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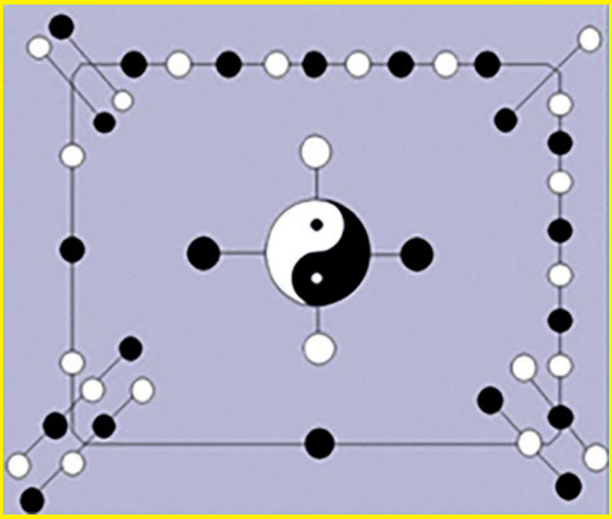

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Mathematics is the tool specially suited for dealing with abstract concepts of any kind and there is no limit to its power in this fieldFor this reason a book on the new physicsif not purely descriptive of experimental workmust be essentially mathematical.

By Paul Adrie Maurice Dirac, an British theoretical physicist

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Surfaces using Smarandache Asymptotic Curves in Galilean Space

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Abstract: In this paper, we consider the problem of constructing surfaces using special Smarandache curves in Galilean 3-space. We give the family of surfaces as a linear combination of the components of this frame, and derive the conditions for coefficients to satisfy both the asymptotic and isoparametric requirements. Finally, we present some examples to verify our method.

Key Words: Galilean space, asymptotic curve, Smarandache curve.

AMS(2010): 53A10.

§1. Introduction

The classical theory of asymptotic curves is one of the most important and efficient methods that can be used to characterize surfaces. Asymptotic curves have a big importance in astronomy, astrophysics and CAD in architecture. A curve on a surface is said to be an asymptotic curve if the curve on a regular surface is given such that the normal curvature is zero in the asymptotic direction. This direction can only happen for non-positive (negative or zero) Gaussian curvature on a surface along the asymptotic curve [5,9]. The notion of the family of surfaces having a given characteristic curve was first investigated by Wang et al. [14] in Euclidean 3-space. Then Bayram et al. [3] extended the work of Wang to how to get a surface pencil from a given spatial asymptotic curve. Moreover, some more studies about the surface family with common asymptotic curves have been given in [12,13].

The classical curve theory is one of the most important research topics in differential geometry. There are many different studies on special curves. Smarandache curves are defined as a regular curve whose position vector is composed of Frenet frame vectors [10]. There has been a lot of researches about Special Smarandache curves and their characterizations in [1,2].

The starting point of our study is to investigate how to characterize parametric surfaces via a given curve as a common isoasymptotic and special Smarandache curves in Galilean 3-space. First, we give the surfaces as a linear combination of the Frenet frame of the given curve and derive the conditions on marching-scale functions to satisfy both isoasymptotic and Smarandache requirements. Finally, we illustrate some examples of these surfaces.

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§2. Preliminaries

The Galilean space $\mathbf{G}_3$ is a Cayley-Klein space equipped with the metric of signature $(0,0,+,+)$, given in [6]. The absolute figure of the Galilean space consists of an ordered triple $\{\omega, f, I\}$ in which ω is the ideal (absolute) plane, f is the line (absolute line) in ω and I is the fixed elliptic involution of f.

In the Galilean space there are just two types of vectors, non-isotropic $\mathbf{x} = (x_1, x_2, x_3)$ (for which holds $x_1 \neq 0$). Otherwise, it is said to be isotropic.

Definition 2.1([11]) *Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be vectors in Galilean space G_3. The Galilean scalar product of x and y is given by*

$$\langle x, y\rangle = \begin{cases} x_1 y_1, & if\ x_1 \neq 0\ or\ y_1 \neq 0 \\ x_2 y_2 + x_3 y_3, & if\ x_1 = 0\ and\ y_1 = 0 \end{cases} . \tag{1}$$

Definition 2.2([8]) *Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be vectors in Galilean space G_3. The Galilean vector product of x and y is given by*

$$x \wedge y = \begin{vmatrix} 0 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} . \tag{2}$$

Let r be an admissible curve of the class C^∞ in G_3, parameterized by the invariant parameter u, given by

$$r(u) = (u, f(u), g(u)).$$

Then the curvature $\kappa(u)$ and the torsion $\tau(u)$ of the curve $r(u)$ can be given by

$$\kappa(u) = \sqrt{f''(u)^2 + g''(u)^2},$$

and

$$\tau(u) = \frac{\det(r'(u), r''(u), r'''(u))}{\kappa^2(u)}$$

and the associated moving trihedron satisfies

$$\begin{cases} T(u) = r'(u) = (1, f'(u), g'(u)) \\ N(u) = \frac{r''}{\kappa(u)} = \frac{1}{\kappa(u)}(0, f''(u), g''(u)) \\ B(u) = \frac{1}{\kappa(u)}(0, -g''(u), f''(u)) \end{cases} ,$$

where T, N and B are called the vectors of the tangent, principal normal and binormal of $r(u)$, respectively.

Frenet formulas are given by

$$\begin{cases} T' = \kappa N \\ N' = \tau B \\ B' = -\tau N \end{cases}, \tag{3}$$

for more information, we refer to [7].

Definition 2.3([1]) *Let $r(u)$ be an admissible curve in G_3 and $\{T, N, B\}$ be its moving Frenet frame. Smarandache TN, TB and TNB curves are respectively, defined by*

$$\begin{aligned} r_{TN} &= \frac{T+N}{\|T+N\|}, \\ r_{TB} &= \frac{T+B}{\|T+B\|}, \\ r_{TNB} &= \frac{T+N+B}{\|T+N+B\|}, \end{aligned}$$

The equation of a surface in G_3 can be given by the parametrization

$$\varphi(u,v) = (\varphi_1(u,v), \varphi_2(u,v), \varphi_3(u,v)),$$

where $\varphi_1(u,v), \varphi_2(u,v)$ and $\varphi_3(u,v) \in \mathbf{C}^3$, in [7].

Let $r(u)$ is an parametric curve on the surface $\varphi(u,v)$, there exists a parameter $v_0 \in [T_1, T_2]$ such that $r(u) = \varphi(u, v_0)$

$$x(u,v_0) = y(u,v_0) = z(u,v_0) = 0, \ L_1 \leq u \leq L_2 \text{ and } T_1 \leq v \leq T_2.$$

Given a curve $r(u)$ on the surface $\varphi(u,v)$ is an asymptotic iff the binormal $B(u)$ of the curve $r(u)$ and the normal $\eta(u,v_0)$ of the surface $\varphi(u,v)$ at any point on the curve $r(u)$ are parallel to each other [4].

§3. Surfaces with Common Smarandache Asymptotic Curves in Galilean Space G_3

Let $\varphi(u,v)$ be a parametric surface. The surface is defined by a given curve $r(u)$ as follows:

$$\varphi(u,v) = r(u) + [x(u,v)T(u) + y(u,v)N(u) + z(u,v)B(u)], \tag{4}$$

$$L_1 \leq u \leq L_2 \quad \text{and} \quad T_1 \leq v \leq T_2,$$

where $x(u,v_0)$, $y(u,v_0)$ and $z(u,v_0)$ which are the values of the marching-scale functions indicate, respectively, the extension-like, flexion-like and retortion-like by the point unit through time v, starting from $r(u)$, $\{T(u), N(u), B(u)\}$ is the frame associated with the curve $r(u)$ in G_3, and values of this functions are C^1 functions. Throughout this paper, we assume that $\kappa \neq 0$.

Our main aim is to get the conditions for which the some special Smarandache curves of the unit space curve $r(u)$ is a parametric curve and an asymptotic curve on the surface $\varphi(u,v)$.

If the curve is both an asymptotic and a parameter curve on φ, then it is called isoasymptotic on a surface φ.

