1$. Note that $\varepsilon_1 = \mathrm{rad}(G)$ and $\varepsilon_k = \mathrm{diam}(G)$. Eccentric sequence is interesting since it provides information on the vertex eccentricities and some structural properties of the graph such as diameter, radius and variability of vertex eccentricities. Call a sequence of positive integer eccentric if it is eccentric sequence of a graph. In a series of papers several properties of eccentric sequences are studied. For instance see surveys [3, 7, 11, 14, 17]. Characterization of eccentric sequences of graphs was first considered by Lesniak [11] who characterized sequences which are the eccentricity sequences of trees.

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Study of graph invariants specially topological indices under graph products is very interested in mathematical literature. Some properties and application are reported in surveys [1, 2, 4 C 6, 8 C 10, 12, 15, 18]. In this paper, we study the eccentric sequence of composite graphs. We obtain explicit formulas of eccentric sequence for some graph product such as: line graph, sum, cartesian product, disjunction, symmetric difference, lexicographic product and corona product. Two important topological indices based on eccentricity of vertices are the total eccentricity and eccentric connectivity index. The *total eccentricity* of a graph G, $\xi(G)$ is the sum of eccentricities of its vertices. Clearly, if

$$es(G) = \{\varepsilon_1^{t_1}, \cdots, \varepsilon_k^{t_k}\}$$

then,

$$\xi(G) = \sum_{i=1}^{k} t_i \varepsilon(v_i).$$

The *eccentric connectivity index* of graph G, ECI(G), introduced by Sharma et al. [16], is defined as

$$\mathrm{ECI}(G) = \sum_{v \in V(G)} \varepsilon(v)\deg(v),$$

where $\deg(v)$ denotes degree of vertex v. These topological indices have been used as mathematical models for the prediction of biological activities of diverse nature. The automorphism group of G is denoted with Aut(G). A graph is called *vertex transitive* if for any pair of vertices u and v, there is an automorphism α such that $\alpha(u) = v$. It is known that an automorphism of a graph preserve the distance function. It follows that vertex-transitive graphs are always regular and self centered graph.

In this paper we construct an infinite family of non-vertex transitive graphs which are self centered. In the rest of the section, some standard graph products are introduced, then eccentric sequence of graphs under these graph products is verified. First, we start with line graph. *Line graph* of G, $L(G)$ is a graph which each vertex of $L(G)$ is associated with an edge of G and two vertices in $L(G)$ is adjacent if and only if the corresponding edges of G have an end vertex in common. The *sum* of two graphs G_1 and G_2, $G_1 + G_2$ is defined as the graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{u_1u_2 \,|u_1 \in V(G_1), u_2 \in V(G_2)\}$. The next binary graph product is cartesian product. The *Cartesian product* $G_1 \Box G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ and (u_1, u_2) is adjacent to (v_1, v_2) if $u_1 = v_1$ and $(u_2v_2) \in E(G_2)$, or $u_2 = v_2$ and $(u_1v_1) \in E(G_1)$.

The *disjunction* $G_1 \vee G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ and

$$E(G_1 \vee G_2) = \{(u_1, u_2)(v_1, v_2)\,|u_1v_1 \in E(G_1) \text{ or } u_2v_2 \in E(G_2) \ \}.$$

For given graphs G_1 and G_2, their *symmetric difference* $G_1 \bigoplus G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $u_1v_1 \in E(G_1)$ or $u_2v_2 \in E(G_2)$

The diameter of disjunction and symmetric difference of two graphs when both of them contain more than one vertex do not exceed of 2. The next binary operation is the *lexicographic product*. The lexicographic product of two graphs G_1 and G_2, $G_1[G_2]$ is the graph with vertex set $V(G_1)\times V(G_2)$ and two vertices $u=(u_1,u_2)$ and $v=(v_1,v_2)$ are adjacent if (u_1 is adjacent with v_1) or ($u_1=v_1$ and u_2 and v_2 are adjacent).The operations sum, disjunction and symmetric difference are symmetric operation and this fact implies that they have symmetric eccentric sequence. But the lexicographic product do not have such property. Let n_i, $i=1,2$ denotes the order of G_i. The *corona product* of two graphs is denoted by $G_1\circ G_2$ and is obtained from one copy of G_1 and n_1 copies of G_2, and then joining all vertices of the i-th copy of G_2 to the i-th vertex of G_1 for $i=1,2,\cdots,n_1$. In [13], application of coronas in chemical modeling was reported.

§2. Main Result

In this section, explicit formulas for eccentric sequence of some composite graphs is given.

2.1 Line Graph of Trees

Eccentric sequence of line graph of a tree can be determined by its eccentric sequence. We present a relation between of total eccentricity of a tree and total eccentricity of its line graph.

Theorem 2.1 *Let T be a tree with* rad(T) = r. *If $es(T)=\{r^{n_0},(r+1)^{n_1},\cdots,(2r)^{n_r}\}$, then eccentric sequence of $L(T)$ is obtained as*

$$es(L(T))=\begin{cases}\{r^{n_1},(r+1)^{n_2},\cdots,(2r-1)^{n_r}\} & \text{if } C(T)=K_1\\ \{(r-1)^1,r^{n_1},\cdots,(2r-2)^{n_{r-1}}\} & \text{if } C(T)=K_2\end{cases}$$

where $n_r\geqslant 0$ and for $0\leqslant i\leqslant r-1$, $n_i\geqslant 1$.

Proof We must to consider two cases.

Case 1. $C(T)=K_1$.

Let p be the unique central vertex. Let $N_i(p)=\{v\in V(T)|d(p,v)=i\}$. Since T is a tree, for a vertex $u\in N_i(p)$, $\varepsilon_T(u)=i+r$. Also there is a unique vertex $v\in N_{i-1}(p)$ that is adjacent to u. Now consider the bijection $f:V(T)-\{p\}\to E(T)$, if $u\in N_i(p)$, $i\geqslant 1$, then $f(u)=uv$ where $v\in N_{i-1}(p)$. For a vertex $u\in N_i(p)$, we have $\varepsilon_T(u)=r+i$ and $\varepsilon_{L(T)}(f(u))=r+i-1$. Thus if

$$es(T)=\{r^1,(r+1)^{n_1},\cdots,(2r)^{n_r}\},$$

the eccentric sequence of $L(T)$ is obtained as

$$es(L(T))=\{r^{n_1},(r+1)^{n_2},\ldots,(2r-1)^{n_r}\}.$$

Case 2. $C(T)=K_2$.

Let p_1 and p_2 be two adjacent central vertices of T. It is easy to see $\varepsilon_{L(T)}(p_1p_2) = r-1$. Let $T-\{p_1p_2\} = T_1 \cup T_2$ which $p_j \in T_j$ for $j=1,2$. Clearly if $u \in T_j$, clearly $\varepsilon(u) = r + d(u, p_j)$. By a similar argument to Case 1, if $u \in N_i(P_j) \cap T_j$, $j=1,2$, then there is a unique vertex $v \in N_{i-1}(P_j)$ that $uv \in E(T)$. Thus we get again a bijection $f : V(T) - \{p_1, p_2\} \to E(T) - \{p_1p_2\}$. Also if $u \in N_i(P_j) \cap T_j$, then $\varepsilon_{L(T)}(f(u)) = \varepsilon_T(u) - 1$. This implies that if

$$es(T) = \{r^2, (r+1)^{n_1}, \cdots, (2r-1)^{n_{r-1}}\},$$

then

$$es(L(T)) = \{(r-1)^1, (r)^{n_1}, \cdots, (2r-2)^{n_{r-1}}\}. \qquad \square$$

Corollary 2.2 *Let T be a tree. $L(T)$ is self centered graph if and only if T is a star graph.*

Proof Clearly the line graph of a star graph is complete graph and then is self centered. Let T be a tree of order n and rad(T) = r. If $L(T)$ is self center graph, by Theorem 2.1, eccentric sequence of T has form $es(T) = \{(r-1)^{n-1}\}$ or $\{r^1, (r+1)^{n-2}\}$. This means that the variability of eccentricity in T is 1 or 2. Since T is a tree, this implies that rad(T) = 1 and the proof is completed. $\square$

Corollary 2.3 *Let T be a tree of order n and radius r. Then*

$$\xi(T) = \xi(L(T)) + n + r - 1.$$

Proof We consider two cases with respect to $es(T)$ and $es(L(T))$.

Case 1. $es(T) = \{r^1, (r+1)^{n_1}, \cdots, (2r)^{n_r}\}$ and $es(L(T)) = \{r^{n_1}, (r+1)^{n_2}, \cdots, (2r-1)^{n_r}\}$.

By a straight calculation we get

$$\xi(T) - \xi(L(T)) = r + \sum_{i=1}^{r} n_i = r + n - 1.$$

Case 2. $es(T) = \{r^2, (r+1)^{n_1}, \cdots, (2r-1)^{n_{r-1}}\}$ and $es(L(T)) = \{r-1^1, (r)^{n_1}, \cdots, (2r-2)^{n_{r-1}}\}$.

Again, in this case a same result is obtained as well.

$$\xi(T) - \xi(L(T)) = 2r - (r-1) + \sum_{i=1}^{r-1} n_i = r + 1 + n - 2 = n + r - 1. \qquad \square$$

2.2 Sum

Theorem 2.4 *Let G_1 and G_2 be simple connected graphs. Then,*

$$es(G_1 + G_2) = \{1^{c_1+c_2}, 2^{n_1+n_2-c_1-c_2}\},$$

where c_i is the number of vertices of eccentricity 1 *in G_i and n_i is the number of vertices of G_i; $i = 1, 2$.*

Proof It is not difficult to see that $\mathrm{diam}(\mathrm{G}_1 + \mathrm{G}_2) \leqslant 2$. For vertex $x \in V(G_i)$ we have $\varepsilon_{G_1+G_2}(x) = 1$ if and only if $\varepsilon_{G_i}(x) = 1$. Let c_i denotes the number vertices of eccentricity 1 in G_i, $i = 1, 2$. Then the eccentric sequence of $G_1 + G_2$ is obtained as $es(G_1 + G_2) = \{1^{c_1+c_2}, 2^{n_1+n_2-c_1-c_2}\}$. □

The eccentric sequence of sum of more than two graphs can be obtained by a reasoning similar to the above.

Corollary 2.5 *Let $G_1, G_2, \cdots, G_k$ be simple connected graphs. Then*

$$es(G_1 + G_2 + \cdots + G_k) = \{1^{\sum_{i=1}^{k} c_i}, 2^{\sum_{i=1}^{k} n_i - c_i}\},$$

where c_i is the number of vertices of eccentricity 1 *(or* 0*) in G_i and n_i is the number of vertices of G_i; $i = 1, 2, \cdots n$.*

Corollary 2.6 *For any integer $n \geqslant 5$, there is a self centered graph and non vertex transitive of diameter 2 and order n.*

Proof It is sufficient to consider the complete bipartite graph $K_{2,n-2} = \bar{K}_2 + \bar{K}_{n-2}$. □

2.3 Cartesian Product

Theorem 2.7 *Let $es(G_1) = \{{\varepsilon_1}^{t_1}, {\varepsilon_2}^{t_2}, \cdots, {\varepsilon_k}^{t_k}\}$ and $es(G_2) = \{{\delta_1}^{s_1}, {\delta_2}^{s_2}, \cdots, {\delta_m}^{s_m}\}$. Then,*

$$es(G_1 \Box G_2) = \left\{(\varepsilon_i + \delta_j)^{t_i s_j}\right\}_{1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant m.}$$

Proof It is known that $d_{G_1 \Box G_2}((x, y), (u, v)) = d_{G_1}(x, u) + d_{G_2}(y, v)$, this implies that $\varepsilon_{G_1 \Box G_2}(x, y) = \varepsilon_{G_1}(x) + \varepsilon_{G_2}(y)$.

Let m_i and n_j be the number of vertices of eccentricity ε_i and δ_j in G_1 and G_2 respectively. Then, $m_i n_j$ vertices of $G_1 \Box G_2$ have eccentricity $\varepsilon_i + \delta_j$. Therefore $es(G_1 \Box G_2) = \left\{(\varepsilon_i + \delta_j)^{t_i s_j}\right\}_{1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant m}$. □

Corollary 2.8 *There are infinite family of non-vertex transitive self centered graph.*

Proof It is sufficient to consider the powers of a non-vertex transitive self centered graph such as $K_{m,n}$ where $m \neq n$ and $m, n \geqslant 2$. □

2.4 Disjunction

First note that if $G = K_1$ then $G \vee H \cong H$ and $G \oplus H \cong H$ as well. Therefore the considered graph for these two graph products are except K_1.

Theorem 2.9 *Let $G_1 \neq K_1 \neq G_2$. Then*

$$es(G_1 \vee G_2) = \{1^{c_1c_2}, 2^{n_1n_2-c_1c_2}\}$$

where c_i is the number of vertices of eccentricity 1 in G_i and n_i is the number of vertices of G_i; $i = 1, 2$.