3.1 Surfaces with a Common Smarandache TN Asymptotic Curve in Galilean Space G_3

Theorem 3.1 *Smarandache TN curve of the curve $r(u)$ is an isoasympotic on a surface $\varphi(u,v)$ if and only if the following relations are satisfied*

$$x\left(u,v_0\right)=y\left(u,v_0\right)=z\left(u,v_0\right)=0, \tag{5}$$

and

$$\tau=0,\quad \frac{\partial x(u,v_0)}{\partial v}\neq 0. \tag{6}$$

Proof Taking account of [4], a parametric surface $\varphi(u,v)$ is given by a Smarandache TN curve of $r(u)$ as follows

$$\varphi(u,v)=\frac{T(u)+N(u)}{\|T(u)+N(u)\|}+\left[x(u,v)T(u)+y(u,v)N(u)+z(u,v)B(u)\right]. \tag{7}$$

Let $r(u)$ be a Smarandache TN curve on surface $\varphi(u,v)$. If Smarandache TN curve is a parametric curve on $\varphi(u,v)$, then there exists a parameter $v=v_0$ such that

$$\varphi(u,v)=\frac{T(u)+N(u)}{\|T(u)+N(u)\|},$$

that is

$$x(u,v_0)=y(u,v_0)=z(u,v_0)=0.$$

The normal $\eta(u,v)$ of the surface is given by

$$\eta(u,v)=\varphi_u\times\varphi_v. \tag{8}$$

From (7),

$$\begin{aligned}\varphi_u \quad &= \quad \frac{\partial x(u,v)}{\partial u}T(u)+\left(\kappa+x(u,v)\kappa+\frac{\partial y(u,v)}{\partial u}-z(u,v)\tau\right)N(u)\\ &\quad +\left(\tau+y(u,v)\tau+\frac{\partial z(u,v)}{\partial u}\right)B(u)\end{aligned}$$

and

$$\varphi_v=\frac{\partial x(u,v)}{\partial v}T(u)+\frac{\partial y(u,v)}{\partial v}N(u)+\frac{\partial z(u,v)}{\partial v}B(u).$$

Using (8), the normal $\eta(u,v)$ can be written as

$$\begin{aligned}\eta(u,v) &= \left[-\frac{\partial x(u,v)}{\partial u}\frac{\partial z(u,v)}{\partial v} + \left(\tau + y(u,v)\tau + \frac{\partial z(u,v)}{\partial u}\right)\frac{\partial x(u,v)}{\partial v}\right] N(u) \\ &\quad + \left[\frac{\partial x(u,v)}{\partial u}\frac{\partial y(u,v)}{\partial v} - \left(\kappa + x(u,v)\kappa + \frac{\partial y(u,v)}{\partial u} - z(u,v)\tau\right)\frac{\partial x(u,v)}{\partial v}\right] B(u)\end{aligned}$$

and from (5), we have

$$\eta(u,v_0) = \left[\tau\frac{\partial x(u,v_0)}{\partial v}\right] N(u) + \left[-\kappa\frac{\partial x(u,v_0)}{\partial v}\right] B(u), \tag{9}$$

which gives that $r(u)$ is an asymptotic curve if and if $B(u) \parallel \eta(u,v_0)$, we obtain

$$\tau\frac{\partial x(u,v_0)}{\partial v} = 0$$

and

$$\kappa\frac{\partial x(u,v_0)}{\partial v} \neq 0.$$

Since $\frac{\partial x(u,v_0)}{\partial v} \neq 0$, we have

$$\begin{aligned}\tau &= 0, \\ \frac{\partial x(u,v_0)}{\partial v} &\neq 0.\end{aligned}$$

This completes the proof. □

Corollary 3.2 *Smarandache TN curve of the curve $r(u)$ is an isoasympotic if and only if $r(u)$ is a plane curve.*

The set of surfaces given by (7) and satisfying (5) and (6) is called the family of surfaces with common Smarandache TN asymptotic curve in Galilean space G_3. The functions $x(u,v_0)$, $y(u,v_0)$ and $z(u,v_0)$ can be given in two different forms:

Case 1. If we take

$$\begin{aligned}x(u,v) &= a(u)X(v), \\ y(u,v) &= b(u)Y(v), \\ z(u,v) &= c(u)Z(v),\end{aligned}$$

then the sufficient condition for which Smarandache TN curve of the curve $r(u)$ is an isoasymptotic on $\varphi(u,v)$ can be given by

$$\begin{aligned}X(v_0) &= Y(v_0) = Z(v_0) = 0, \\ a(u) &\neq 0 \quad \text{and} \quad \frac{dX(v_0)}{dv} \neq 0,\end{aligned} \tag{10}$$

where $a(u), b(u), c(u), X(v), Y(v)$ and $Z(v)$ are C^1 functions and $a(u), b(u)$ and $c(u)$ are not identically zero. Also $r(u)$ should be a plane curve.

Case 2. If we take

$$\begin{aligned} x(u,v) &= f(a(u)X(v)), \\ y(u,v) &= g(b(u)Y(v)), \\ z(u,v) &= h(c(u)Z(v)), \end{aligned}$$

then the sufficient condition for which Smarandache TN curve of the curve $r(u)$ is an isoasymptotic and a plane curve on $\varphi(u,v)$ can be expressed as

$$\begin{aligned} X(v_0) &= Y(v_0) = Z(v_0) = 0, \\ f(0) &= g(0) = h(0) = 0, \\ a(u) &\neq 0, \ \frac{dX(v_0)}{dv} \neq 0 \text{ and } f'(0) \neq 0, \end{aligned} \tag{11}$$

where $a(u), b(u), c(u), X(v), Y(v)$ and $Z(v)$ are C^1functions and $a(u), b(u)$ and $c(u)$ are not identically zero. Also, $r(u)$ should be a plane curve.

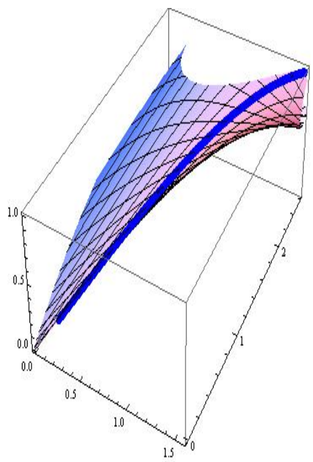

Figure 1 $\varphi(u,v)$ surface with curve $r(u)$.

Example 3.3 Let $r(u) = (u, u + \sin u, \sin u)$ be a curve and if we take $0 < u < \pi$, It is easy to show that

$$\begin{aligned} T(u) &= (1, 1 + \cos u, \cos u), \\ N(u) &= (0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), \\ B(u) &= (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), \end{aligned}$$

where $\tau = 0$ is the torsion and $\kappa = \sqrt{2}\sin u$ and $u \neq \frac{k\pi}{2}$ $(k = 0, 2, \cdots, 2n.)$ is the curvature of the curve in G_3. We obtain the family of surfaces with this isoasymptotic curve. If we choose

$$x(u,v) = v,\ y(u,v) = \sin(uv),\ z(u,v) = u(1-\cos v),$$

and $v_0 = 0$ such that equation (10) is satisfied, a member of this family in G_3 is presented by (see Figure 1 for details)

$$\varphi(u,v) = \begin{pmatrix} u+v, u+v+\sin u + v\cos u + \dfrac{u-\sin(uv)-u\cos v}{\sqrt{2}}, \\ \sin u + v\cos u + \dfrac{u\cos v - u - \sin(uv)}{\sqrt{2}} \end{pmatrix}. \quad (12)$$

If we take

$$x(u,v) = v,\ y(u,v) = \sin(uv),\ z(u,v) = u(1-\cos v)$$

and $v_0 = 0$, then (5) and (6) are satisfied. Thus, we obtain a member of the surfaces with this common Smarandache TN isoasymptotic curve as (see Figure 2 for details)

$$\varphi_{TN}(u,v) = \begin{pmatrix} 1+v, (1+v)(\cos u + 1) + \frac{u-1-\sin(uv)-u\cos v}{\sqrt{2}}, \\ (1+v)\cos u + \frac{u\cos v - \sin(uv) - u - 1}{\sqrt{2}} \end{pmatrix} \quad (13)$$

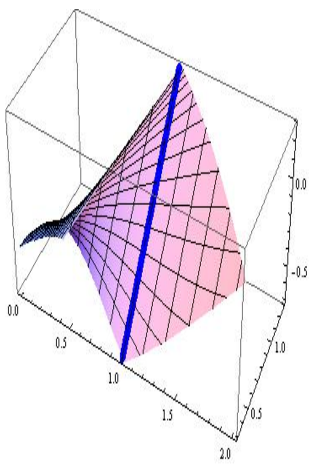

Figure 2 Surfaces $\varphi_{TN}(u,v)$ and its Smarandache TN asymptotic curve of $r(u)$.

3.2 Surfaces with a Common Smarandache TB AsymptoticCurve in Galilean Space G_3

Theorem 3.4 *Smarandache TB curve of the curve $r(u)$ is an isoasympotic on a surface $\varphi(u,v)$ if and only if the following relations are satisfied*

$$x(u,v_0) = y(u,v_0) = z(u,v_0) = 0, \quad (14)$$

$$\kappa \neq \tau \quad \text{and} \quad \frac{\partial x(u,v_0)}{\partial v} \neq 0 \quad (15)$$

Proof By using (4), the parametric surface $\varphi(u,v)$ is defined by a given Smarandache TB curve of $r(u)$ as follows

$$\varphi(u,v)=\frac{T(u)+B(u)}{\|T(u)+B(u)\|}+\left[x(u,v)T(u)+y(u,v)N(u)+z(u,v)B(u)\right]. \tag{16}$$

Let $r(u)$ be a Smarandache TB curve on surface $\varphi(u,v)$. If Smarandache TB curve is a parametric curve on this surface, then there exists a parameter $v=v_0$ such that

$$r(u)=\frac{T(u)+B(u)}{\|T(u)+B(u)\|},$$

that is

$$x(u,v_0)=y(u,v_0)=z(u,v_0)=0.$$

By using (16),

$$\begin{aligned}\varphi_u &= \frac{\partial x(u,v)}{\partial u}T(u)+\left(\kappa-\tau+x(u,v)\kappa+\frac{\partial y(u,v)}{\partial u}-z(u,v)\tau\right)N(u)\\ &\quad+\left(y(u,v)\tau+\frac{\partial z(u,v)}{\partial u}\right)B(u)\end{aligned}$$

and

$$\varphi_v=\frac{\partial x(u,v)}{\partial v}T(u)+\frac{\partial y(u,v)}{\partial v}N(u)+\tau+\frac{\partial z(u,v)}{\partial v}B(u).$$