Proof Let (x, y) and (u, v) be two vertices of $G_1 \vee G_2$ and $xx' \in E(G_1)$ and $vv' \in E(G_2)$. Since (x, y) and (u, v) both are adjacent to (x', v') then $d((x, y), (u, v)) \leqslant 2$. Therefore for each vertex $(x, y) \in G_1 \vee G_2$, we have $\varepsilon_{G_1 \vee G_2}(x, y) \leqslant 2$. If $\varepsilon_{G_1}(x) > 1$, and $d_{G_1}(x, y) \geqslant 2$ then $d((x, u), (y, u)) = 2$. Hence, $\varepsilon_{G_1 \vee G_2}(x, y) = 1$ if and only if $\varepsilon_{G_1}(x) = 1 = \varepsilon_{G_2}(y)$. Let c_i vertices of G_i are of eccentricity 1 for $i = 1, 2$. Then c_1c_2 vertices of $G_1 \vee G_2$ have eccentricity 1 and the other vertices are of eccentricity 2. This implies that $es(G_1 \vee G_2) = \{1^{c_1c_2}, 2^{n_1n_2-c_1c_2}\}$. □

Corollary 2.10 *Let G_1 and G_2 be two graphs of radius at least 2. Then $G_1 \vee G_2$ is a self centered graph and $es(G_1 \vee G_2) = \{2^{n_1n_2}\}$.*

2.5 Symmetric Difference

Lemma 2.11([9]) *Let G_1 and G_2 be two simple connected graphs. The number of vertices of G_i is denoted by n_i for $i = 1, 2$. Then $deg_{G_1 \oplus G_2}((u, v)) = n_2 deg_{G_1}(u) + n_1 deg_{G_2}(v) - 2deg_{G_1}(u)deg_{G_2}(v)$.*

Theorem 2.12 *Let $G_1 \neq K_1 \neq G_2$. Then $es(G_1 \oplus G_2) = \{2^{n_1n_2}\}$.*

Proof Let (x, y) and (u, v) be two vertices of $G_1 \vee G_2$ and $xx' \in E(G_1)$ and $vv' \in E(G_2)$. Since (x, y) and (u, v) are adjacent to (x, v') then $d((x, y), (u, v)) \leqslant 2$. On the other hand, (x, y) and (x', v') are not adjacent in $G_1 \oplus G_2$. Therefore each vertex $(x, y) \in G_1 \oplus G_2$ we have $\varepsilon_{G_1 \oplus G_2}(x, y) = 2$. This concludes that $es(G_1 \oplus G_2) = \{2^{n_1n_2}\}$. □

Corollary 2.13 *For any positive integer d there is an infinite family of non vertex transitive graphs which are self centered with diameter d.*

Proof Let $\oplus_{i=1}^{k} G_i = G_1 \oplus G_2 \oplus \cdots \oplus G_k$. For any positive integers $n, k \geqslant 3$, let $G_{n,k} = \oplus_{i=1}^{k} P_n$. Using Lemma 2.11 and Theorem 2.12 we get that $G_{n,k}$ is a self centered graph of diameter 2 and it is non vertex transitive because it is not regular. Now consider the graph $H_{n,k,d} = G_{n,k} \Box C_{2(d-2)}$ which is self center graph of diameter d. Since $G_{n,k}$ is not regular graph then $H_{n,k,d}$ is not regular and consequently is not vertex transitive graph as well. Clearly diameter of $H_{n,k,d}$ is d and the proof is completed. □

2.6 Lexicographic Product

For the lexicographic product of graphs, the distance of pair vertices is determined by the following lemma.

Lemma 2.14([9]) *The distance of pair vertices in* $G_1[G_2]$ *is*

$$d_{G_1[G_2]}\left((u_1,v_1),(u_2,v_2)\right)=\begin{cases} d_{G_1}(u_1,u_2) & v_1=v_2 \\ 1 & u_1=u_2, v_1v_2\in E(G_2) \\ 2 & otherwise \end{cases}$$

Therefore, the eccentricity of vertex $(u,v)\in V(G_1[G_2])$ is determined by

$$\varepsilon(u,v)=\begin{cases} 1 & \text{if } \varepsilon_{G_1}(u)=\varepsilon_{G_2}(v)=1 \\ 2 & \text{if } \varepsilon_{G_1}(u)=1 \ \text{ and } \varepsilon_{G_2}(v)\geqslant 2 \\ \varepsilon_{G_1}(u) & \text{if } \varepsilon(u)\geqslant 2 \end{cases}$$

Now, all conditions are ready to obtain the eccentric sequence of $G_1[G_2]$.

Theorem 2.15 *Let* $es(G_1)=\{\varepsilon_1^{t_1},\ldots,\varepsilon_k^{t_k}\}$ *and* n_i *and* c_i, $i=1,2$ *be the order and the number of vertices of eccentricity* 1 *of* G_i *respectively. Then*

$$es(G_1[G_2])=\{1^{c_1c_2},2^{c_1(n_2-c_2)+t_2n_2},\varepsilon_3^{t_3n_2},\cdots,\varepsilon_k^{t_kn_2}\}.$$

2.7 Corona

Remark 2.16 If $G_1=K_1$, then $G_1oG_2=K_1+G_2$ and

$$es(G_1oG_2)=\{1^{1+c_2},2^{n_2-c_2}\}$$

.

Theorem 2.17 *Let* $G_1\neq K_1$ *and* $es(G_1)=\left\{\varepsilon_i^{t_i}\right\}_{i=1}^{k}$. *Then*

$$es(G_1oG_2)=\left\{\varepsilon_2^{t_1},\varepsilon_3^{t_2+n_2t_1},\varepsilon_4^{t_3+n_2t_2},\cdots,\varepsilon_{k+1}^{t_k+n_2t_{k-1}},\varepsilon_{k+2}^{n_2t_k}\right\},$$

where $n_2=|V(G_2)|$ *and* $\varepsilon_{i+1}=\varepsilon_i+1$.

Proof Let $V(G_1)=\{v_1,v_2,\ldots,v_n\}$ and $G_{2,i}$ be the copy of G_2 associated to v_i. From the structure of corona product of graphs one can see that $\varepsilon_{G_1oG_2}(v_i)=\varepsilon_{G_1}(v_i)+1$ and if $x\in V(G_{2,i})$, $\varepsilon_{G_1oG_2}(x)=\varepsilon_{G_1}(v_i)+2, 1\leqslant i\leqslant n$. Then for $x\in V(G_1oG_2)$,

$$\varepsilon_2=\varepsilon_1+1\leqslant\varepsilon(x)\leqslant\varepsilon_k+2=\varepsilon_{k+2}.$$

Let v be a central vertex of G_1 and $\varepsilon_{G_1}(v)=\varepsilon_1$, then $\varepsilon_{G_1oG_2}(v)=\varepsilon_1+1=\varepsilon_2$. This follows that center of G_1 coincides center of G_1oG_2. For $i\geqslant 2$, the set of vertices of G_1 having eccentricity ε_i and the vertices of $G_{2,t}$ which $\varepsilon(V_t)=\varepsilon_{i-1}$ are of eccentricity ε_{i+1} in G_1oG_2. Thus for $i\geqslant 2$, the number of vertices that have eccentricity ε_{i+1} is $t_i+n_2t_{i-1}$. The proof is completed. □

§3. Examples and Concluding Remarks

In this section, our theorems for eccentric sequence are illustrated for several more particular composite graphs. We first give the expressions for suspensions.

Corollary 2.18 *Let G be a graph on n vertices. Then*

$$es(K_1 + G) = \{1^{c+1}, 2^{n-c}\},$$

where c is the number of vertices of eccentricity 1 in G.

Next, the eccentric sequence for the fan graph $K_1 + P_n$ and the wheel graph $W_n = K_1 + C_n$ are presented by

Corollary 2.19 *For the fan graph $K_1 + P_n$ and the wheel graph $W_n = K_1 + C_n$,*

$$es(K_1 + P_n) = \begin{cases} \{1^2\} & \textit{if } n = 1, \\ \{1^3\} & \textit{if } n = 2, \\ \{1^2, 2^2\} & \textit{if } n = 3, \\ \{1^1, 2^n\} & \textit{if } n \geqslant 4 \end{cases}$$

and

$$es(W_n) = \begin{cases} \{1^4\} & \textit{if } n = 3 \\ \{1^1, 2^n\} & \textit{if } n \geqslant 4. \end{cases}$$

By composing paths and cycles with various small graphs, we can obtain different classes of polymer like graphs. For example, we state the eccentric sequence for the fence graph $P_n[K_2]$ and the closed fence $C_n[K_2]$ in the following conclusion.

Corollary 2.20 *For the fence graph $P_n[K_2]$ and the closed fence $C_n[K_2]$,*

$$es(P_n[K_2]) = \begin{cases} \{1^2\} & \textit{if } n = 1, \\ \{1^4\} & \textit{if } n = 2, \\ \{1^2, 2^4\} & \textit{if } n = 3, \\ \{2^{2t_2}, 3^{2t_3}, \cdots k^{2t_k}\} & \textit{if } n \geqslant 4, \end{cases}$$

where $es(P_n) = \{2^{t_2}, 3^{t_3}, \cdots k^{t_k}\}$ for $n \geqslant 4$ and

$$es(C_n[K_2]) = \begin{cases} \{1^6\} & \textit{if } n = 3, \\ \{[\frac{n}{2}]^{2n}\} & \textit{if } n \geqslant 4. \end{cases}$$

The t-thorny graph of a given graph G is obtained as $Go\bar{K_n}$, where $\bar{K_n}$ denotes the empty graph on n vertices. For the t-thorny path and t-thorny cycle we get the following eccentric sequence.

Corollary 2.21 *For the t-thorny graph $P_n o\bar{K}_t$,*

$$es(P_n o\bar{K}_t) = \begin{cases} \{1^1, 2^t\} & n = 1, \\ \{2^2, 3^{2t}\} & n = 2, \\ \{2^1, 3^{t+2}, 4^{2t}\} & n = 3. \end{cases}$$

If $n \geqslant 4$ and it is even

$$es(P_n o\bar{K}_t) = \{(\frac{n}{2}+1)^2, (\frac{n}{2}+2)^{2t+2}, \cdots, n^{2t+2}, (n+1)^{2t}\}$$

and if $n \geqslant 5$ and it is odd then

$$es(P_n o\bar{K}_t) = \{(\frac{n+1}{2})^1, (\frac{n+1}{2}+1)^{t+2}, (\frac{n+1}{2}+2)^{2t+2}, \cdots, n^{2t+2}, (n+1)^{2t}\}.$$

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F-Centroidal Mean Labeling of Graphs Obtained From Paths

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Abstract: A function f is called an F-centroidal mean labeling of a graph $G(V,E)$ with p vertices and q edges if $f : V(G) \rightarrow \{1,2,3,\cdots,q+1\}$ is injective and the induced function $f^* : E(G) \rightarrow \{1,2,3,\cdots,q\}$ defined as

$$f^*(uv) = \left\lfloor \frac{2\left[f(u)^2 + f(u)f(v) + f(v)^2\right]}{3\left[f(u)+f(v)\right]} \right\rfloor,$$

for all $uv \in E(G)$, is bijective. A graph that admits an F-centroidal mean labeling is called an F-centroidal mean graph. In this paper, we have discussed the F-centroidal meanness of the graph $P_n(X_1, X_2, \cdots X_n)$, the twig graph $TW(P_n)$, the graph $P_n \circ S_m$ for $m \leq 4$, planar grid $P_m \times P_n$ for $m \leq 3$, the ladder graph L_n, the graph $P_n \circ K_2$, the graph P_a^b for $a \geq 2$ and $b \leq 3$, the middle graph of the path, total graph of the path and the square graph of the path, the splitting graph of the path and the graph $P(1,2,\cdots,n-1)$.

Key Words: Labeling, F-centroidal mean labeling, F-centroidal mean graph, Smarandachely F-centroidal mean labeling.

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§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected and simple graph. Let $G(V,E)$ be a graph with p vertices and q edges. For notations and terminology, we follow [8]. For a detailed survey on graph labeling, we refer [7].

Path on n vertices is denoted by P_n. The graph $P_n(X_1, X_2, \ldots X_n)$, is a tree obtained from a path on n vertices by attaching X_i pendent vertices at each i^{th} vertex of the path, for $1 \leq i \leq n$. A Twig $TW(P_n), n \geq 4$ is a graph obtained from a path by attaching exactly two pendant vertices to each internal vertices of the path P_n. The graph $G \circ S_m$ is obtained from G by attaching m pendant vertices to each vertex of G. Let G_1 and G_2 be any two graphs with p_1 and p_2 vertices respectively. Then the Cartesian product $G_1 \times G_2$ has p_1p_2 vertices which are $\{(u,v) : u \in G_1, v \in G_2\}$ and the edges are obtained as follows: (u_1,v_1) and (u_2,v_2) are adjacent in $G_1 \times G_2$ if either $u_1 = u_2$ and v_1 and v_2 are adjacent in G_2 or u_1 and u_2 are adjacent

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in G_1 and $v_1 = v_2$. The product $P_m \times P_n$ and is called a planar grid and $P_2 \times P_n$ is called a ladder, denoted by L_n.

Let a and b be integers such that $a \geq 2$ and $b \geq 2$. Let $y_1, y_2, \ldots y_a$ be the $'a'$ fixed vertices. Connect y_i and y_{i+1} by means of b internally disjoint paths P_i^j of length $'i+1'$ each, for $1 \leq i \leq a-1$ and $1 \leq j \leq b$. The resulting graph embedded in the plane is denoted by P_a^b. The middle graph $M(G)$ of a graph G is the graph whose vertex set is $\{v : v \in V(G)\} \cup \{e : e \in E(G)\}$ and the edge set is $\{e_1e_2 : e_1, e_2 \in E(G)$ and e_1 and e_2 are adjacent edges of $G\} \cup \{ve : v \in V(G), e \in E(G)$ and e is incident with $v\}$. The total graph $T(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent if and only if either they are adjacent vertices of G or adjacent edges of G or one is a vertex of G and the other one is an edge incident on it. Square of a graph G, denoted by G^2, has the vertex set as in G and two vertices are adjacent in G^2 if they are at a distance either 1 or 2 apart in G. For each vertex v of the graph G, take a new vertex v' to these vertices of G adjacent to v. The graph thus obtained is called the splitting graph G and it is denoted by $S'(G)$. An arbitrary super subdivision $P(m_1, m_2, \cdots, m_{n-1})$ of a path P_n is a graph obtained by replacing each i^{th} edge of P_n by identifying its end vertices of the edge with a partition of K_{2,m_i} having 2 elements, where m_i is any positive integer.

Durai Baskar and Arockiaraj defined the F-harmonic mean labeling [6] and discussed its meanness of some standard graphs. The concept of F-geometric mean labeling was introduced by Durai Baskar and Arockiaraj [5] and it was developed [4]. The concept of F-root square mean labeling was introduced by Arockiaraj et al., [1] and they studied the F-root square mean labeling of some standard graphs [2]. Durai Baskar and Manivannan were introduced F-heronian mean labeling [3]. Motivated by the works of so many authors in the area of graph labeling, we introduced a new type of labeling called an F-centroidal mean labeling.

A function f is called an F-centroidal mean labeling of a graph $G(V, E)$ with p vertices and q edges if $f : V(G) \to \{1, 2, 3, \cdots, q+1\}$ is injective and the induced function $f^* : E(G) \to \{1, 2, 3, \cdots, q\}$ defined by

$$f^*(uv) = \left\lfloor \frac{2\,[f(u)^2 + f(u)f(v) + f(v)^2]}{3\,[f(u) + f(v)]} \right\rfloor,$$

for all $uv \in E(G)$, is bijective. Otherwise, it is called a Smarandachely F-centroidal mean labeling of G if there is a number $k \in \{1, 2, 3, \cdots, q\}$ such that the inverse f^{-*} of f^* holds with $|f^{-*}(k)| \geq 2$. A graph that admits an F-centroidal mean labeling is called an F-centroidal mean graph.

An F-centroidal mean labeling of cycle C_4 is given in Figure 1.

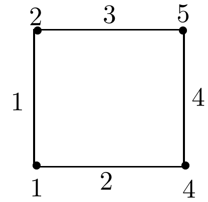

Figure 1 An F-centroidal mean labeling labeling of C_4

In this paper, we have discussed the F-centroidal meanness of the graph $P_n(X_1, X_2, \cdots X_n)$, the twig graph $TW(P_n)$, the graph $P_n \circ S_m$ for $m \leq 4$, planar grid $P_m \times P_n$ for $m \leq 3$, the ladder graph L_n, the graph $P_n \circ K_2$, the graph P_a^b for $a \geq 2$ and $b \leq 3$, the middle graph of the path, total graph of the path and the square graph of the path, the splitting graph of the path and the graph $P(1, 2, \cdots, n-1)$.

§2. Main Results

Theorem 2.1 *The graph $P_n(X_1, X_2, \cdots X_n)$ is an F-centroidal mean graph, for $1 \leq X_i \leq 3$ and $|X_i - X_{i+1}| \leq 1$, for $1 \leq i \leq n-1$.*

Proof Let $u_1, u_2, \cdots, u_n$ be the vertices of the path P_n. Let $v_i^{(1)}, v_i^{(2)}, \cdots, v_i^{(X_i)}$ be the pendant vertices attached at u_i, for $1 \leq i \leq n$.

Define $f : V(P_n(X_1, X_2, \cdots X_n)) \rightarrow \{1, 2, 3, \cdots, \sum\limits_{i=1}^{n} X_i + n\}$ as follows:

$$f(v_i^{(1)}) = \begin{cases} 2, & X_1 = 1, \\ 1, & X_1 \neq 1. \end{cases}$$

For $2 \leq i \leq n$,

$$f(v_i^{(1)}) = \begin{cases} \sum\limits_{k=1}^{i-1} X_k + i, & X_i = 2, 3, \\ \sum\limits_{k=1}^{i-1} X_k + i + 1, & X_i = 1. \end{cases}$$

For $1 \leq i \leq n$,

$$f(v_i^{(j)}) = \begin{cases} f(v_i^{(1)}) + 2, & j = 2 \\ f(v_i^{(1)}) + 3, & X_i = 3 \text{ and } j = 3 \end{cases}$$

and

$$f(u_i) = \begin{cases} f(v_i^{(1)}) + 1, & X_i = 2, 3, \\ f(v_i^{(1)}) - 1, & X_i = 1. \end{cases}$$

Then the induced edge labeling f^* is obtained as follows:

For $1 \leq i \leq n-1$,

$$f^*(u_i u_{i+1}) = \begin{cases} f(u_i) + 1, & X_i = 1, 2, \\ f(u_i) + 2, & X_i = 3 \end{cases}$$

and $f^*(v_1^{(1)} u_1) = 1$.

For $1 \leq i \leq n$,

$$f^*(v_i^{(1)}u_i) = \begin{cases} f(v_i^{(1)}) + 1, & X_i = 2, 3, \\ f(v_i^{(1)}) - 1, & X_i = 1 \end{cases}$$

and

$$f^*(v_i^{(j)}u_i) = \begin{cases} f(u_i), & X_i = 2, 3 \text{ and } j = 2, \\ f(u_i) + 1, & X_i = 3 \text{ and } j = 3. \end{cases}$$

Hence f is an F-centroidal mean labeling of $P_n(X_1, X_2, \cdots X_n)$. Thus the graph $P_n(X_1, X_2, \cdots X_n)$ is an F-centroidal mean graph, for $1 \leq X_i \leq 3$ and $|X_i - X_{i+1}| \leq 1$, for $1 \leq i \leq n-1$. □

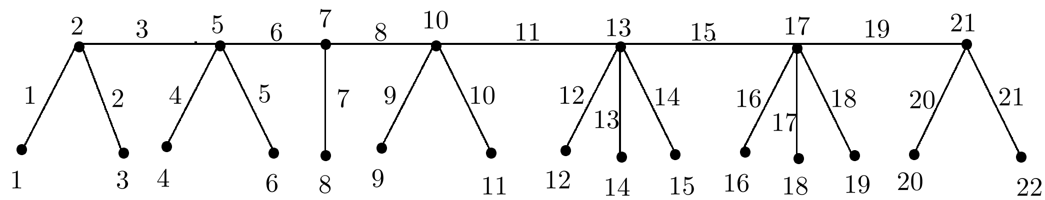

Figure 2 An F-centroidal mean labeling of $P_n(2, 2, 1, 2, 3, 3, 2)$

Corollary 2.2 *The twig graph $TW(P_n)$ of the path P_n is an F-centroidal mean graph, for $n \geq 4$.*

Theorem 2.3 *The graph $P_n \circ S_m$ is an F-centroidal mean graph, for $n \geq 1$ and $m \leq 4$.*

Proof Let $v_1, v_2, v_3, \cdots, v_n$ be the vertices of the path P_n and $u_1^{(i)}, u_2^{(i)}, u_3^{(i)}, \cdots, u_m^{(i)}$ be the pendant vertices at each v_i, for $1 \leq i \leq n$.

Case 1. $m = 4$.

Define $f : V(P_n \circ S_4) \to \{1, 2, 3, \ldots, 5n\}$ as follows:

$$\begin{aligned} f(v_1) &= 2, \\ f(v_i) &= 5i - 2, \text{ for } 1 \leq i \leq n, \\ f(u_1^{(1)}) &= 1, \\ f(u_1^{(i)}) &= 5i - 5, \text{ for } 2 \leq i \leq n, \\ f(u_2^{(1)}) &= 3, \\ f(u_2^{(i)}) &= 5i - 3, \text{ for } 2 \leq i \leq n, \\ f(u_3^{(i)}) &= 5i - 1, \text{ for } 1 \leq i \leq n, \\ f(u_4^{(i)}) &= 5i + 1, \text{ for } 1 \leq i \leq n-1, \\ f(u_4^{(n)}) &= 5n. \end{aligned}$$

Then the induced edge labeling f^* is obtained as follows:

$$f^*(v_iv_{i+1}) = 5i, \text{ for } 1 \le i \le n-1,$$
$$f^*(v_iu_1^{(i)}) = 5i-4, \text{ for } 1 \le i \le n,$$
$$f^*(v_iu_2^{(i)}) = 5i-3, \text{ for } 1 \le i \le n,$$
$$f^*(v_iu_3^{(i)}) = 5i-2, \text{ for } 1 \le i \le n$$
$$f^*(v_iu_4^{(i)}) = 5i-1, \text{ for } 1 \le i \le n.$$

Case 2. $1 \le m \le 3$.

By Theorem 2.1, the results follows in this case.

Hence f is an F-centroidal mean labeling of $P_n \circ S_m$, for $n \ge 1$ and $m \le 4$. Thus the graph $P_n \circ S_m$ is an F-centroidal mean graph, for $n \ge 1$ and $m \le 4$. □

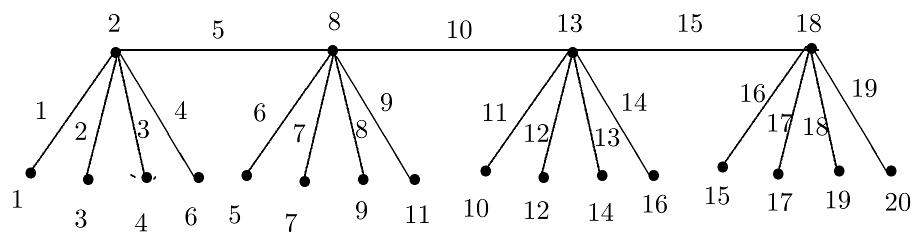

Figure 3 An F-centroidal mean labeling of $P_4 \circ S_4$

Theorem 2.4 *The planar grid $P_m \times P_n$, is an F-centroidal mean graph, for $m \le 3$ and $n \ge 2$.*

Proof Let $V(P_m \times P_n) = \{v_{ij} : 1 \le i \le m,\ 1 \le j \le n\}$ and $E(P_m \times P_n) = \{v_{ij}v_{(i+1)j} : 1 \le i \le m-1,\ 1 \le j \le n\} \cup \{v_{ij}v_{i(j+1)} : 1 \le i \le m,\ 1 \le j \le n-1\}$ be the vertex set and edge set of the graph $P_m \times P_n$.

Case 1. $m = 2$.

Define $f : V(P_2 \times P_n) \to \{1, 2, 3, \cdots, 3n-1\}$ as follows:

$$f(v_{ij}) = i + 3(j-1), \text{ for } 1 \le i \le 2 \text{ and } 1 \le j \le n.$$

Then the induced edge labeling f^* is obtained as follows:

$$f^*(v_{1j}v_{2j}) = 3j-2, \text{ for } 1 \le i \le n,$$
$$f^*(v_{ij}v_{i(j+1)}) = i + 3j - 2, \text{ for } 1 \le i \le 2 \text{ and } 1 \le j \le n-1.$$

Case 2. $m = 3$.

Define $f : V(P_3 \times P_n) \to \{1, 2, 3, \cdots, 5n-2\}$ as follows:

$$f(v_{i1}) = i, \text{ for } 1 \le i \le 3,$$

$$f(v_{i2}) = \begin{cases} i+4, & i = 1, \\ i+5, & 2 \le i \le 3, \end{cases}$$

$$f(v_{ij}) = i + 5(j-1), \text{ for } 1 \le i \le 3 \text{ and } 3 \le j \le n.$$

Then the induced edge labeling f^* is obtained as follows:

$$f^*(v_{i1}v_{(i+1)1}) = i, \text{ for } 1 \le i \le 2,$$

$$f^*(v_{i1}v_{i2}) = i + 2, \text{ for } 1 \le i \le 3,$$

$$f^*(v_{ij}v_{(i+1)j}) = \begin{cases} 2i + 3(j-1), & 1 \le i \le 2 \text{ and } j = 2, \\ i + 5(j-1), & 1 \le i \le 2 \text{ and } 3 \le j \le n, \end{cases}$$

$$f^*(v_{ij}v_{i(j+1)}) = i + 5j - 3, \text{ for } 1 \le i \le 3 \text{ and } 2 \le j \le n-1.$$

Hence the graph $P_m \times P_n$ admits an F-centroidal mean labeling. Thus the graph $P_m \times P_n$ is an F-centroidal meangraph for $m \le 3$.