Using (8), the normal $\eta(u,v)$ can be expressed as

$$\begin{aligned}\eta(u,v) &= \left[-\frac{\partial x(u,v)}{\partial u}\frac{\partial z(u,v)}{\partial v}+\left(y(u,v)\tau+\frac{\partial z(u,v)}{\partial u}\right)\frac{\partial x(u,v)}{\partial v}\right]N(u)\\ &\quad+\left[\frac{\partial x(u,v)}{\partial u}\frac{\partial y(u,v)}{\partial v}-\left(\kappa-\tau+x(u,v)\kappa+\frac{\partial y(u,v)}{\partial u}-z(u,v)\tau\right)\frac{\partial x(u,v)}{\partial v}\right]B(u)\end{aligned}$$

and from (15), we have

$$\eta(u,v_0)=\left[(\tau-\kappa)\frac{\partial x(u,v_0)}{\partial v}\right]B(u). \tag{17}$$

We know that $r(u)$ is an asymptotic curve if and only if

$$(\tau-\kappa)\frac{\partial x(u,v_0)}{\partial v}\neq 0.$$

That is

$$\kappa\neq\tau \quad \text{and} \quad \frac{\partial x(u,v_0)}{\partial v}\neq 0. \tag{18}$$

This completes the proof. □

The set of surfaces given by (16) and satisfying (14) and (15) is called the family of surfaces with common Smarandache TB asymptotic curve in Galilean space G_3. The marching-scale functions can be given in two different forms:

Case 1. If we take

$$\begin{aligned} x(u,v) &= a(u)X(v), \\ y(u,v) &= b(u)Y(v), \\ z(u,v) &= c(u)Z(v), \end{aligned}$$

then the sufficient condition for which Smarandache TB curve of the curve $r(u)$ is an isoasymptotic on the surface $\varphi(u,v)$ can be expressed as

$$\begin{aligned} X(v_0) &= Y(v_0) = Z(v_0) = 0 \text{ and } \kappa \neq \tau, \\ a(u) &\neq 0 \quad \text{and} \quad \frac{dX(v_0)}{dv} \neq 0, \end{aligned} \tag{19}$$

where $a(u), b(u), c(u), X(v), Y(v)$ and $Z(v)$ are C^1functions and $a(u), b(u)$ and $c(u)$ are not identically zero.

Case 2. If we take

$$\begin{aligned} x(u,v) &= f(a(u)X(v)), \\ y(u,v) &= g(b(u)Y(v)), \\ z(u,v) &= h(c(u)Z(v)), \end{aligned}$$

then the sufficient condition for which Smarandache TB curve of the curve $r(u)$ is an isoasymptotic on the surface $\varphi(u,v)$ can be expressed as

$$\begin{aligned} X(v_0) &= Y(v_0) = Z(v_0) = 0 \quad \text{and} \quad \kappa \neq \tau, \\ f(0) &= g(0) = h(0) = 0, \\ a(u) &\neq 0, \ \frac{dX(v_0)}{dv} \neq 0, \ f'(0) \neq 0, \end{aligned} \tag{20}$$

where $a(u), b(u), c(u), X(v), Y(v)$ and $Z(v)$ are C^1 functions and $a(u), b(u)$ and $c(u)$ are not identically zero.

Example 3.5 Let $r(u) = (u, \ 2\sin u, 2\cos u)$ be a curve. It is easy to show that

$$\begin{aligned} T(u) &= (1, 2\cos u, -2\sin u), \\ N(u) &= (0, -\sin u, -\cos u), \\ B(u) &= (0, \cos u, -\sin u), \end{aligned}$$

where $\tau = -1$ is the torsion and $\kappa = 2$ is the curvature of the curve in G_3. We will give the family of surfaces with this isoasymptotic curve. If we choose

$$x(u,v) = v, \ y(u,v) = \cos v - 1, \ z(u,v) = \sin(uv)$$

and $v_0 = 0$ such that equation (19) is satisfied, a member of this family in G_3 is obtained by

(See Figure 3 for details)

$$\begin{aligned}\varphi(u,v) &= (u+v, 2v\cos u + \sin u(3-\cos v) + \cos u \sin(uv),\\ &\quad \cos u(3-\cos v) - 2v\sin u - \sin u \sin(uv)).\end{aligned} \tag{21}$$

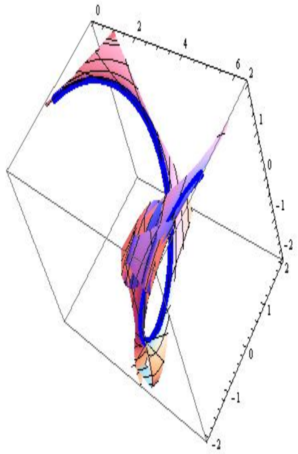

Figure 3 $\varphi(u,v)$ surface and the curve $r(u)$.

If we take

$$x(u,v) = v,\ \ y(u,v) = \cos v - 1,\ \ z(u,v) = \sin(uv)$$

and $v_0 = 0$ then the equality (14) and (15) are satisfied. Thus, we obtain a member of the surfaces with this common Smarandache TB isoasymptotic curve as (Figure 4)

$$\begin{aligned}\varphi_{TB}(u,v) &= (v+1, \cos u(2v + \sin(uv) + 3) + \sin u(1-\cos v),\\ &\quad \cos u(1-\cos v) - \sin u(2v + \sin(uv) + 3).\end{aligned} \tag{22}$$

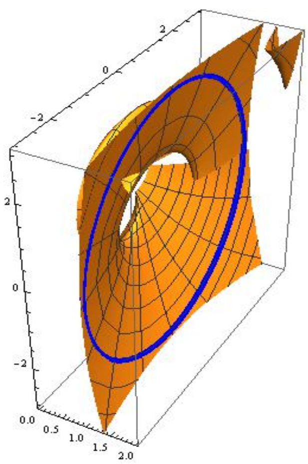

Figure 4 $\varphi_{TB}(u,v)$ surface and its Smarandache TB asymptotic curve of $r(u)$.

3.3 Surfaces with Common Smarandache TNB Asymptotic Curve in Galilean Space G_3

Theorem 3.6 *Smarandache TNB curve of the curve $r(u)$ is an isoasympotic on a surface $\varphi(u,v)$ if and only if the following conditions are satisfied:*

$$x(u,v_0) = y(u,v_0) = z(u,v_0) = 0, \tag{23}$$

$$\tau = 0 \text{ and } \frac{\partial x(u,v_0)}{\partial v} \neq 0 \tag{24}$$

Proof From (4), a parametric surface $\varphi(u,v)$ is defined by a given Smarandache TNB curve of $r(u)$ as follows

$$\varphi(u,v) = \frac{T(u)+N(u)+B(u)}{\|T(u)+N(u)+B(u)\|} + [x(u,v)T(u) + y(u,v)N(u) + z(u,v)B(u)]\,. \tag{25}$$

Let $r(u)$ be a Smarandache TNB curve on surface $\varphi(u,v)$. If Smarandache TNB curve is an parametric curve on this surface, then there exists a parameter $v = v_0$ such that

$$r(u) = \frac{T(u)+N(u)+B(u)}{\|T(u)+N(u)+B(u)\|},$$

that is

$$x(u,v_0) = y(u,v_0) = z(u,v_0) = 0.$$

From

$$\begin{aligned}\varphi_u &= \frac{\partial x(u,v)}{\partial u}T(u) + \left(\kappa - \tau + x(u,v)\kappa + \frac{\partial y(u,v)}{\partial u} - z(u,v)\tau\right)N(u) \\ &\quad + \left(y(u,v)\tau + \frac{\partial z(u,v)}{\partial u} + \tau\right)B(u)\end{aligned}$$

and

$$\varphi_v = \frac{\partial x(u,v)}{\partial v}T(u) + \frac{\partial y(u,v)}{\partial v}N(u) + \frac{\partial z(u,v)}{\partial v}B(u).$$

Using (8), the normal $\eta(u,v)$ can be written as

$$\begin{aligned}\eta(u,v) &= \left[-\frac{\partial x(u,v)}{\partial u}\frac{\partial z(u,v)}{\partial v} + \left(y(u,v)\tau + \frac{\partial z(u,v)}{\partial u} + \tau\right)\frac{\partial x(u,v)}{\partial v}\right]N(u) \\ &\quad + \left[\frac{\partial x(u,v)}{\partial u}\frac{\partial y(u,v)}{\partial v} - \left(\kappa - \tau + x(u,v)\kappa + \frac{\partial y(u,v)}{\partial u} - z(u,v)\tau\right)\frac{\partial x(u,v)}{\partial v}\right]B(u)\end{aligned}$$

and from (23), we have

$$\eta(u,v_0) = \left[\frac{\partial x(u,v_0)}{\partial v}\tau\right]N(u) + \left[(\tau - \kappa)\frac{\partial x(u,v_0)}{\partial v}\right]B(u). \tag{26}$$