For $n = 2$, an F-centroidal mean labeling of $P_2 \times P_4$ is as shown in Figure 4. □

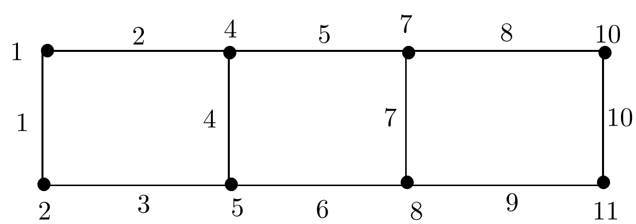

Figure 4 An F-centroidal mean labeling of $P_2 \times P_4$

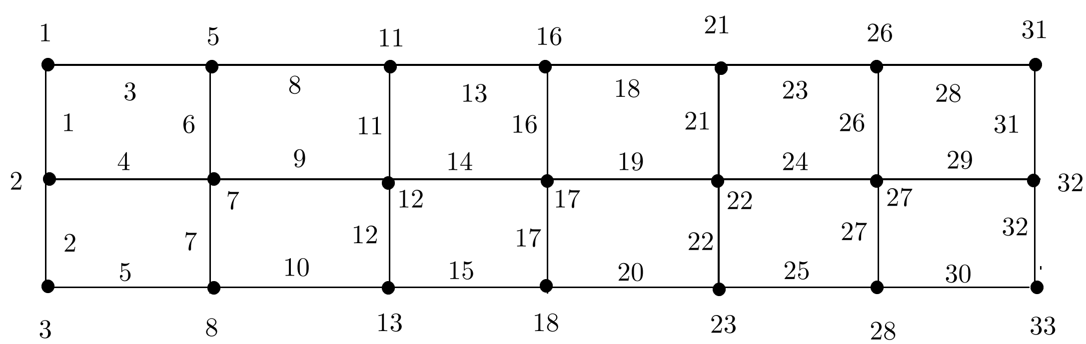

Figure 5 An F-centroidal mean labeling of $P_3 \times P_7$

Corollary 2.5 *Every Ladder graph $L_n = P_2 \times P_n$ is an F-centroidal mean graph for $n \ge 2$.*

Theorem 2.6 *The graph $P_n \circ K_2$ is an F-centroidal mean graph for $n \ge 1$.*

Proof Let $v_1, v_2, v_3, \cdots, v_n$ be the vertices of the path P_n and $u_i^{(1)}, u_i^{(2)}$ be the vertices of i^{th} copy of K_2 attached with v_i, for $1 \le i \le n$. Define $f : V(P_n \circ K_2) \to \{1, 2, 3, \cdots, 4n\}$ as follows:

$$\begin{aligned} f(v_i) &= 4i - 2, \text{ for } 1 \le i \le n, \\ f(u_i^{(1)}) &= 4i - 3, \text{ for } 1 \le i \le n, \\ f(u_i^{(2)}) &= 4i, \text{ for } 1 \le i \le n. \end{aligned}$$

Then the induced edge labeling f^* is obtained as follows:

$$\begin{aligned} f^*(v_i v_{i+1}) &= 4i, \text{ for } 1 \le i \le n-1, \\ f^*(u_i^{(1)} u_i^{(2)}) &= 4i - 2, \text{ for } 1 \le i \le n, \\ f^*(u_i^{(1)} v_1) &= 4i - 3, \text{ for } 1 \le i \le n, \\ f^*(u_i^{(2)} v_i) &= 4i - 1, \text{ for } 1 \le i \le n. \end{aligned}$$

Hence f is an F-centroidal mean labeling of $P_n \circ K_2$ for $n \ge 1$. Thus the graph $P_n \circ K_2$ is an F-centroidal mean graph, for $n \ge 1$. □

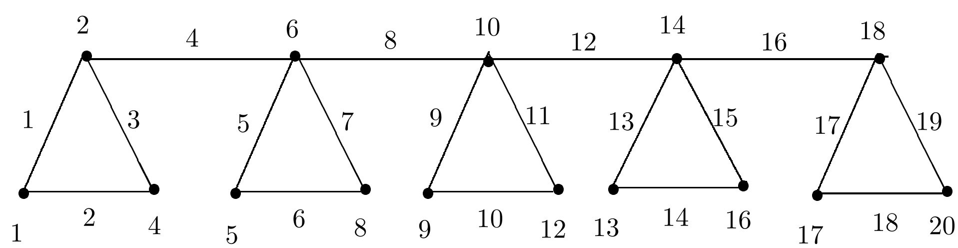

Figure 6 An F-centroidal mean labeling of $P_5 \circ K_2$

Theorem 2.7 *The graph P_a^b is an F-centroidal mean graph, for $a \ge 2$ and $b \le 3$.*

Proof Let $y_i, x_{ij1}, x_{ij2}, \ldots x_{iji}, y_{i+1}$ be the vertices of the path P_i^j, where $1 \le i \le a-1$ and $1 \le j \le b$. Let $V(P_a^b) = \{y_i : 1 \le i \le a\} \cup \bigcup_{i=1}^{a-1} \bigcup_{j=1}^{b} \{x_{ijk} : 1 \le k \le i\}$ and $E(P_a^b) = \bigcup_{i=1}^{a-1} \{y_i x_{ij1} : 1 \le i \le b\} \cup \bigcup_{i=1}^{a-1} \bigcup_{j=1}^{b} \{x_{ijk} x_{ij(k+1)} : 1 \le k \le i-1\} \cup \bigcup_{i=1}^{a-1} \{x_{iji} y_{i+1} : 1 \le j \le b\}$ be the vertex set and edge set of the graph P_a^b.

Case 1. $b = 2$.

Define $f : V(P_a^2) \to \{1, 2, 3, \cdots, (a-1)(a+2)+1\}$ as follows:

$$\begin{aligned} f(y_1) &= 1, \\ f(y_i) &= (i-1)(i+2) + 1, \text{ for } 2 \le i \le a, \\ f(x_{1j1}) &= j + 1, \text{ for } 1 \le j \le 2 \text{ and for } 2 \le i \le a-1, \end{aligned}$$

$$f(x_{ijk}) = (i-1)(i+2) + 2k + j - 1, \text{ for } 1 \le k \le i \text{ and } 1 \le j \le 2.$$

Then the induced edge labeling f^* is obtained as follows:

$$\begin{aligned} f^*(y_1 x_{1j1}) &= j, \text{ for } 1 \le j \le 2, \\ f^*(x_{1j1} y_2) &= j + 2, \text{ for } 1 \le j \le 2, \\ f^*(y_i x_{ij1}) &= (i-1)(i+2) + j, \text{ for } 2 \le i \le a-1 \text{ and } 1 \le j \le 2, \\ f^*(x_{ijk} x_{ij(k+1)}) &= (i-1)(i+2) + j + 2k, \text{ for } 2 \le i \le a-1, \ 1 \le k \le i-1 \text{ and } 1 \le j \le 2, \\ f^*(x_{iji} y_{i+1}) &= i(i+3) + j - 2, \text{ for } 2 \le i \le a-1 \text{ and } 1 \le j \le 2. \end{aligned}$$

Case 2. $b = 3$.

Define $f : V(P_a^3) \to \{1, 2, 3, \cdots, \frac{3(a-1)(a+2)}{2} + 1\}$ as follows:

$$f(y_1) = 1, \ f(y_2) = 5, \ f(x_{111}) = 2, \ f(x_{1j1}) = 4j - 5, \text{ for } 2 \le j \le 3,$$

$$f(y_i) = \frac{3(i-1)(i+2)}{2} + 1, \text{ for } 3 \le i \le a, \ f(x_{21k}) = \begin{cases} 4k+5, & k = 1, \\ 4k+4, & k = 2, \end{cases}$$

$$f(x_{22k}) = \begin{cases} 7k+6, & k = 1, \\ 7k-4, & k = 2, \end{cases} \quad f(x_{23k}) = \begin{cases} 5k+6, & k = 1, \\ 5k+4, & k = 2. \end{cases}$$

For $3 \le i \le a-1$,

$$f(x_{ij1}) = \begin{cases} \frac{3(i-1)(i+2)}{2} + j + 1, & 1 \le j \le 2, \\ \frac{3(i-1)(i+2)}{2} + 2j, & j = 3 \ \text{ and} \end{cases}$$

$$f(x_{ijk}) = \begin{cases} \frac{3(i-1)(i+2)}{2} + 2j + 3k - 1, & 1 \le j \le 2, 2 \le k \le i-1, \text{ and } k \text{ is even}, \\ \frac{3(i-1)(i+2)}{2} + 3k - 2, & j = 3, 2 \le k \le i-1 \text{ and } k \text{ is even}, \\ \frac{3(i-1)(i+2)}{2} + 2j + 3k - 3, & 1 \le j \le 3, 2 \le k \le i-1 \text{ and } k \text{ is odd}, \\ \frac{3(i-1)(i+2)}{2} + 3k - 1, & j = 1, k = i \text{ and } k \text{ is odd}, \\ \frac{3(i-1)(i+2)}{2} + 3k + j - 1, & j = 2, k = i \text{ and } k \text{ is odd}, \\ \frac{3(i-1)(i+2)}{2} + 3k + j, & j = 3, k = i \text{ and } k \text{ is odd}, \\ \frac{3(i-1)(i+2)}{2} + 3k + j, & 1 \le j \le 2, k = i, \text{ and } k \text{ is even}, \\ \frac{3(i-1)(i+2)}{2} + 3k - 1, & j = 3, k = i \text{ and } k \text{ is even}. \end{cases}$$

Then the induced edge labeling f^* is obtained as follows:

$$f^*(y_1x_{1j1}) = \begin{cases} j, & 1 \le j \le 2, \\ 5, & j = 3, \end{cases}$$

$$f^*(y_ix_{ij1}) = \begin{cases} \frac{3(i-1)(i+2)}{2} + j, & j = 1 \text{ and } 2 \le i \le a-1, \\ \frac{3(i-1)(i+2)}{2} + j + 1, & j = 2 \text{ and } i = 2, \\ \frac{3(i-1)(i+2)}{2} + j - 1, & j = 3 \text{ and } i = 2, \\ \frac{3(i-1)(i+2)}{2} + j, & j = 2, 3 \text{ and } 3 \le i \le a-1, \end{cases}$$

$$f^*(x_{1j1}y_2) = \begin{cases} 3, & j = 1, \\ 2j, & 2 \le j \le 3, \end{cases} \qquad f^*(x_{2j2}y_3) = \begin{cases} 14, & j = 1, \\ 13, & j = 2, \\ 15, & j = 3, \end{cases}$$

$$f^*(x_{2j1}x_{2j2}) = j + 9 \text{ for } 1 \le j \le 3 \text{ and } 3 \le i \le a-1,$$

$$f^*(x_{ijk}x_{ij(k+1)}) = \begin{cases} \frac{3(i-1)(i+2)}{2} + 3k + 2(j-1) + 1, & 1 \le k \le i-1, \\ & \text{and } 1 \le j \le 2, \\ \frac{3(i-1)(i+2)}{2} + 3k + 2, & 1 \le k \le i-1, \\ & \text{and } j = 3, \end{cases}$$

$$\text{and } f^*(x_{iji}y_{i+1}) = \begin{cases} \frac{3i(i+3)}{2} + j - 3, & 1 \le j \le 3 \text{ and } i \text{ is odd}, \\ \frac{3i(i+3)}{2} + j - 2, & 1 \le j \le 2 \text{ and } i \text{ is even}, \\ \frac{3i(i+3)}{2} - 2, & j = 3 \text{ and } i \text{ is even}. \end{cases}$$

Hence f is an F-centroidal mean labeling of P_a^b for $a \ge 2$ and $b \le 3$. Thus the graph P_a^b is an F-centroidal mean graph, for $a \ge 2$ and $b \le 3$. □

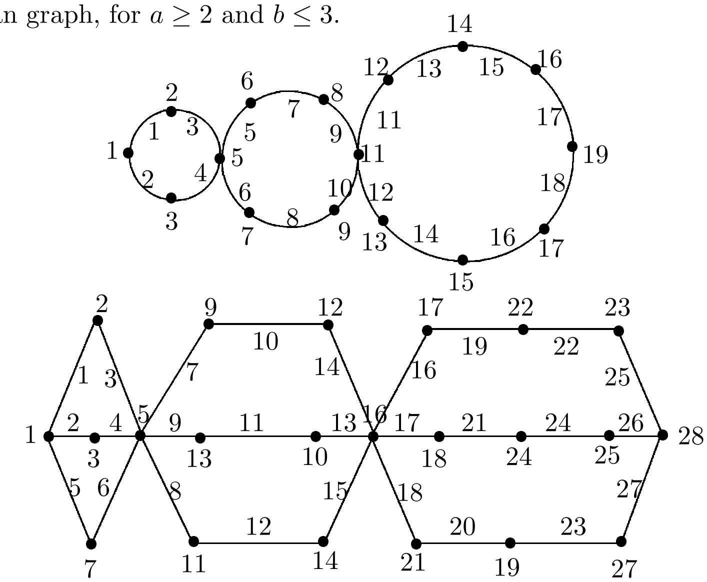

Figure 7 An F-centroidal mean labeling of P_4^2 and P_4^3

Theorem 2.8 *The middle graph $M(P_n)$ of a path P_n is an F-centroidal mean graph.*

Proof Let $V(P_n) = \{v_1, v_2, v_3, \cdots, v_n\}$ and $E(P_n) = \{e_i = v_iv_{i+1} : 1 \leq i \leq n-1\}$ be the vertex set and edge set of the path P_n. Then,

$$V(M(P_n)) = \{v_1, v_2, v_3, \ldots, v_n, e_1, e_2, e_3, \cdots, e_{n-1}\},$$
$$E(M(P_n)) = \{v_ie_i, e_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{e_ie_{i+1} : 1 \leq i \leq n-2\}.$$

Define $f : V(M(P_n)) \to \{1, 2, 3, \cdots, 3n-3\}$ as follows:

$$f(v_i) = \begin{cases} 1, & \text{for } i = 1, \\ 3i-3, & \text{for } 2 \leq i \leq n, \end{cases}$$
$$f(e_i) = 3i-1, \text{ for } 1 \leq i \leq n-1.$$

Then the induced edge labeling f^* is obtained as follows:

$$f^*(v_ie_i) = 3i-2, \text{ for } 1 \leq i \leq n-1,$$
$$f^*(e_iv_{i+1}) = 3i-1, \text{ for } 1 \leq i \leq n-1,$$
$$f^*(e_ie_{i+1}) = 3i, \text{ for } 1 \leq i \leq n-2.$$

Hence f is an F-centroidal mean labeling of the graph $M(P_n)$. Thus the graph $M(P_n)$ is an F-centroidal mean graph. □

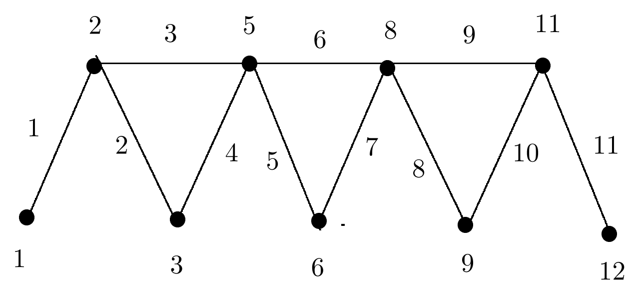

Figure 8 An F-centroidal mean labeling of $M(P_5)$

Theorem 2.9 *The total graph $T(P_n)$ of a path P_n is an F-centroidal mean graph for $n \geq 1$.*

Proof Let $V(P_n) = \{v_1, v_2, v_3, \cdots, v_n\}$ and $E(P_n) = \{e_i = v_iv_{i+1} : 1 \leq i \leq n-1\}$ be the vertex set and edge set of the path P_n. Then $V(T(P_n)) = \{v_1, v_2, v_3, \cdots, v_n, e_1, e_2, e_3, \cdots, e_{n-1}\}$ and $E(T(P_n)) = \{v_iv_{i+1}, e_iv_i, e_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{e_ie_{i+1} : 1 \leq i \leq n-2\}$.

Define $f : V(T(P_n)) \to \{1, 2, 3, \ldots, 4(n-1)\}$ as follows:

$$f(v_1) = 1,$$
$$f(v_i) = 4i-4, \text{ for } 2 \leq i \leq n,$$
$$f(e_i) = 4i-2, \text{ for } 1 \leq i \leq n-1.$$

Then the induced edge labeling f^* is obtained as follows:

$$f^*(v_iv_{i+1}) = 4i-2, \text{ for } 1 \le i \le n-1,$$
$$f^*(e_ie_{i+1}) = 4i, \text{ for } 1 \le i \le n-2,$$
$$f^*(v_ie_i) = 4i-3, \text{ for } 1 \le i \le n-1,$$
$$f^*(e_iv_{i+1}) = 4i-1, \text{ for } 1 \le i \le n-1.$$

Hence f is an F-centroidal mean labeling of the graph $T(P_n)$. Thus the graph $T(P_n)$ is an F-centroidal mean graph. □

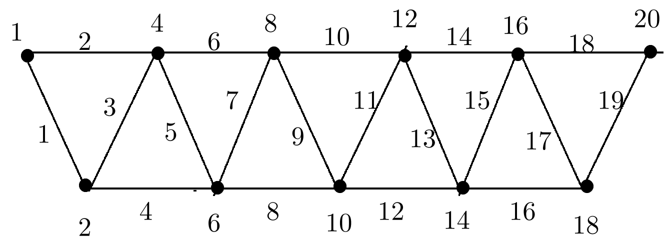

Figure 9 An F-centroidal mean labeling of $T(P_6)$

Theorem 2.10 *The square graph P_n^2 of the path P_n is an F-centroidal mean graph for $n \ge 1$.*

Proof Let $v_1, v_2, v_3, \cdots, v_n$ be the vertices of the path P_n. Define $f : V\left(P_n^2\right) \to \{1, 2, 3, \cdots, 2(n-1)\}$ as follows:

$$f(v_1) = 1,$$
$$f(v_i) = 2i-2, \text{ for } 2 \le i \le n.$$

Then the induced edge labeling f^* is obtained as follows:

$$f^*(v_iv_{i+1}) = 2i-1, \text{ for } 1 \le i \le n-1,$$
$$f^*(v_iv_{i+2}) = 2i, \text{ for } 1 \le i \le n-2.$$

Hence f is an F-centroidal mean labeling of the graph P_n^2. Thus the graph P_n^2 is an F-centroidal mean graph. □

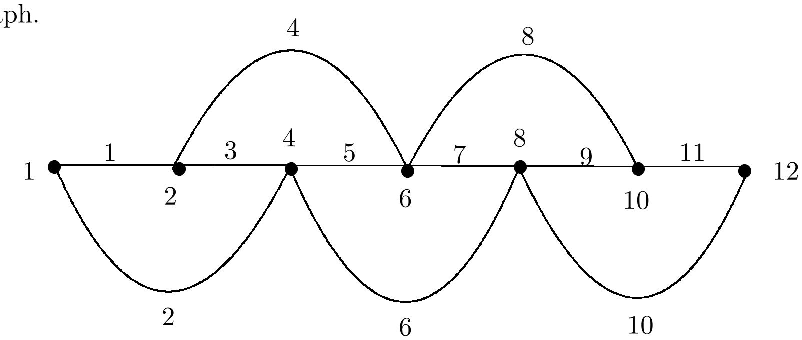

Figure 10 An F-centroidal mean labeling of P_7^2

Theorem 2.11 *The splitting graph $S'(P_n)$ is an F-centroidal mean graph for $n \geq 2$.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of the path P_n. Let $v_1, v_2, \cdots, v_n, v'_1, v'_2, \cdots, v'_n$ be the vertices of the graph $S'(P_n)$. Let $V(S'(P_n)) = \{v_i, v'_i : 1 \leq i \leq n\}$ and $E(S'(P_n)) = \{v_iv_{i+1}, v_iv'_{i+1}, v'_iv_{i+1} : 1 \leq i \leq n-1\}$ be the vertex set and edge set of the splitting graph $S'(P_n)$.

Case 1. n is odd.

Define $f : V(S'(P_n)) \to \{1, 2, 3, \cdots, 3n-2\}$ as follows:

$$f(v_i) = \begin{cases} 4i-3, & 1 \leq i \leq 2, \\ 3, & i = 3, \\ 3i-4, & 4 \leq i \leq n \text{ and } i \text{ is odd}, \\ 3i, & 4 \leq i \leq n \text{ and } i \text{ is even}, \end{cases}$$

$$f(v'_i) = \begin{cases} 6, & i = 1, \\ 2, & i = 2, \\ 3i-2, & 3 \leq i \leq n. \end{cases}$$

Then the induced edge labeling f^* is obtained as follows:

$$f^*(v_iv_{i+1}) = \begin{cases} i+2, & 1 \leq i \leq 2, \\ 3i-1, & 3 \leq i \leq n-1, \end{cases}$$

$$f^*(v_iv'_{i+1}) = \begin{cases} 5i-4, & 1 \leq i \leq 2, \\ 3i-2, & 3 \leq i \leq n-1 \text{ and } i \text{ is odd}, \\ 3i, & 3 \leq i \leq n-1 \text{ and } i \text{ is even}, \end{cases}$$

$$f^*(v'_iv_{i+1}) = \begin{cases} 5, & i = 1, \\ 2, & i = 2, \\ 3i, & 3 \leq i \leq n-1 \text{ and } i \text{ is odd}, \\ 3i-2, & 3 \leq i \leq n-1 \text{ and } i \text{ is even}. \end{cases}$$

Case 2. n is even.

Define $f : V(S'(P_n)) \to \{1, 2, 3, \cdots, 3n-2\}$ as follows:

$$f(v_i) = \begin{cases} 4-i, & 1 \leq i \leq 2, \\ 3i-1, & 3 \leq i \leq n \text{ and } i \text{ is odd}, \\ 3i-3, & 3 \leq i \leq n \text{ and } i \text{ is even}, \end{cases}$$

$$f(v_i') = \begin{cases} 1, & i = 1, \\ 3i-3, & 2 \le i \le n \text{ and } i \text{ is odd}, \\ 3i-2, & 2 \le i \le n \text{ and } i \text{ is even}. \end{cases}$$

Then the induced edge labeling f^* is obtained as follows:

$$f^*(v_iv_{i+1}) = 3i-1, \text{ for } 1 \le i \le n-1,$$

$$f^*(v_iv_{i+1}') = \begin{cases} 3i, & 3 \le i \le n-1 \text{ and } i \text{ is odd}, \\ 3i-2, & 3 \le i \le n-1 \text{ and } i \text{ is even}, \end{cases}$$

$$f(v_i'v_{i+1}) = \begin{cases} 3i-2, & 1 \le i \le n-1 \text{ and } i \text{ is odd}, \\ 3i, & 1 \le i \le n-1 \text{ and } i \text{ is even}. \end{cases}$$

Hence f is an F-centroidal mean labeling of $S'(P_n)$. Thus the splitting graph $S'(P_n)$ is an F-centroidal mean graph for $n \ge 2$. □

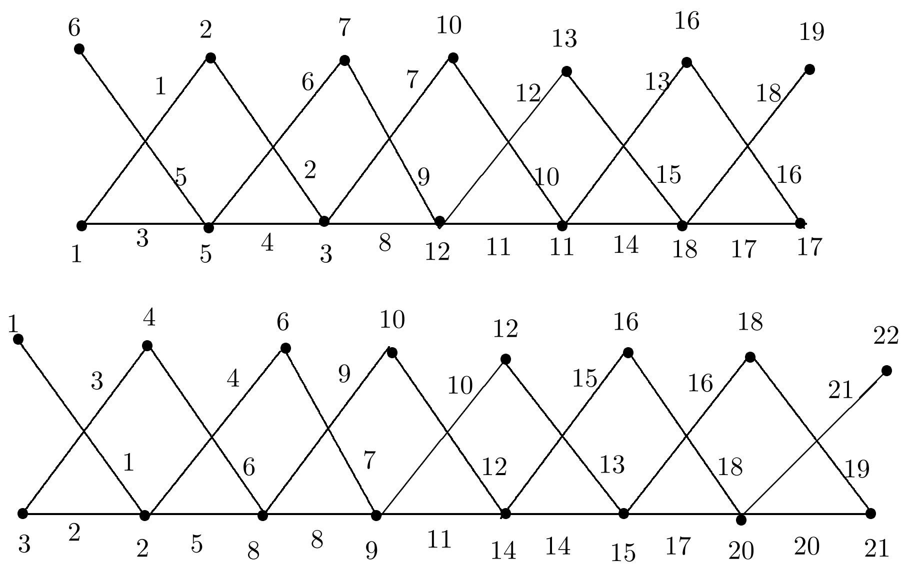

Figure 11 An F-centroidal mean labeling of $S'(P_7)$ and $S'(P_8)$

Theorem 2.12 *The graph $P(1, 2, \cdots, n-1)$ is an F-centroidal mean graph for $n \ge 2$.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of the path P_n and let u_{ij} be the vertices of the partition of K_{2,m_i} with cardinality m_i, $1 \le i \le n-1$ and $1 \le j \le m_i$. Define $f : V(P(1, 2, \cdots, n-1)) \to \{1, 2, 3, \cdots, n(n-1)+1\}$ as follows:

$$f(v_i) = i(i-1)+1, \text{ for } 1 \le i \le n,$$
$$f(u_{ij}) = i(i-1)+2j, \text{ for } 1 \le j \le i, \text{ and } 1 \le j \le n-1.$$

Then the induced edge labeling f^* is obtained as follows:

$$f^*(v_i u_{ij}) = i(i-1) + j, \text{ for } 1 \leq j \leq i \text{ and } 1 \leq i \leq n-1,$$
$$f(u_{ij} v_{i+1}) = i^2 + j, \text{ for } 1 \leq j \leq i \text{ and } 1 \leq i \leq n-1.$$

Hence f is an F-centroidal mean labeling of $P(1,2,\ldots,n-1)$. Thus the graph $P(1,2,\ldots,n-1)$ is an F-centroidal mean graph for $n \geq 2$. □

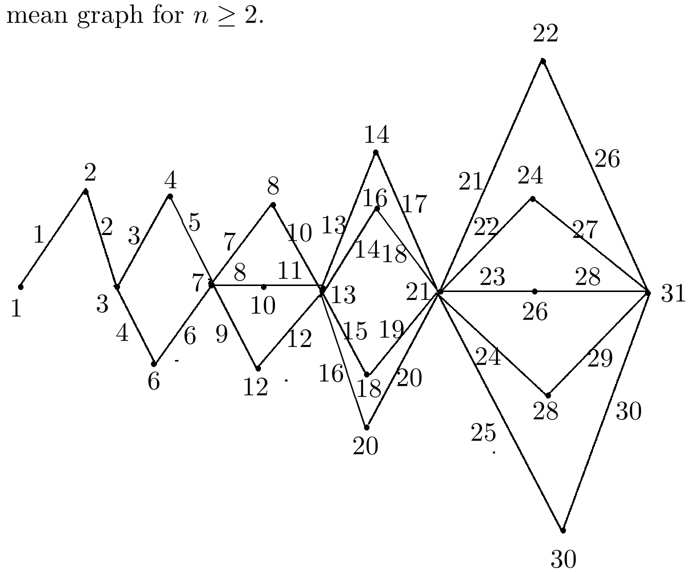

Figure 12 An F-centroidal mean labeling of $P(1,2,3,4,5)$

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Further Results on Analytic Odd Mean Labeling of Graphs

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Abstract: Let $G = (V, E)$ be a graph with p vertices and q edges. A graph G is analytic odd mean if there exist an injective function $f : V \to \{0, 1, 3, 5 \cdots , 2q - 1\}$ with an induce edge labeling $f^* : E \to Z$ such that for each edge uv with $f(u) < f(v)$,

$$f^*(uv) = \begin{cases} \left\lceil \frac{f(v)^2-(f(u)+1)^2}{2} \right\rceil, & \text{if } f(u) \neq 0 \\ \left\lceil \frac{f(v)^2}{2} \right\rceil, & \text{if } f(u) = 0 \end{cases}$$

is injective. We say that f is an analytic odd mean labeling of G. In this paper we prove that sun graph S_n, prism D_n, helm graph H_n, the graph $C_n \circ P_2$, banana tree, bamboo tree, perfect binary tree, the graph PC_n, unicyclic graph, the caterpillar $P_k(n_0, n_1, \cdots , n_{k-1})$ and spider graph are analytic odd mean graph.

Key Words: Mean labeling, analytic mean labeling, analytic odd mean labeling, Smarandachely analytic odd mean labeling, analytic odd mean graph.