We know that $r(u)$ is an asymptotic curve if and only if

$$\frac{\partial x(u,v_0)}{\partial v}\tau = 0$$

and

$$(\tau-\kappa)\frac{\partial x(u,v_0)}{\partial v} \neq 0.$$

Consequently, we have

$$\frac{\partial x(u,v_0)}{\partial v} \neq 0, \tau = 0. \tag{27}$$

This completes the proof. □

Corollary 3.7 *Smarandache TNB curve of the curve $r(u)$ is an isoasympotic if and only if $r(u)$ is a plane curve.*

The set of surfaces given by (25) and satisfying (23) and (24) is called the family of surfaces with common Smarandache TNB asymptotic curve in Galilean space G_3. The marching-scale functions $x(u,v_0)$, $y(u,v_0)$ and $z(u,v_0)$ can be given two different forms:

Case 1. If we take

$$\begin{aligned} x(u,v) &= a(u)X(v),\\ y(u,v) &= b(u)Y(v),\\ z(u,v) &= c(u)Z(v), \end{aligned}$$

then the sufficient condition for which Smarandache TNB curve of the curve $r(u)$ is isoasymptotic on the surface $\varphi(u,v)$ can be expressed as

$$\begin{aligned} X(v_0) &= Y(v_0) = Z(v_0) = 0,\\ a(u) &\neq 0, \ \frac{dX(v_0)}{dv} \neq 0, \end{aligned} \tag{28}$$

where $a(u), b(u), c(u), X(v), Y(v)$ and $Z(v)$ are C^1 functions and $a(u), b(u)$ and $c(u)$ are not identically zero. Also $r(u)$ should be a plane curve.

Case 2. If we take

$$\begin{aligned} x(u,v) &= f(a(u)X(v)),\\ y(u,v) &= g(b(u)Y(v)),\\ z(u,v) &= h(c(u)Z(v)), \end{aligned}$$

then the sufficient condition for which Smarandache TNB curve of the curve $r(u)$ is isoasymptotic on the surface $\varphi(u,v)$ can be expressed as

$$\begin{aligned} X(v_0) &= Y(v_0) = Z(v_0) = 0, \\ f(0) &= g(0) = h(0) = 0, \\ a(u) &\neq 0, \ \frac{dX(v_0)}{dv} \neq 0, \ f\'(0) \neq 0, \end{aligned} \tag{29}$$

where $a(u), b(u), c(u), X(v), Y(v)$ and $Z(v)$ are C^1 functions and $a(u), b(u)$ and $c(u)$ are not identically zero. Also $r(u)$ should be a plane curve.

Example 3.8 Let $r(u) = (u, \ \cos u, u + \cos u)$ be a curve and if we take $0 \leq u \leq \pi$, It is easy to show that

$$\begin{aligned} T(u) &= (1, -\sin u, 1 - \sin u), \\ N(u) &= (0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), \\ B(u) &= (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), \end{aligned}$$

where $\tau = 0$, $\kappa = \sqrt{2}\cos u$ and $u \neq \frac{k\pi}{2}$ $(k = 1, 3, \cdots, 2n-1.)$ in G_3. We will give the family of surfaces with this isoasymptotic curve. If we choose

$$x(u, v) = v, \ y(u, v) = \sin v, \ z(u, v) = e^{uv} - 1.$$

and $v_0 = 0$ such that equation (28) is satisfied, a member of this family in G_3 is obtained by (see Figure 5 for details)

$$\varphi(u, v) = \begin{pmatrix} u + v, \cos u - v \sin u + \dfrac{e^{uv} - 1 - \sin v}{\sqrt{2}}, \\ u + v + \cos u - v \sin u + \dfrac{1 - e^{uv} - \sin v}{\sqrt{2}} \end{pmatrix}.$$

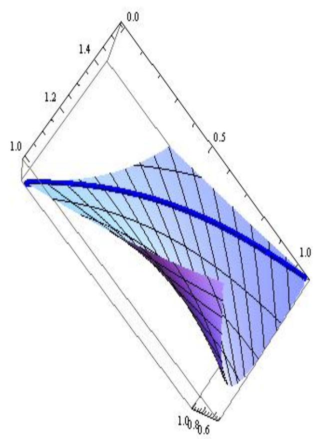

Figure 5 $\varphi(u, v)$ surface with the curve $r(u)$.

If we take

$$x(u,v) = v,\ y(u,v) = \sin v,\ z(u,v) = e^{uv} - 1$$

and $v_0 = 0$ then (23) and (24) are satisfied. Thus, we obtain a member of the surfaces with this common Smarandache TNB isoasymptotic curve as (see Figure 6 for details)

$$\varphi_{TNB}(u,v) = \begin{pmatrix} v+1, \dfrac{e^{uv} - 1 - \sin v}{\sqrt{2}} - v\sin u - \sin u, \\ v - v\sin u + \dfrac{1 - e^{uv} - \sin v}{\sqrt{2}} + 1 - \sqrt{2} - \sin u \end{pmatrix}.$$

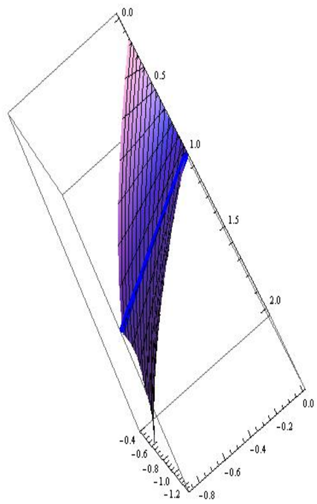

Figure 6 $\varphi_{TNB}(u,v)$ surface and its Smarandache TNB asymptotic curve of $r(u)$.

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On n-Polynomial P-Function and Related Inequalities

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Abstract: In this paper, we introduce and study the concept of n-polynomial P-function and establish Hermite-Hadamard's inequalities for this type of functions. In addition, we obtain some new Hermite-Hadamard type inequalities for functions whose first derivative in absolute value is n -polynomial P-function by using Hölder and power-mean integral inequalities. Some applications to special means of real numbers are also given.

Key Words: n-polynomial convexity, n-polynomial P-function, Hermite-Hadamard inequality.

AMS(2010): 26A51, 26D10, 26D15.

§1. Preliminaries

Let $f : I \to \mathbb{R}$ be a convex function. Then the following inequalities hold

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

for all $a, b \in I$ with $a < b$. Both inequalities hold in the reversed direction if the function f is concave. This double inequality is well known as the Hermite-Hadamard inequality []5. Note that some of the classical inequalities for means can be derived from Hermite-Hadamard integral inequalities for appropriate particular selections of the mapping f.

In [4], Dragomir et al. gave the following definition and related Hermite-Hadamard integral inequalities as follow:

Definition 1.1 *A nonnegative function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be P-function if the inequality*

$$f(tx + (1-t)y) \leq f(x) + f(y)$$

holds for all $x, y \in I$ and $t \in (0, 1)$.

Theorem 1.1 *Let $f \in P(I)$, $a, b \in I$ with $a < b$ and $f \in L_1[a, b]$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_a^b f(x)\,dx \leq 2\left[f(a) + f(b)\right]. \tag{1.2}$$

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In [10], Tekin et al. gave the following definition and related Hermite-Hadamard integral inequalities as follow:

Definition 1.2 *Let $n \in \mathbb{N}$. A non-negative function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called n-polynomial convex function if for every $x, y \in I$ and $t \in [0,1]$,*

$$f\left(tx+(1-t)y\right) \leq \frac{1}{n}\sum_{s=1}^{n}\left[1-(1-t)^{s}\right]f(x)+\frac{1}{n}\sum_{s=1}^{n}\left[1-t^{s}\right]f(y).$$

Theorem 1.2([10]) *Let $f : [a,b] \to \mathbb{R}$ be a n-polynomial convex function. If $a < b$ and $f \in L\,[a,b]$, then the following Hermite-Hadaamrd type inequalities hold:*

$$\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_{a}^{b}f(x)dx \leq \left(\frac{f\left(a\right)+f(b)}{n}\right)\sum_{s=1}^{n}\frac{s}{s+1}.$$

The main purpose of this paper is to introduce the concept of n-polynomial P-function which is connected with the concepts of P-function and n-polynomial convex function and establish some new Hermite-Hadamard type inequality for these classes of functions. In recent years many authors have studied error estimations Hermite-Hadamard type inequalities; for refinements, counterparts, generalizations, for some related papers see [1, 2, 3, 4, 6, 7, 8, 9, 10].

§2. Definition of n-Polynomial P-Function

In this section, we introduce a new concept, which is called n-polynomial P-function and we give by setting some algebraic properties for the n-polynomial P-function, as follows:

Definition 2.1 *Let $n \in \mathbb{N}$. A non-negative function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called n-polynomial P-function if for every $x, y \in I$ and $t \in [0,1]$,*

$$f\left(tx+(1-t)y\right) \leq \frac{1}{n}\sum_{s=1}^{n}\left[2-t^{s}-(1-t)^{s}\right]\left[f(x)+f(y)\right]. \tag{2.1}$$

We will denote by $POLP\,(I)$ the class of all n-polynomial P-functions on interval I. Notice that every n-polynomial P-function is a h-convex function with the function

$$h(t)=\frac{1}{n}\sum_{s=1}^{n}\left[2-t^{s}-(1-t)^{s}\right].$$

Therefore, if $f, g \in POLC\,(I)$, then

(i) *$f+g \in POLCP\,(I)$ and for $c \in \mathbb{R}$ $(c \geq 0)$ $cf \in POLCP\,(I)$* (see [11], Proposition 9).