AMS(2010): 05C78.

§1. Introduction

Throughout this paper we consider only finite, simple and undirected graph $G = (V, E)$ with p vertices and q edges and notations not defined here are used in the sense of Harary [2]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. There are several types of labeling. An excellent survey of graph labeling is available in [1]. The concept of analytic mean labeling was introduced in [7]. A graph G is analytic mean graph if it admits a bijection $f : V \to \{0, 1, 2, \cdots , p - 1\}$ such that the induced edge labeling $f^* : E \to Z$ given by $f^*(uv) = \left\lceil \frac{f(u)^2-f(v)^2}{2} \right\rceil$ with $f(u) > f(v)$ is injective. Motivated by the results in [7], we introduced a new mean labeling called analytic odd mean labeling in [3]. A graph G is an analytic odd mean if there exist an injective function $f : V \to \{0, 1, 3, 5 \cdots , 2q - 1\}$ with an induce edge labeling $f^* : E \to Z$ such that for each edge

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uv with $f(u) < f(v)$,

$$f^*(uv) = \begin{cases} \left\lceil \frac{f(v)^2-(f(u)+1)^2}{2} \right\rceil, & if \quad f(u) \neq 0 \\ \left\lceil \frac{f(v)^2}{2} \right\rceil, & if \quad f(u) = 0 \end{cases}$$

is injective. We say such an f is an analytic odd mean labeling of G. Otherwise, a Smarandachely analytic odd mean labeling of G if there exists an integer $0 \leq k \leq q$ holding with $|f^{-*}(k)| \geq 2$, where f^{-*} is the inverse of f^*. We proved that cycle C_n, path P_n, n-bistar, comb $P_n \odot K_1$, graph $L_n \odot K_1$, wheel graph W_n, flower graph Fl_n, some splitting graphs, multiple of graphs, quadrilateral snake $Q(n)$, double quadrilateral snake $DQ(n)$, coconut tree, fire cracker and some star graphs, splitting graph $spl(G)$, $P_n(1, 2, 3, \cdots, n)$, the complete bipartite graph $K_{m,n}$, the graph $C_k \odot \bar{K_n}$, the square graph of P_n, C_n, $B_{n,n}$, H-graph and $H \odot mK_1$ are analytic odd mean graphs in [4], [5] and [6].

We use the following definitions in the subsequent section to prove the results.

Definition 1.1 *A sun graph S_n is a cycle C_n with a pendent edge attached to each vertex of a cycle C_n.*

Definition 1.2 *The prism $D_n, n \geq 3$ is a cubic graph which can be represented as a Cartesian product $P_2 \times C_n$ of a path on two vertices with a cycle on n vertices.*

Definition 1.3 *A banana tree is a tree obtained from a family of stars by joining one end vertex of each star to a new vertex.*

Definition 1.4 *A tree is called a spider if it has a center vertex c of degree $R > 1$ and all the other vertex is either a leaf or with degree 2. Thus a spider is an amalgamation of k paths with various lengths. If it has x_1's path of length a_1, x_2's path of length a_2,$\cdots$. We denote the spider graph by $SP({a_1}^{x_1}, {a_2}^{x_2}, \cdots, {a_m}^{x_m})$ where $a_1 < a_2 < \cdots < a_m$ and $x_1 + x_2 + \cdots + x_m = R$.*

Definition 1.5 *A helm $H_n, n \geq 3$ is obtained from the wheel graph W_n by adding a pendent edge at each vertex on the wheel's rim.*

Definition 1.6 *A bamboo tree $(P_n \otimes K_{1,n_i})_{i=1}^{k}$ is an one point union of $P_n \otimes K_{1,n_i}$ where $1 \leq i \leq k$.*

Definition 1.7 *A caterpillar $P_k(n_1, n_2, \cdots, n_k)$ is a tree in which all the vertices are within distance 1 of a central path P_k for $k \geq 1$. When $k \geq 2$, a caterpillar is obtained from a path $P_k = u_1u_2u_3 \cdots u_k$ attaching $n_i \geq 0$ pendent vertices $v_i^j (1 \leq j \leq n_i)$ to each u_i.*

Definition 1.8 *A perfect binary tree is a full binary tree in which all the leaves are at the same level and in which every parent has two children.*

§2. Main Results

In this section we prove that sun graph S_n, prism D_n, helm graph H_n, path union of $n-3$

copies of C_n,the graph $C_n \circ P_2$, banaba tree, bamboo tree, the graph PC_n, unicyclic graph, perfect binary tree, the caterpillar graph $P_k(n_0, n_1, \cdots, n_{k-1})$ and spider graph are analytic odd mean graphs.

Theorem 2.1 *For every positive integer $n \geq 3$, the sun graph S_n is an analytic odd mean graph.*

Proof Let the vertex set and edge set of the sun graph be $V(S_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(S_n) = \{u_iu_{i+1} : 1 \leq i \leq n-1\} \bigcup \{u_iv_i : 1 \leq i \leq n\} \bigcup \{u_nu_1\}$
Now $|V(G)| = 2n = |E(G)|$. We define an injective map $f : V(S_n) \to \{0, 1, 3, 5, \cdots, 4n-1\}$ by $f(u_i) = 2i-1$ for $1 \leq i \leq n$ and $f(v_i) = 2n+2i-1$ for $1 \leq i \leq n$.

The induced edge labeling f^* is defined as follows:

$f^*(u_iu_{i+1}) = 2i+1$ for $1 \leq i \leq n-1$,
$f^*(u_1u_n) = 2n^2 - 2n - 1$,
$f^*(u_iv_i) = 2n(n-1) + 2i(2n-1) + 1$ for $1 \leq i \leq n$.

We observe that the edge labels of u_iu_{i+1} are $3, 5, \cdots 2n-1$ as i increases and the edge labels of u_iv_i are increased by $4n-2$ as i increases from 1 to n. Hence all the edge labels are distinct and odd . Hence S_n admits an analytic odd mean labeling. □

An analytic odd mean labeling of S_8 is shown in Figure 1.

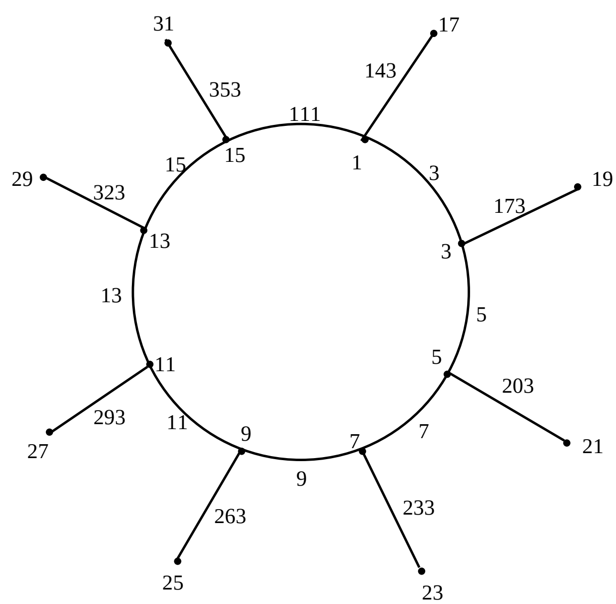

Figure 1

Theorem 2.2 *For every positive $n \geq 3$, the prism D_n is an analytic odd mean graph.*

Proof Let the vertex set and edge set of the prism be $V(D_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(D_n) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\} \bigcup \{u_iv_i : 1 \leq i \leq n\} \bigcup \{u_nv_1, v_nv_1\}$. Now $|V(D_n)| = 2n$ and $|E(D_n)| = 3n$. We define an injective map $f : V(D_n) \to \{0, 1, 3, 5, \cdots, 6n-1\}$ by $f(u_i) = 2i-1$ for $1 \leq i \leq n$ and $f(v_i) = 2n+2i-1$ for $1 \leq i \leq n$.

The induced edge labeling f^* is defined as follows:

$f^*(u_iu_{i+1}) = 2i + 1$ for $1 \leq i \leq n-1$,
$f^*(u_1u_n) = 2n^2 - 2n - 1$,
$f^*(u_iv_i) = 2n(n-1) + 2i(2n-1) + 1$ for $1 \leq i \leq n$,
$f^*(v_iv_{i+1}) = 2n + 2i + 1$ for $1 \leq i \leq n-1$,
$f^*(v_1v_n) = 6n^2 - 8n - 1$.

We observe that the edge labels of u_iu_{i+1} and v_iv_{i+1} are $3, 5, \ldots, 2n-1$ and $2n+3, 2n+5, \cdots, 2n+2n-1 = 4n-1$ respectively as i increases and the edge labels of u_iv_i are increased by $4n-2$ as i increases from 1 to n. So all the edge labels are distinct and odd. Hence D_n admits an analytic odd mean labeling. □

An analytic odd mean labeling of D_8 is shown in Figure 2.

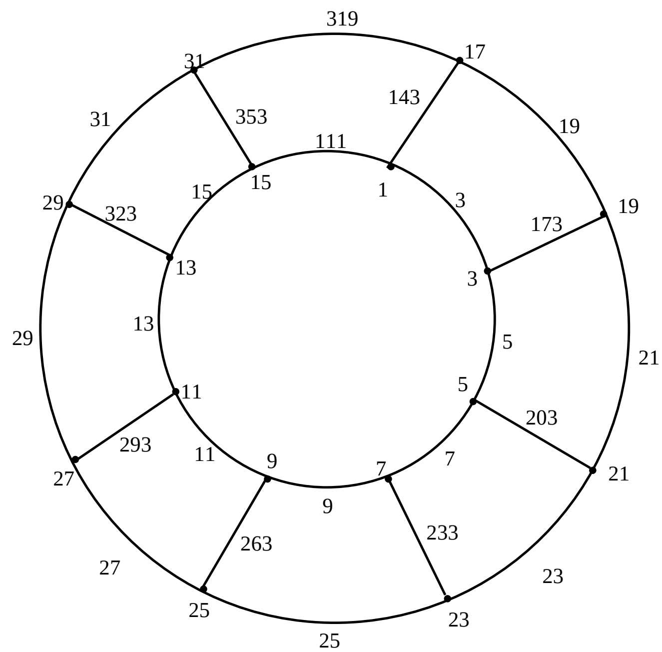

Figure 2

Theorem 2.3 *The helm graph $H_n, n \geq 3$ is an analytic odd mean graph.*

Proof Let $V(H_n) = \{v, v_i, u_i : 1 \leq i \leq n\}$ and $E(H_n) = \{vv_i, v_iu_i : 1 \leq i \leq n\} \cup \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_nv_1\}$. Now $|V(G)| = 2n+1$ and $|E(G)| = 3n$. We define an injective map $f : V(G) \to \{0, 1, 3, 5, \cdots, 6n-1\}$ by $f(v) = 0$, $f(v_i) = 4i - 3$ for $1 \leq i \leq n$ and $f(u_i) = 4n + 2i - 1$ for $1 \leq i \leq n$.

The induced edge labeling f^* is defined as follows:

$f^*(vv_i) = 8i^2 - 12i + 5$ for $1 \leq i \leq n$,
$f^*(v_iv_{i+1}) = 12i - 1$ for $1 \leq i \leq n-1$,
$f^*(v_nv_1) = 8n^2 - 12n + 3$,
$f^*(v_iu_i) = 4n(2n-1) - 6i(i-1) + 8ni - 1$ for $1 \leq i \leq n$.

We observe that when i increases, the difference of edges v_iu_i are decreased by 12. Clearly all the edge labels are odd and distinct. Hence H_n admits an analytic odd mean labeling. □

An analytic odd mean labeling of H_{10} is shown in Figure 3.

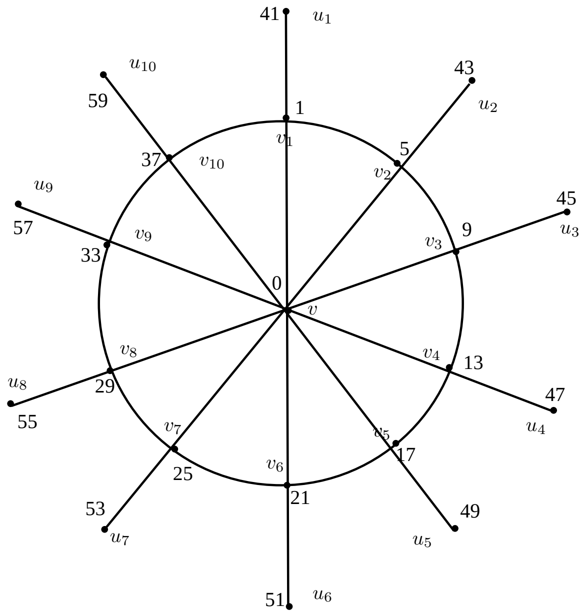

Figure 3

Theorem 2.4 *The graph $PC_n (n \geq 4$ and n is even) is obtained from $C_n = v_1v_2 \cdots v_nv_1$ by adding the chords v_i and v_{n-i+2} for $2 \leq i \leq l$ where $l = n/2$. Then the graph PC_n is an analytic odd mean graph.*

Proof Let $G = PC_n$. Let the vertex set and edge set of G be $V(G) = \{v_i, : \ 1 \leq i \leq n\}$ and $E(G) = \{v_iv_{i+1} : \ 1 \leq i \leq n-1\} \cup \{v_iv_{n-i+2} : \ 2 \leq i \leq l\} \cup \{v_nv_1\}$. Then there are n vertices and $3n/2 - 1$ edges. We define an injective map $f : V(G) = \{0, 1, 3, \cdots 3n-3\}$ by $f(v_i) = 2i - 1$ for $1 \leq i \leq n$.

The induced edge labeling f^* is defined as follows:

$f^*(v_iv_{i+1}) = 2i + 1$ for $1 \leq i \leq n$,

$f^*(v_nv_1) = 2n^2 - 2n - 1$,

$f^*(v_iv_{n-i+2}) = 2n(n+3) - 2i(2n+3) + 5$ for $2 \leq i \leq l$.

It can be easily verified that f is an analytic odd mean labeling and hence PC_n is an analytic odd mean graph. □

An analytic odd mean labeling of PC_{10} is shown in Figure 4.

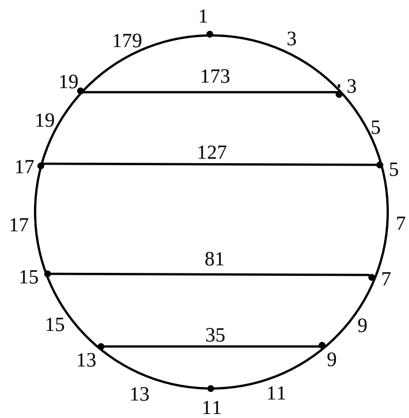

Figure 4

Theorem 2.5 *Let G be a unicycle graph with a cycle $C_k = a_1a_2\cdots a_ka_1$ such that the vertex a_i is attached to n_i pendent vertices. Then the unicycle graph admits an analytic odd mean labeling.*

Proof Let $V(G) = \{a_i, a_i^j : 1 \leq i \leq k$ and $1 \leq j \leq n_i\}$ and $E(G) = \{a_ia_{i+1} : 1 \leq i \leq k-1\} \cup \{a_ia_i^j : 1 \leq i \leq k$ and $1 \leq j \leq n_i\} \cup \{a_ka_1\}$. We assume $n_0 = 0$. Hence $|V| = n_1 + n_2 + \cdots + n_k + k = |E|$.

We define an injective map on the vertex set by $f(a_i) = 2i - 1$ for $1 \leq i \leq k$ and $f(a_i^j) = 2k - 1 + 2\sum_{r=0}^{i-1} n_r + 2j$ for $1 \leq i \leq k$ and $1 \leq j \leq n_i$.

The induced edge labeling f^* is defined as follows:

$f^*(a_ia_{i+1}) = 2i + 1$ for $1 \leq i \leq k-1$,

$f^*(a_ka_1) = 2k^2 - 2k - 1$,

$f^*(a_ia_i^j) = 2(k^2 + j^2 - i^2) + 2(\sum_{r=0}^{i-1} n_r)^2 + 2(2k + 2j - 1)\sum_{i=1}^{j-1} n_i + 4kj - 2(k + j) + 1$ for $1 \leq i \leq k$ and $1 \leq j \leq n_i$.

Clearly the edge labels are odd and distinct. Hence the unicycle graph admits an analytic odd mean labeling. □

An analytic odd mean labeling of the unicycle graph with $k = 6$ is shown in Figure 5.

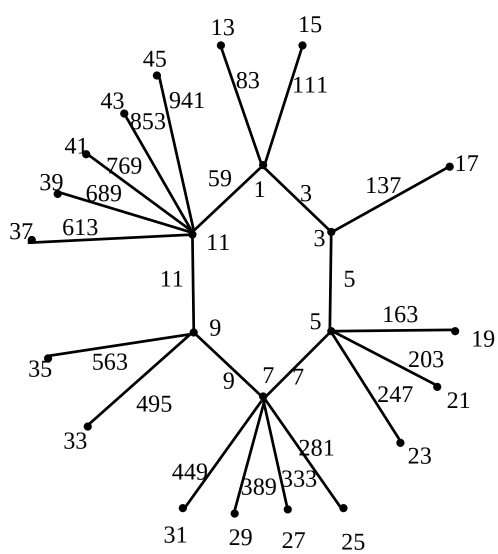

Figure 5

Theorem 2.6 *The caterpillar $P_k(n_0, n_1, \cdots, n_{k-1})$ is an analytic odd mean graph for $k \geq 2, n_i \geq 0$.*

Proof Let $V(G) = \{v_i, v_i^j : 0 \leq i \leq k-1 \text{ and } 1 \leq j \leq n_i\}$ and $E(G) = \{v_{i-1}v_i : 1 \leq i \leq k-1\} \cup \{v_iv_i^j : 0 \leq i \leq k-1 \text{ and } 1 \leq j \leq n_i\}$. Hence $|V| = n_0 + n_1 + \ldots + n_k + k$ and $|E| = n_0 + n_1 + \cdots + n_k + k - 1$. We define an injective map on the vertex set by

$f(v_0) = 0, f(v_i) = 2i - 1$ for $1 \leq i \leq k-1$,
$f(v_0^j) = 2k + 2j - 3$ for $1 \leq j \leq n_i$,
$f(v_i^j) = 2k - 3 + 2\sum_{r=0}^{i-1} n_r + 2j$ for $1 \leq i \leq k-1$ and $1 \leq j \leq n_i$.

The induced edge labeling f^* is defined as follows:

$f^*(v_{i-1}v_i) = 2i - 1$ for $1 \leq i \leq k-1$,
$f^*(v_0v_0^j) = 2k(k-3) + 2j(j-3) + 4kj + 5$ for $1 \leq j \leq n_i$,
$f^*(v_iv_i^j) = \left[\left(2k - 3 + 2\sum_{r=0}^{i-1} n_r + 2j - 2i\right)\left(2k - 3 + 2\sum_{r=0}^{i-1} n_r + 2j + 2i\right) + 1\right] \div 2$ for $1 \leq i \leq k-1$ and $1 \leq j \leq n_i$.

Clearly the edge labels are odd and distinct. Hence $P_k(n_0, n_1, ..., n_{k-1})$ admits an analytic odd mean labeling. □

An analytic odd mean labeling of $P_5(3, 2, 1, 5, 4)$ is shown in Figure 6.

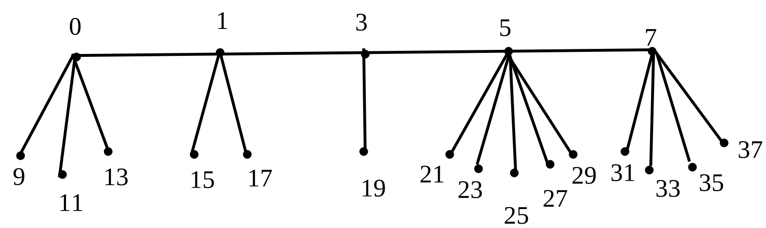

Figure 6

Theorem 2.7 *The graph $C_n \circ P_2$ is an analytic odd mean graph.*

Proof Let G be the graph $C_n \circ P_2$. Let $V(G) = \{v_i, v_{i,1}, v_{i,2} : 1 \leq i \leq n\}$ and $E(G) = \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_nv_1\} \cup \{v_iv_{i,1}, v_iv_{i,2}, v_{i,1}v_{i,2} : 1 \leq i \leq n\}$. Now $|V(G)| = 3n$ and $|E(G)| = 4n$.

We define an injective map $f : V(G) \to \{0, 1, 3, 5, \cdots, 8n-1\}$ by

$f(v_i) = 2i - 1$ for $1 \leq i \leq n$,
$f(v_{i,1}) = 4i + 2n - 3$ for $1 \leq i \leq n$,
$f(v_{i,2}) = 4i + 2n - 1$ for $1 \leq i \leq n$.

The induced edge labeling f^* is defined as follows:

$f^*(v_iv_{i+1}) = 2i + 1$ for $1 \leq i \leq n-1$,
$f^*(v_nv_1) = 2n^2 - 2n + 1$,
$f^*(v_iv_{i,1}) = 6i(i-2) + 2n(n-3) + 8ni + 5$ for $1 \leq i \leq n$,
$f^*(v_iv_{i,2}) = 2i(3i-2) + 2n(n-1) + 8ni + 1$ for $1 \leq i \leq n$,
$f^*(v_{i,1}v_{i,2}) = 4i + 2n - 1$ for $1 \leq i \leq n$.

Clearly the edge labels are odd and distinct. Hence $C_n \circ P_2$ admits an analytic odd mean labeling. □

An analytic odd mean labeling of $C_5 \circ P_2$ is shown in Figure 7.

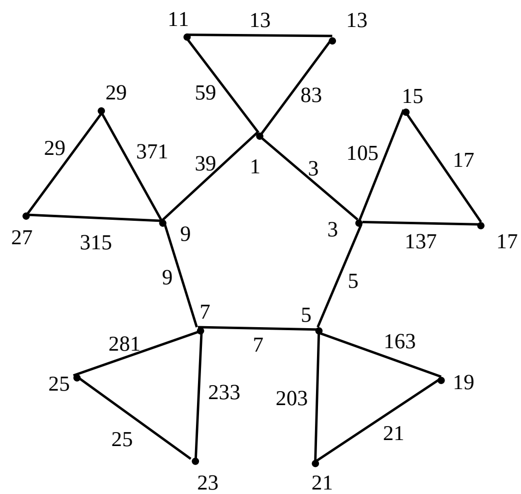

Figure 7

Theorem 2.8 *Let $K_{1,n_1}, K_{1,n_2}, \cdots, K_{1,n_k}$ be a family of stars with vertex sets $V(K_{1,n_j}) = \{a_j, a_j^1, a_j^2, \cdots, a_j^{n_j}\}$, and $deg(a_j) = n_j, 1 \le j \le k$. Let $BT(n_1, n_2, \cdots, n_k)$ be a banana tree obtained by adding a new vertex a and joning it to $a_1^1, a_2^1, a_3^1, \cdots, a_k^1$. Then $BT(n_1, n_2, \cdots, n_k)$ admits an analytic mean labeling where n_j is any positive integer.*

Proof Let the vertex set and edge set be $V = \{a, a_j, a_j^r : 1 \le j \le k, 1 \le r \le n_j\}$ and $E = \{aa_j^1 : 1 \le j \le k\} \cup \{a_j a_j^r : 1 \le j \le k, 1 \le r \le n_j\}$ respectively. We assume $n_0 = 0$. Hence $|V| = n_1 + n_2 + \cdots + n_k + k + 1$ and $|E| = n_1 + n_2 + \cdots + n_k + k$.

We define an injective function f on the vertex set of banana tree as follows :

$f(a) = 0$, $f(a_j) = 2j - 1$ for $1 \le j \le k$,
$f(a_j^r) = 2k - 1 + 2\sum_{i=0}^{j-1} n_i + 2r$ for $1 \le r \le n_j, 1 \le j \le k$.

The induced edge labeling f^* is defined as follows:

$$f^*(a_j a_j^r) = 2(k^2 + r^2 - j^2) + 2(\sum_{i=0}^{j-1} n_i + j)(2k - 1 + \sum_{i=0}^{j-1} n_i) + 2j\sum_{i=0}^{j-1} n_i - 2k + 1$$

for $1 \le r \le n_j, 1 \le j \le k$ and

$$\begin{aligned} f^*(aa_j^1) &= \frac{\left[\left(2k - 1 + 2\sum_{i=0}^{j-1} n_i + 2\right)^2 + 1\right]}{2} \\ &= 2k(k+1) + 2\sum_{i=0}^{j-1} n_i(2k+1) + 2(\sum_{i=0}^{j-1} n_i)^2 + 1 \end{aligned}$$

for $1 \leq j \leq k$.

Clearly the edge labels are odd and distinct. Therefore f is an analytic odd mean labeling and hence the banana tree is an analytic odd mean graph. □

An analytic odd mean labeling of $BT(3,5,6,4)$ is shown in Figure 8.

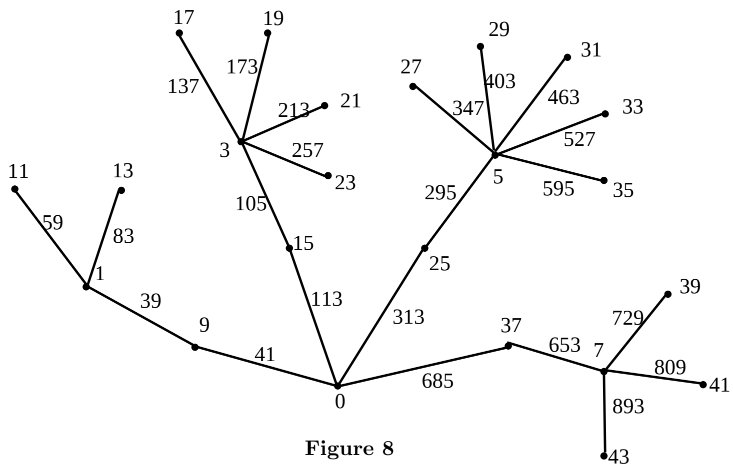

Figure 8

Theorem 2.9 *Let $K_{1,n_1}, K_{1,n_2}, \cdots, K_{1,n_k}, K_{1,k}$ be a family of stars with vertex sets $V(K_{1,n_j}) = \{v_j^r : 0 \leq r \leq n_j\}$ and $deg(v_j^0) = n_j$, for $1 \leq j \leq k, V(K_{1,k}) = \{v, v_1, v_2, \cdots, v_k\}$. The bamboo tree $(P_2 \otimes K_{1,n_j})$ for $j = 1, 2, \ldots, k$ is obtained by joining the vertex $v_1, v_2, \ldots, v_k$ with $v_1^1, v_2^1, \cdots, v_k^1$ respectively. Clearly the number of vertices and edges of the bamboo tree are $n_1 + n_2 + \cdots + n_k + k + 1$ and $n_1 + n_2 + \ldots + n_k + k$ respectively. Then the bamboo tree $(P_2 \otimes K_{1,n_j})$ for $j = 1, 2, \cdots, k$ is an analytic odd mean graph.*

Proof Let the vertex set and edge set be $V = \{v, v_j, v_j^r : 1 \leq j \leq k, 0 \leq r \leq n_j\}$ and $E = \{vv_j : 1 \leq j \leq k\} \cup \{v_j v_j^1 : 1 \leq j \leq k\} \cup \{v_j^0 v_j^r : 1 \leq j \leq k, 1 \leq r \leq n_j\}$ respectively. We assume $n_0 = 0$. Hence $|V| = n_1 + n_2 + \ldots + n_k + 2k + 1$ and $|E| = n_1 + n_2 + \cdots + n_k + 2k$.