(ii) If f and g be a similarly ordered functions on I , then $fg \in POLCP\,(I)$.(see [11], Proposition 10).

Also, if $f : I \to J$ is a convex and $g \in POLCP(J)$ and nondecreasing, then $g \circ f \in POLCP(I)$ (see [11], Theorem 15).

Remark 2.1 We note that if f satisfies (2.1), then f is a nonnegative function. Indeed, if we rewrite the inequality (2.1) for $t = 0$, then

$$f(y) \leq f(x) + f(y)$$

for every $x, y \in I$. Thus we have $f(x) \geq 0$ for all $x \in I$.

Proposition 2.1 *Every nonnegative P-function is also a n-polynomial P-function.*

Proof The proof is clear from the following inequalities

$$t \leq \frac{1}{n}\sum_{s=1}^{n}\left[1-(1-t)^s\right] \text{ and } 1-t \leq \frac{1}{n}\sum_{s=1}^{n}\left[1-t^s\right]$$

for all $t \in [0,1]$ and $n \in \mathbb{N}$. In this case, we can write

$$1 \leq \frac{1}{n}\sum_{s=1}^{n}\left[2-t^s-(1-t)^s\right].$$

Therefore, the desired result is obtained. □

We can give the following corollary for every nonnegative convex function is also a P-function.

Corollary 2.1 *Every nonnegative convex function is also a n-polynomial P-function.*

Theorem 2.1 *Let $b > a$ and $f_\alpha : [a,b] \to \mathbb{R}$ be an arbitrary family of n-polynomial P-function and let $f(x) = \sup_\alpha f_\alpha(x)$. If $J = \{u \in [a,b] : f(u) < \infty\}$ is nonempty, then J is an interval and f is a n-polynomial P-function on J.*

Proof Let $t \in [0,1]$ and $x, y \in J$ be arbitrary. Then

$$\begin{aligned}
f(tx+(1-t)y) &= \sup_\alpha f_\alpha(tx+(1-t)y) \\
&\leq \sup_\alpha\left[\frac{1}{n}\sum_{s=1}^{n}\left(2-t^s-(1-t)^s\right)\left[f_\alpha(x)+f_\alpha(y)\right]\right] \\
&\leq \frac{1}{n}\sum_{s=1}^{n}\left(2-t^s-(1-t)^s\right)\left[\sup_\alpha f_\alpha(x)+\sup_\alpha f_\alpha(y)\right] \\
&= \frac{1}{n}\sum_{s=1}^{n}\left(2-t^s-(1-t)^s\right)\left[f(x)+f(y)\right] < \infty.
\end{aligned}$$

This shows simultaneously that J is an interval, since it contains every point between any two of its points, and that f is a n-polynomial P-function on J. □

§3. Hermite-Hadamard Inequality for n-Polynomial P-Functions

The goal of this paper is to establish some inequalities of Hermite-Hadamard type for n-polynomial P-functions. In this section, we will denote by $L[a,b]$ the space of (Lebesgue) integrable functions on $[a,b]$.

Theorem 3.1 *Let $f:[a,b]\to\mathbb{R}$ be a n-polynomial P-function. If $a<b$ and $f\in L[a,b]$, then the following Hermite-Hadamard type inequalities hold*

$$\frac{1}{4}\left(\frac{n}{n+2^{-n}-1}\right)f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\left(\frac{f(a)+f(b)}{n}\right)\sum_{s=1}^{n}\frac{2s}{s+1}. \tag{3.1}$$

Proof From the property of the n-polynomial P-function of f, we get

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &= f\left(\frac{[ta+(1-t)b]+[(1-t)a+tb]}{2}\right)\\ &= f\left(\frac{1}{2}[ta+(1-t)b]+\frac{1}{2}[(1-t)a+tb]\right)\\ &\le \frac{1}{n}\sum_{s=1}^{n}\left[2-2\left(\frac{1}{2}\right)^s\right][f(ta+(1-t)b)+f((1-t)a+tb)].\end{aligned}$$

By taking integral in the last inequality with respect to $t\in[0,1]$, we deduce that

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &\le \frac{1}{n}\sum_{s=1}^{n}\left[2-2\left(\frac{1}{2}\right)^s\right]\left[\int_0^1 f(ta+(1-t)b)\,dt+\int_0^1 f((1-t)a+tb)\,dt\right]\\ &= \frac{4}{b-a}\left(\frac{n+2^{-n}-1}{n}\right)\int_a^b f(x)dx.\end{aligned}$$

By using the property of the n-polynomial P-function of f, if the variable is changed as $x=ta+(1-t)b$, then

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)du &= \int_0^1 f(ta+(1-t)b)\,dt\\ &\le \int_0^1\left[\frac{1}{n}\sum_{s=1}^{n}[2-t^s-(1-t)^s][f(a)+f(b)]\right]dt\\ &= \frac{f(a)+f(b)}{n}\sum_{s=1}^{n}\int_0^1[2-t^s-(1-t)^s]\,dt\\ &= \left[\frac{f(a)+f(b)}{n}\right]\sum_{s=1}^{n}\frac{2s}{s+1},\end{aligned}$$

where

$$\int_0^1[2-t^s-(1-t)^s]\,dt=\frac{2s}{s+1}.$$

This completes the proof of theorem. □

Remark 3.1 In case of $n = 1$, the inequality (3.1) coincides with the the inequality (1.2).

§4. New Inequalities for n-Polynomial P-Functions

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard inequality for functions whose first derivative in absolute value is n-polynomial P-function. Dragomir and Agarwal [3] used the following lemma.

Lemma 4.1([3]) *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $f' \in L[a, b]$, then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)\,dt.$$

Theorem 4.1 it Let $f : I \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and assume that $f' \in L[a, b]$. If $|f'|$ is n-polynomial P-function on interval $[a, b]$, then the following inequality

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{n}\sum_{s=1}^{n}\left[\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^s}\right] A\left(\left|f'(a)\right|, \left|f'(b)\right|\right) \quad (4.1)$$

holds for $t \in [0, 1]$.

Proof Using Lemma 4.1 and the inequality

$$\left|f'(ta+(1-t)b)\right| \leq \frac{1}{n}\sum_{s=1}^{n}\left(2-t^s-(1-t)^s\right)\left[|f'(a)|+|f'(b)|\right],$$

we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \left|\frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)\,dt\right| \\
&\leq \frac{b-a}{2}\int_0^1 |1-2t|\left(\frac{1}{n}\sum_{s=1}^{n}\left(2-t^s-(1-t)^s\right)\left[|f'(a)|+|f'(b)|\right]\right)dt \\
&\leq \frac{b-a}{2n}\left[|f'(a)|+|f'(b)|\right]\sum_{s=1}^{n}\int_0^1 |1-2t|\left(2-t^s-(1-t)^s\right)dt \\
&= \frac{b-a}{n}\sum_{s=1}^{n}\left[\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^s}\right] A\left(\left|f'(a)\right|, \left|f'(b)\right|\right),
\end{aligned}$$

where

$$\int_0^1 |1-2t|\left(2-t^s-(1-t)^s\right)dt = \frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^s}$$

and A is the arithmetic mean. This completes the proof of theorem. □

Corollary 4.1 *If we take $n = 1$ in the inequality (4.1), we get the following inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2} A\left(\left|f'(a)\right|, \left|f'(b)\right|\right). \tag{4.2}$$

Theorem 4.2 *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and assume that $f' \in L[a,b]$. If $|f'|^q$, $q > 1$, is an n-polynomial P-function on interval $[a,b]$, then the following inequality holds for $t \in [0,1]$.*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\leq \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{4}{n}\sum_{s=1}^{n}\frac{s}{s+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right),\end{aligned} \tag{4.3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and A is the arithmetic mean.

Proof Using Lemma 4.1, Hölder's integral inequality and the following inequality

$$|f'(ta+(1-t)b)|^q \leq \frac{1}{n}\sum_{s=1}^{n}\left(2 - t^s - (1-t)^s\right)\left[|f'(a)|^q + |f'(b)|^q\right]$$

which is the n-polynomial P-function of $|f'|^q$, we get

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \\ &\leq \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q + |f'(b)|^q}{n}\int_0^1\sum_{s=1}^{n}\left[2 - t^s - (1-t)^s\right]dt\right)^{\frac{1}{q}}\end{aligned}$$

$$= \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{4}{n}\sum_{s=1}^{n}\frac{s}{s+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right)$$

where

$$\int_0^1 |1-2t|^p\,dt = \frac{1}{p+1}, \quad \int_0^1 \left[2 - t^s - (1-t)^s\right]dt = \frac{2s}{s+1}$$

This completes the proof of theorem. □

Corollary 4.2 *If we take $n=1$ in the inequality (4.3), we get the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq (b-a)\left[\frac{1}{2(p+1)}\right]^{\frac{1}{p}} A^{\frac{1}{q}}\left(|f'(a)|^q,|f'(b)|^q\right).$$

Theorem 4.3 *Let $f: I\subseteq\mathbb{R}\rightarrow\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$ and assume that $f'\in L[a,b]$. If $|f'|^q, q\geq 1$, is an n-polynomial P-function on the interval $[a,b]$, then the following inequality holds for $t\in[0,1]$.*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\leq \frac{b-a}{2^{2-\frac{1}{q}}}\left(\frac{1}{n}\sum_{s=1}^{n}\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s-1}}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(|f'(a)|^q,|f'(b)|^q\right). \qquad (4.4)\end{aligned}$$

Proof From Lemma 4.1, well known power-mean integral inequality and the property of the n-polynomial P-function of $|f'|^q$, we obtain

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\leq \frac{b-a}{2}\left(\int_0^1|1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1|1-2t|\,|f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \\ &\leq \frac{b-a}{2^{2-\frac{1}{q}}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{n}\int_0^1\sum_{s=1}^{n}|1-2t|\left[2-t^s-(1-t)^s\right]dt\right)^{\frac{1}{q}} \\ &= \frac{b-a}{2^{2-\frac{1}{q}}}\left(\frac{1}{n}\sum_{s=1}^{n}\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s-1}}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(|f'(a)|^q,|f'(b)|^q\right)\end{aligned}$$

where

$$\begin{aligned}\int_0^1|1-2t|dt &= \frac{1}{2}, \\ \int_0^1|1-2t|\left[1-(1-t)^s\right]dt &= \frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^s}.\end{aligned}$$

This completes the proof of theorem. □

Corollary 4.3 *Under the assumption of Theorem 4.3, If we take $q=1$ in the inequality (4.4), then we get the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{n}\left(\sum_{s=1}^{n}\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^s}\right) A\left(|f'(a)|,|f'(b)|\right)$$

This inequality coincides with the inequality (4.1).

Corollary 4.4 *Under the assumption of Theorem* 4.3*, If we take* $n = 1$ *in the inequality (*4.4*), then we get the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2^{2-\frac{1}{q}}}A^{\frac{1}{q}}\left(\left|f'(a)\right|^q, \left|f'(b)\right|^q\right).$$

which is identical to the inequality in [1, Theorem 2.3].

Corollary 4.5 *Under the assumption of Theorem* 4.3*, If we take* $n = 1$ *and* $q = 1$ *in the inequality (*4.4*), then we get the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}A\left(\left|f'(a)\right|, \left|f'(b)\right|\right).$$

*This inequality coincides with the inequality (*4.2*).*

§5. Applications for Special Means

Throughout this section, for shortness, the following notations will be used for special means of two nonnegative numbers a, b with $b > a$.

1. The arithmetic mean

$$A := A(a,b) = \frac{a+b}{2}, \quad a, b \geq 0,$$

2. The geometric mean

$$G := G(a,b) = \sqrt{ab}, \quad a, b \geq 0$$

3. The harmonic mean

$$H := H(a,b) = \frac{2ab}{a+b}, \quad a, b > 0,$$

4. The logarithmic mean

$$L := L(a,b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & a \neq b \\ a, & a = b \end{cases} \quad ; \quad a, b > 0$$

5. The p-logaritmic mean

$$L_p := L_p(a,b) = \begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & a \neq b, p \in \mathbb{R}\backslash\{-1,0\} \\ a, & a = b \end{cases} \quad ; \quad a, b > 0.$$

6. The identric mean

$$I := I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, \quad a,b > 0,$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships

$$H \leq G \leq L \leq I \leq A.$$

are known in the literature. It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 5.1 *Let $a,b \in [0,\infty)$ with $a < b$ and $n \in (-\infty,0) \cup [1,\infty) \setminus \{-1\}$. Then, the following inequalities are obtained:*

$$\frac{1}{4}\left(\frac{n}{n+2^{-n}-1}\right)A^n(a,b) \leq L_n^n(a,b) \leq A(a^n,b^n)\frac{2}{n}\sum_{s=1}^{n}\frac{2s}{s+1}.$$

Proof The assertion follows from the inequalities (3.1) for the function

$$f(x) = x^n, \quad x \in [0,\infty).$$ □

Proposition 5.2 *Let $a,b \in (0,\infty)$ with $a < b$. Then, the following inequalities are obtained*

$$\frac{1}{4}\left(\frac{n}{n+2^{-n}-1}\right)A^{-1}(a,b) \leq L^{-1}(a,b) \leq \frac{2}{n}H^{-1}(a,b)\sum_{s=1}^{n}\frac{2s}{s+1}.$$

Proof The assertion follows from the inequalities (3.1) for the function

$$f(x) = x^{-1}, \quad x \in (0,\infty).$$ □

Proposition 5.3 *Let $a,b \in (0,1]$ with $a < b$. Then, the following inequalities are obtained*

$$\frac{2\ln G(a,b)}{n}\sum_{s=1}^{n}\frac{2s}{s+1} \leq \ln I(a,b) \leq \frac{1}{4}\left(\frac{n}{n+2^{-n}-1}\right)\ln A(a,b).$$

Proof The assertion follows from the inequalities (3.1) for the function

$$f(x) = -\ln x, \quad x \in (0,1].$$ □

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Number of Spanning Trees of Some of the Families of Sequence Graphs Generated by Triangle Graph

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Abstract: The number of spanning trees is an important quantity characterizing the reliability of a network. In this paper, we find the explicit formulas of the number of spanning trees of some of the families of sequence graphs generated by triangle graph with its special feature in iteration. Using the electrically equivalent transformations, we obtain the weights of corresponding equivalent graphs and we further derive relationships for spanning trees between these graphs and transformed graphs. Finally, we compare the entropy of our graphs together and with other studied graphs of average degree.

Key Words: Number of spanning trees, electrically equivalent transformations, entropy, sequence graphs.

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§1. Introduction

Calculating number of spanning trees in a graph is one of the well studied problems in Graph Theory. A spanning tree of a connected graph G with n vertices is a connected $(n-1)$-edge subgraph of G. The number of spanning trees of a graph G denoted by $\tau(G)$, also called the complexity of G [1], is a well-studied quantity in graph theory, and appears in a number of applications. Most memorable application fields are network reliability [18], recounting certain chemical isomers [2], and counting the number of Eulerian circuits in a graph [1]. In particular, counting spanning trees is an essential step in many methods for computing, bounding, and approximating network reliability [4]. In a network modeled by a graph, intercommunication between all nodes of the network implies that the graph must contain a spanning tree and thus maximizing the number of spanning trees is a way of maximizing reliability.

In 1847, a classical result of Kirchoff [16] can be used to determine the number of spanning trees for a connected graph $G = (V, E)$ with n vertices $\{v_1, v_2, \cdots, v_n\}$, and the Kirchoff matrix L is defined as $n \times n$ characteristic matrix $L = D - A$, where D is the diagonal matrix of the

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degrees of G and A is the adjacency matrix of G, $L = [a_{ij}]$, where

$$a_{ij} = \begin{cases} deg(v_j), \text{ if } i = j \\ -1, \text{ if } (v_i, v_j) \in E(G) \\ 0, \text{ if } (v_i, v_j) \notin E(G) \end{cases}$$

All of co-factors of L are equal to the number of spanning trees of the graph G.

Another method for calculating $\tau(G)$ is described as follows:

Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ denote the eigenvalues of L matrix of a graph G with n vertices. In 1974, Kelmans and Chelnokov [15] has shown that

$$\tau(G) = \frac{1}{n} \prod_{i=1}^{n-1} \mu_i. \tag{1}$$

One common method for finding the number of spanning trees, $\tau(G)$, is the deletion-contraction method. This method is a dependable method which allows to calculate the number of spanning trees of a multigraph G. This method uses the fact that

$$\tau(G) = \tau(G - e) + \tau(G/e), \tag{2}$$

where $G - e$ denotes the graph obtained by deleting an arbitrary edge e and G/e denotes the graph obtained by contracting an arbitrary edge e [1,5]. For more methods and other techniques see [6]-[22].

§2. Electrically Equivalent Transformations

An electrical network is an interconnection of electrical components (eg. inductors, capacitors, batteries, resistors, switches, etc.).

In this section, we provide the relationships between electrical networks and spanning trees. Let G be an edge weighted graph, G' be the corresponding electrically equivalent graph and $\tau(G)$ denote the weighted number of spanning trees of G. Using the results in [20,21], we have the following transformation rules:

Parallel edges: If two parallel edges with conductances a and b in G are merged into a single edge with conductances $a + b$ in G', then $\tau(G') = \tau(G)$.

Serial edges: If two serial edges with conductances a and b in G are merged into a single edge with conductance $\frac{ab}{a+b}$ in G', then $\tau(G') = \frac{1}{a+b}\tau(G)$.

Delta-Wye transformation: If a tiangle with conductances a, b and c in G is changed into an electrically equivalent star graph with conductances $r = \frac{ab+bc+ac}{a}$, $s = \frac{ab+bc+ac}{b}$ and $r = \frac{ab+bc+ac}{c}$ in G', then $\tau(G') = \frac{(ab+bc+ac)^2}{abc}\tau(G)$.

Wye-Delta transformation: If a star graph with conductances a, b and c in G is changed into an electrically equivalent triangle with conductances $r = \frac{bc}{a+b+c}$, $s = \frac{ac}{a+b+c}$ and $t = \frac{ab}{a+b+c}$

in G', then $\tau(G') = \frac{1}{a+b+c}\tau(G)$.

§3. Main Results

In Mathematics, one always tries to get new structure from given once. This also applies to the realm of graphs, where one can generate many new graphs from a given set of graphs.