We define an injective function f on the vertex set of bamboo tree as follows:

$f(v) = 0$, $f(v_j) = 2j - 1$ for $1 \leq j \leq k$,
$f(v_j^r) = 4k - 1 + 2\sum_{i=0}^{j-1} n_i + 2r$ for $1 \leq r \leq n_j, 1 \leq j \leq k$,
$f(v_j^0) = 2k + 2j - 1$ for $1 \leq j \leq k$.

The induced edge labeling f^* is defined as follows:

$f^*(vv_j) = 2j^2 - 2j + 1$ for $1 \leq j \leq k$,
$f^*(v_j v_j^1) = \left[\left(4k + 1 + 2\sum_{i=0}^{j-1} n_i\right)^2 + 1\right] \div 2 - 2j^2$ for $1 \leq j \leq k$,
$f^*(v_j^0 v_j^r) = \left[\left(4k - 1 + 2\sum_{i=0}^{j-1} n_i + 2r\right)^2 + 1\right] \div 2 - 2(k^2 + j^2) - 4kj$ for $1 \leq r \leq n_j, 1 \leq j \leq k$.

Clearly the edge labels are odd and distinct. Therefore f is an analytic odd mean labeling and hence the bamboo tree is an analytic odd mean graph. □

An analytic odd mean labeling of $(P_2 \otimes K_{1,n_j})$ for $j = 1, 2, 3, 4$ with $n_1 = 5, n_2 = 4, n_3 = 3, n_4 = 7$ is shown in Figure 9.

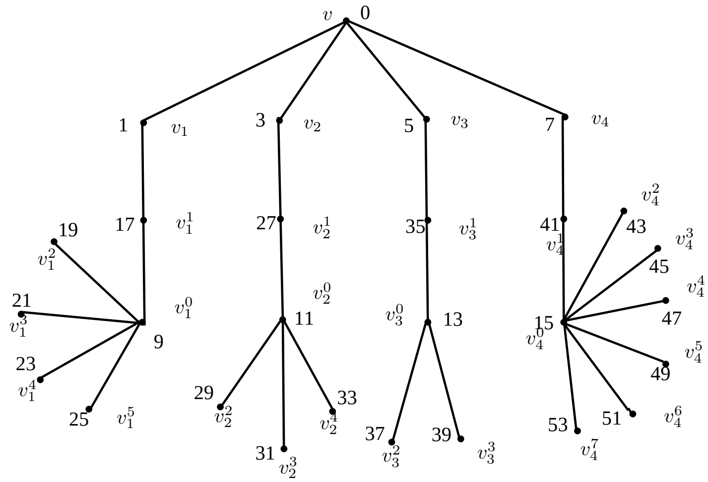

Figure 9

Theorem 2.10 *The perfect binary tree T of order p is an analytic odd mean graph.*

Proof Let $V(T) = \{v_i : \ 1 \le i \le p\}$ and $E(T) = \{v_i v_{2i}, v_i v_{2i+1} : \ 1 \le i \le q/2\}$. Hence $|V| = p$ and $|E| = p - 1$. We define an injective map $f : V(G) = \{0, 1, 3, \cdots, 2p-3\}$ by $f(v_1) = 0$ and $f(v_i) = 2i - 3$ for $2 \le i \le p$. The induced edge labeling f^* is defined as follows:

$f^*(v_1 v_2) = 1,$
$f^*(v_1 v_3) = 5,$
$f^*(v_i v_{2i}) = 6i^2 - 8i + 3$ for $2 \le i \le q/2,$
$f^*(v_i v_{2i+1}) = 6i^2 - 1$ for $2 \le i \le q/2.$

We observe that the difference of edge labels of $v_i v_{2i}$ and $v_i v_{2i+1}$ is $8i - 4$ as i increases from 1 to $\frac{q}{2}$. Therefore the edge labels are odd and distinct. Hence the binary tree admits an analytic odd mean labeling. □

An analytic odd mean labeling of T of order 15 is shown in Figure 10.

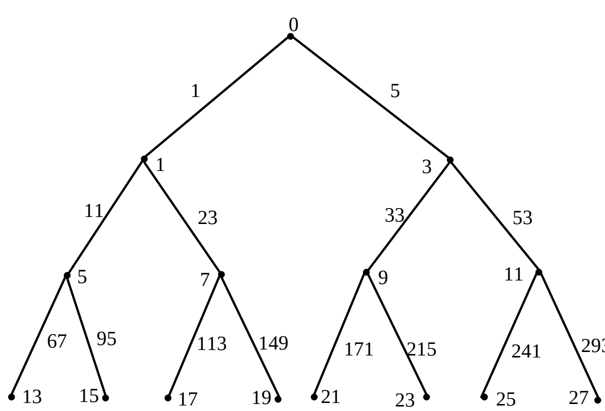

Figure 10

Theorem 2.11 *The spider graph $SP(1^n, k^m), n \geq 1,$ and $k, m \geq 2$ is an analytic odd mean graph.*

Proof Let $V(SP(1^n, k^m)) = \{v\} \cup \{w_a, : 1 \leq a \leq n\} \cup \{v_i^j : 1 \leq i \leq k$ and $1 \leq j \leq m\}$ and $E(SP(1^n, k^m)) = \{vv_1^j, v_i^j v_{i+1}^j : 1 \leq i \leq k-1$ and $1 \leq j \leq m\} \cup \{vw_a : 1 \leq a \leq n\}$. We define an injective map $f : V(SP(1^n, k^m)) \to \{0, 1, 3, 5, \cdots, 2(km+n)-1\}$ by $f(v) = 0, f(v_i^j) = (j-1)2k + 2i - 1$ for $1 \leq i \leq k$, $1 \leq j \leq m$ and $f(w_a) = 2mk + 2a - 1$ for $1 \leq a \leq n$. The induced edge labeling f^* is defined as follows:

$f^*(vv_1^j) = 2k(j-1)(k(j-1)+1) + 1$ for $1 \leq j \leq m$,
$f^*(v_i^j v_{i+1}^j) = 2k(j-1) + 2i + 1$ for $1 \leq i \leq k-1$ and $1 \leq j \leq m$,
$f^*(vw_a) = 2mk(mk-1) + 2a(a-1) + 4mak + 1$ for $1 \leq a \leq n$.

It can be verified that the edge labels are odd and distinct. Hence $SP(1^n, k^m)$ is an analytic odd mean graph. □

An analytic odd mean labeling of $SP(1^n, 6^5)$ is shown in Figure 11.

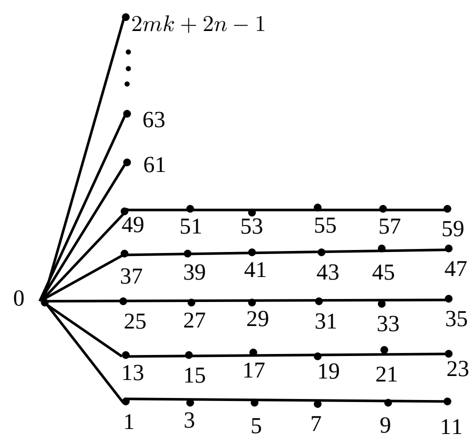

Figure 11

Theorem 2.12 *The spider graph $SP(2^n, k^m), n \geq 1, k, m \geq 2$ is an analytic odd mean graph if (a) k is even and m is any integer and (b) k is odd and m is even.*

Proof Let $V(SP(2^n, k^m)) = \{v\} \cup \{w_{a,1}, w_{a,2} : 1 \leq a \leq n\} \cup \{v_i^j : 1 \leq i \leq k$ and $1 \leq j \leq m\}$ and $E(SP(2^n, k^m)) = \{vv_1^j, v_i^j v_{i+1}^j : 1 \leq i \leq k$ ans $1 \leq j \leq m\} \cup \{vw_{a,1}, w_{a,1}w_{a,2} : 1 \leq a \leq n\}$. We define an injective map $f : V(SP(1^n, k^m)) \to \{0, 1, 3, 5, \cdots, 2km + 4n - 1\}$ by $f(v) = 0, f(v_i^j) = (j-1)2k + 2i - 1$ for $1 \leq i \leq k$, $1 \leq j \leq m$; $f(w_{a,1}) = 2mk + 4a - 3$ for $1 \leq a \leq n$ and $f(w_{a,2}) = 2mk + 4a - 1$ for $1 \leq a \leq n$.

The induced edge labeling f^* is defined as follows:

$f^*(vv_1^j) = 2k(j-1)(k(j-1)+1) + 1$ for $1 \leq j \leq m$,
$f^*(v_i^j v_{i+1}^j) = 2k(j-1) + 2i + 1$ for $1 \leq i \leq k-1$ and $1 \leq j \leq m$,
$f^*(vw_{a,1}) = 2mk(mk-3) + 4a(2a-3) + 8mak + 5$ for $1 \leq a \leq n$,
$f^*(w_{a,1}w_{a,2}) = 2mk + 4a - 1$ for $1 \leq a \leq n$.

It can be verified that the edge labels are odd and distinct. Hence $SP(2^n, k^m)$ is an analytic odd mean graph. □

An analytic odd mean labeling of $SP(2^n, 5^4)$ is shown in Figure 12.

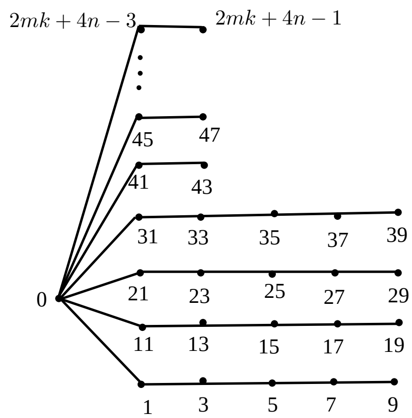

Figure 12

Theorem 2.13 *The spider graph $SP(1^s, 2^n, k^m), n, s \geq 1, k, m \geq 2$ is an analytic odd mean graph if (a) k is even and m is any integer and (b) k is odd and m is even.*

Proof Let $V(SP(1^s, 2^n, k^m)) = \{v\} \cup \{w_{a,1}, w_{a,2} : 1 \leq a \leq n\} \cup \{v_i^j : 1 \leq i \leq k$ and $1 \leq j \leq m\} \cup \{u_r : 1 \leq r \leq s\}$ and $E(SP(1^s, 2^n, k^m)) = \{vv_1^j, v_i^j v_{i+1}^j : 1 \leq i \leq k$ and $1 \leq j \leq m\} \cup \{vw_{a,1}, w_{a,1}w_{a,2} : 1 \leq a \leq n\}$. We define an injective map $f : V(SP(1^n, k^m)) \to \{0, 1, 3, 5, \cdots, 2km + 4n + 2s - 1\}$ by $f(u_r) = 2mk + 4n + 2r - 1$ for $1 \leq r \leq s$ and $f(v), f(v_i^j), f(w_{a,1})$ and $f(w_{a,2})$ are defined as in Theorem 2.12. Then the induced edge labeling $f^*(vv_1^j)$, $f^*(v_i^j v_{i+1}^j)$, $f^*(vw_{a,1})$ and $f^*(w_{a,1}w_{a,2})$ are as in Theorem 2.12 and

$$f^*(vu_r) = \frac{(2mk + 2r + 4n - 1)^2 + 1}{2}$$

for $1 \leq r \leq s$.

It can be verified that the edge labels are odd and distinct. Hence $SP(1^s, 2^n, k^m)$ is an analytic odd mean graph. □

An analytic odd mean labeling of $SP(1^s, 2^n, 3^4)$ is shown in Figure 13.

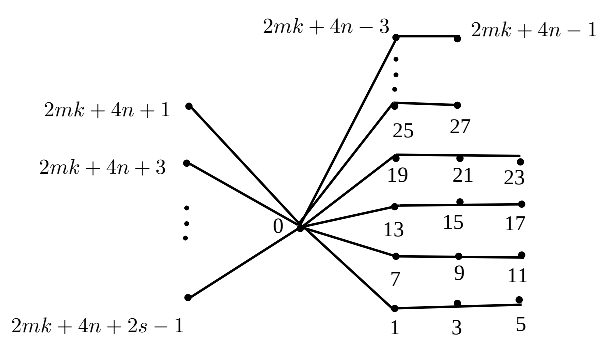

Figure 13

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Famous Words

The reality of a thing T is the behavior with motivation of an abstracted complex network in the microcosmic level. Certainly, there are more microcosmic observing datum on the units, cells or microcosmic particles of matters by scientific instruments. A microcosmic science is such a science established on the microcosmic datum of matters, including theory and experimental subjects, which must be established over 1-dimensional skeleton or in other words, topological graphs. *Could we establish such a mathematics over topological graphs for microcosmic science*? The answer is positive inspired by the traditional Chinese medicine, i.e., 12 meridians theory. (Extracted from the paper: Science's Dilemma - a Review on Science with Applications, *Progress in Physics*, Vol.15, 2(2019), 78-85.)

By Dr.Linfan MAO, a Chinese mathematician, philosophical critic.

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Books

[4]Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, InfoQuest Press, 2009.
[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.
[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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Contents

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