Consider the sequence of graphs $G_1, G_2, \cdots, G_n$ constructed as shown in Fig.1. According to the construction, the number of total vertices $|V(G_n)|$ and edges $|E(G_n)|$ are $|V(G_n)| = 6n-3$ and $|E(G_n)| = 15n-12,\ n = 1, 2, \cdots,$. It is clear that the average degree is approximately 5 for a large n.

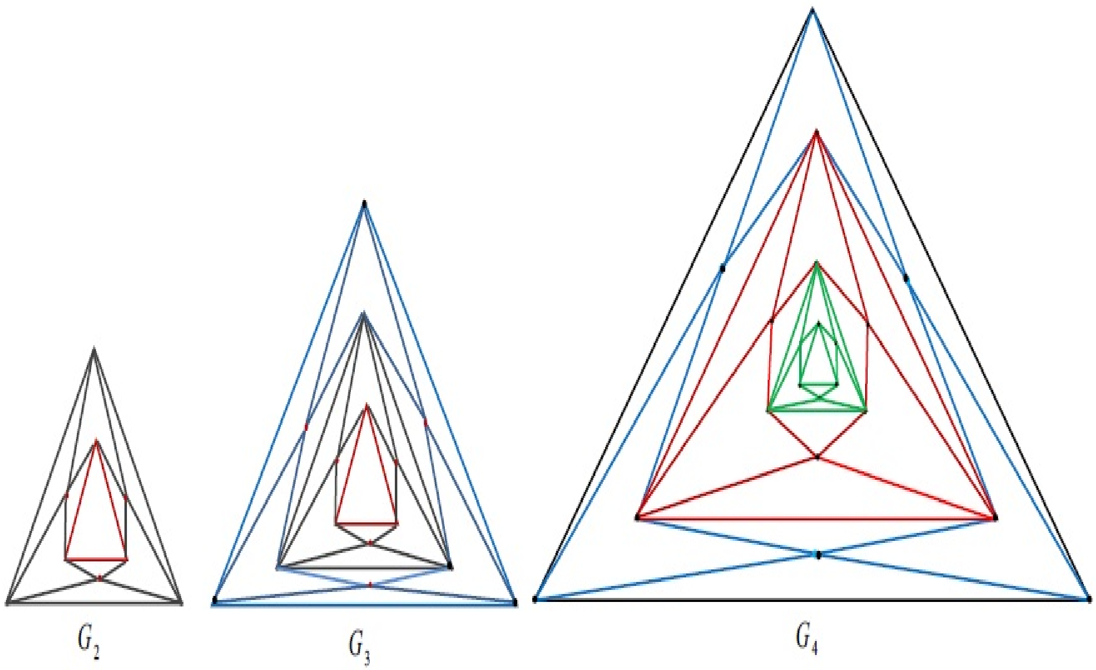

Figure 1

Theorem 3.1 *For $n \geq 1$, the number of spanning trees in the sequence of the graph G_n is given by $\frac{A}{B}$, where*

$$\begin{aligned} A &= 4^{n-4}((98-15\sqrt{42})(13+2\sqrt{42})^n + (13-2\sqrt{42})^n(98+15\sqrt{42}))^2 \\ &\quad \times((135+23\sqrt{42})(337+52\sqrt{42})^n - 11(1173+181\sqrt{42}))^2 \\ B &= 147(3707+572\sqrt{42}+(17+2\sqrt{42})(337+52\sqrt{42})^n)^2. \end{aligned}$$

Proof We use the electrically equivalent transformation to transform G_i to G_{i-1}. Fig.2 illustrates the transformation process from G_2 to G_1.

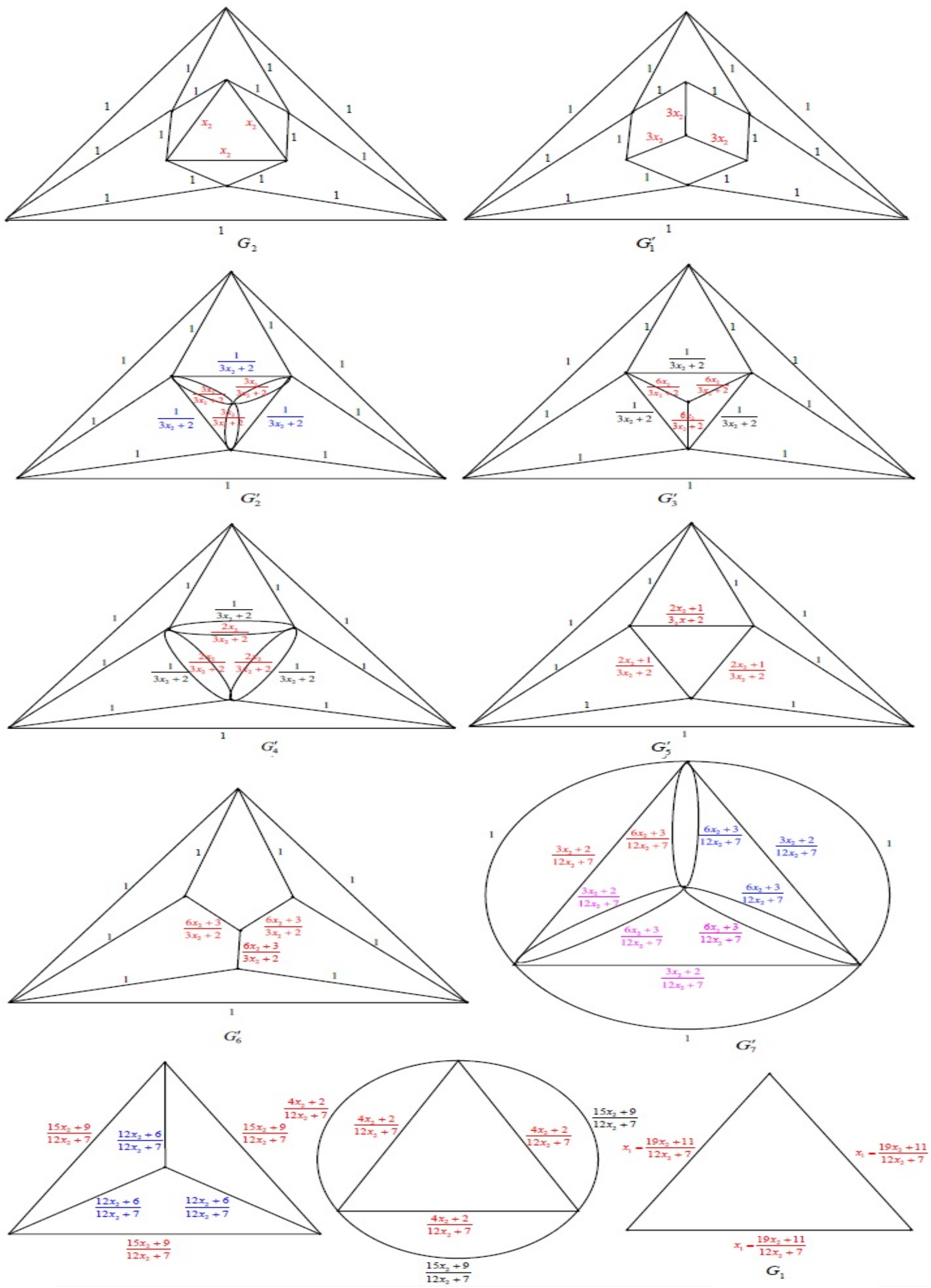

Figure 2

Using the properties given in section 2, we have the following transformations

$$\begin{aligned}
\tau(G_1') &= 9x_2\tau(G_2), & \tau(G_2') &= \frac{1}{(3x_2+2)^3}\tau(G_1'),\\
\tau(G_3') &= \tau(G_2'), & \tau(G_4') &= \frac{3x_2+2}{18x_2}\tau(G_3'),\\
\tau(G_5') &= \tau(G_4'), & \tau(G_6') &= 9\left(\frac{2x_2+1}{3x_2+2}\right)\tau(G_5'),\\
\tau(G_7') &= \left(\frac{3x_2+2}{12x_2+7}\right)^3\tau(G_6'), & \tau(G_8') &= \tau(G_7'),\\
\tau(G_9') &= \frac{12x_2+7}{3(12x_2+6)}\tau(G_8'), & \tau(G_1) &= \tau(G_9').
\end{aligned}$$

Combining these ten transformations, we have

$$\tau(G_2) = 4(12x_2+7)^2\tau(G_1) \tag{3}$$

and further,

$$\tau(G_n) = \prod_{i=2}^{n} 4(12x_i+7)^2\tau(G_1) = 3\times 4^{n-1}x_i^2\left[\prod_{i=2}^{n}(12x_i+7)\right]^2, \tag{4}$$

where $x_{i-1} = \frac{19x_i+11}{12x_i+7}, 2 \le i \le n$. Its characteristic equation is $12u^2 - 12u - 11 = 0$ which have two roots $u_1 = \frac{3-\sqrt{42}}{6}$ and $u_2 = \frac{3+\sqrt{42}}{6}$.

Subtracting these two roots into both sides of $x_{i-1} = \frac{19x_i+11}{12a_i+7}$, we get

$$x_{i-1} - \frac{(3-\sqrt{42})}{6} = \frac{19x_i+11}{12x_i+7} - \frac{3-\sqrt{42}}{6} = (13+2\sqrt{42})\left[\frac{x_i - \left(\frac{1}{2}-\sqrt{\frac{7}{6}}\right)}{12x_i+7}\right] \tag{5}$$

$$x_{i-1} - \frac{(3+\sqrt{42})}{6} = \frac{19x_i+11}{12x_i+7} - \frac{3+\sqrt{42}}{6} = (13-2\sqrt{42})\left[\frac{x_i - \left(\frac{1}{2}+\sqrt{\frac{7}{6}}\right)}{12x_i+7}\right] \tag{6}$$

Let $y_i = \frac{x_i-\left(\frac{1}{2}-\sqrt{\frac{7}{6}}\right)}{x_i-\left(\frac{1}{2}+\sqrt{\frac{7}{6}}\right)}$. Then by equations (5) and (6), we get

$$y_{i-1} = (337+52\sqrt{42})y_i, \quad y_i = (337+52\sqrt{42})^{n-i}y_n.$$

Therefore,

$$x_i = \frac{(337+52\sqrt{42})^{n-i}\left(\frac{1}{2}+\sqrt{\frac{7}{6}}\right)y_n - \left(\frac{1}{2}-\sqrt{\frac{7}{6}}\right)}{(337+52\sqrt{42})^{n-i}y_n - 1}.$$

Thus

$$x_1 = \frac{(337+52\sqrt{42})^{n-1}\left(\frac{1}{2}+\sqrt{\frac{7}{6}}\right)y_n - \left(\frac{1}{2}-\sqrt{\frac{7}{6}}\right)}{(337+52\sqrt{42})^{n-1}y_n - 1}. \tag{7}$$

Using the expression $x_{n-1} = \frac{19x_n+11}{12x_n+7}$ and denoting the coefficients of $19x_n+11$ and $12x_n+7$ as h_n and k_n, we have

$$\begin{aligned}
12x_n + 7 &= h_0(19x_n + 11) + k_0(12x_n + 7), \\
12x_{n-1} + 7 &= \frac{h_1(19x_n + 11) + k_1(12x_n + 7)}{h_0(19x_n + 11) + k_0(12x_n + 7)}, \\
12x_{n-2} + 7 &= \frac{h_2(19x_n + 11) + k_2(12x_n + 7)}{h_1(19x_n + 11) + k_1(12x_n + 7)}, \\
&\vdots \\
12x_{n-i} + 7 &= \frac{h_i(19x_n + 11) + k_i(12x_n + 7)}{h_{i-1}(19x_n + 11) + k_{i-1}(12x_n + 7)}, \qquad (8) \\
12x_{n-(i+1)} + 11 &= \frac{h_{i+1}(19x_n + 11) + k_{i+1}(12x_n + 7)}{h_i(19x_n + 11) + k_i(12x_n + 7)}, \qquad (9) \\
&\vdots \\
12x_2 + 7 &= \frac{h_{n-2}(19x_n + 11) + k_{n-2}(12x_n + 7)}{h_{n-3}(19x_n + 11) + k_{n-3}(12x_n + 7)}. \qquad (10)
\end{aligned}$$

Thus, we obtain

$$\tau(G_n) = 3 \times 4^{n-1} x_i^2 \left[h_{n-2}(19x_n + 11) + k_{n-2}(12x_n + 7)\right]^2, \qquad (11)$$

where $h_0 = 0,\ k_0 = 1$ and $h_1 = 12,\ k_1 = 7$. By the expression $x_{n-1} = \frac{19x_n+11}{12x_n+7}$ and using equations (8) and (9), we have

$$h_{i+1} = 26h_i - h_{i-1}; k_{i+1} = 26k_i - k_{i-1}. \qquad (12)$$

The characteristic equation of (12) is $\nu^2 - 26\nu + 1 = 0$ which have two roots $\nu_1 = 13 + 2\sqrt{42}$ and $\nu_2 = 13 - 2\sqrt{42}$. The general solutions of equation (12) are $h_i = a_1\nu_1^i + a_2\nu_2^i;\ k_i = b_1\nu_1^i + b_2\nu_2^i$.

Using the initial conditions $h_0 = 0,\ k_0 = 1$ and $h_1 = 12,\ k_1 = 7$, yields

$$h_i = \frac{\sqrt{42}}{14}(13 + 2\sqrt{42})^i - \frac{\sqrt{42}}{14}(13 - 2\sqrt{42})^i,$$

$$k_i = \left(\frac{42 - 3\sqrt{42}}{84}\right)(13 + 2\sqrt{42})^i + \left(\frac{42 + 3\sqrt{42}}{84}\right)(13 - 2\sqrt{42})^i. \qquad (13)$$

If $x_n = 1$, it means that G_n without any electrically equivalent transformation. Plugging equation (13) into equation (11), we have

$$\begin{aligned}
\tau(G_n) &= 3 \times 4^{n-1} x_i^2 \\
&\times \left[\left(\frac{798 + 123\sqrt{42}}{84}\right)(13 + 2\sqrt{42})^{n-2} + \left(\frac{798 - 123\sqrt{42}}{84}\right)(13 - 2\sqrt{42})^{n-2}\right]^2 \qquad (14)
\end{aligned}$$

if $n \geq 2$. When $n = 1$, $\tau(G_1) = 3$ which satisfies Eq.(14) Therefore, the number of spanning trees in the sequence of the graph G_n is given by

$$\tau(G_n) = 3 \times 4^{n-1} x_i^2 \times \left[\left(\frac{798 + 123\sqrt{42}}{84} \right) (13 + 2\sqrt{42})^{n-2} + \left(\frac{798 - 123\sqrt{42}}{84} \right) (13 - 2\sqrt{42})^{n-2} \right]^2, \quad (15)$$

if $n \geq 1$, where

$$x_i = \frac{(337 + 52\sqrt{42})^{n-1}(135 + 32\sqrt{42}) + 66\left(\frac{1}{2} - \sqrt{\frac{7}{6}}\right)}{6(337 + 52\sqrt{42})^{n-1}(17 + 2\sqrt{42}) + 66}, \ n \geq 2. \quad (16)$$

Inserting Eq.(16) into Eq.(15) we obtain the result.

Consider the sequence of graphs $H_1, H_2, \cdots, H_n$ constructed as shown in see Fig.3.

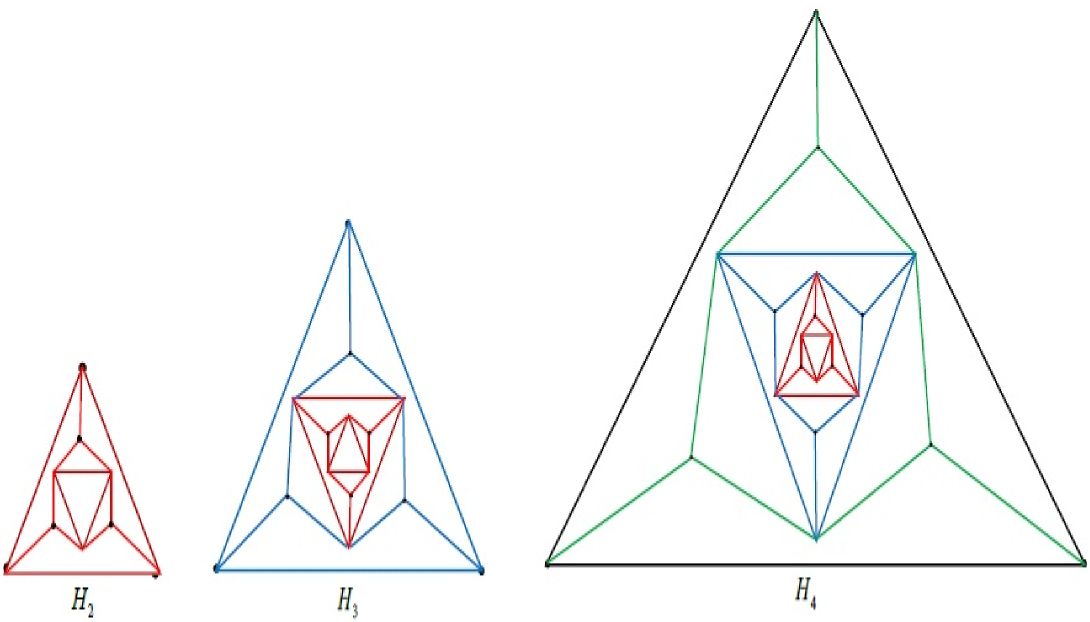

Figure 3

According to the construction, the number of total vertices $|V(H_n)|$ and edges $|E(H_n)|$ are $|V(H_n)| = 6n - 3$ and $|E(H_n)| = 12n - 9, \ n \geq 1$. It is clear that the average degree is approximately 4 for a large n. □

Theorem 3.2 *For $n \geq 1$, the number of spanning trees in the sequence of the graph H_n is given by*

$$\frac{3 \times 2^{n-3} \left(2(8 + 3\sqrt{7})^n(98 + 37\sqrt{7}) - 3(8 - 3\sqrt{7})^n(1897 + 717\sqrt{7}) + (7 + 5\sqrt{7})(2024 + 765\sqrt{7})^n\right)^2}{49\left(381 + 144\sqrt{7} + (11 + 4\sqrt{7})(127 + 48\sqrt{7})^n\right)^2}.$$

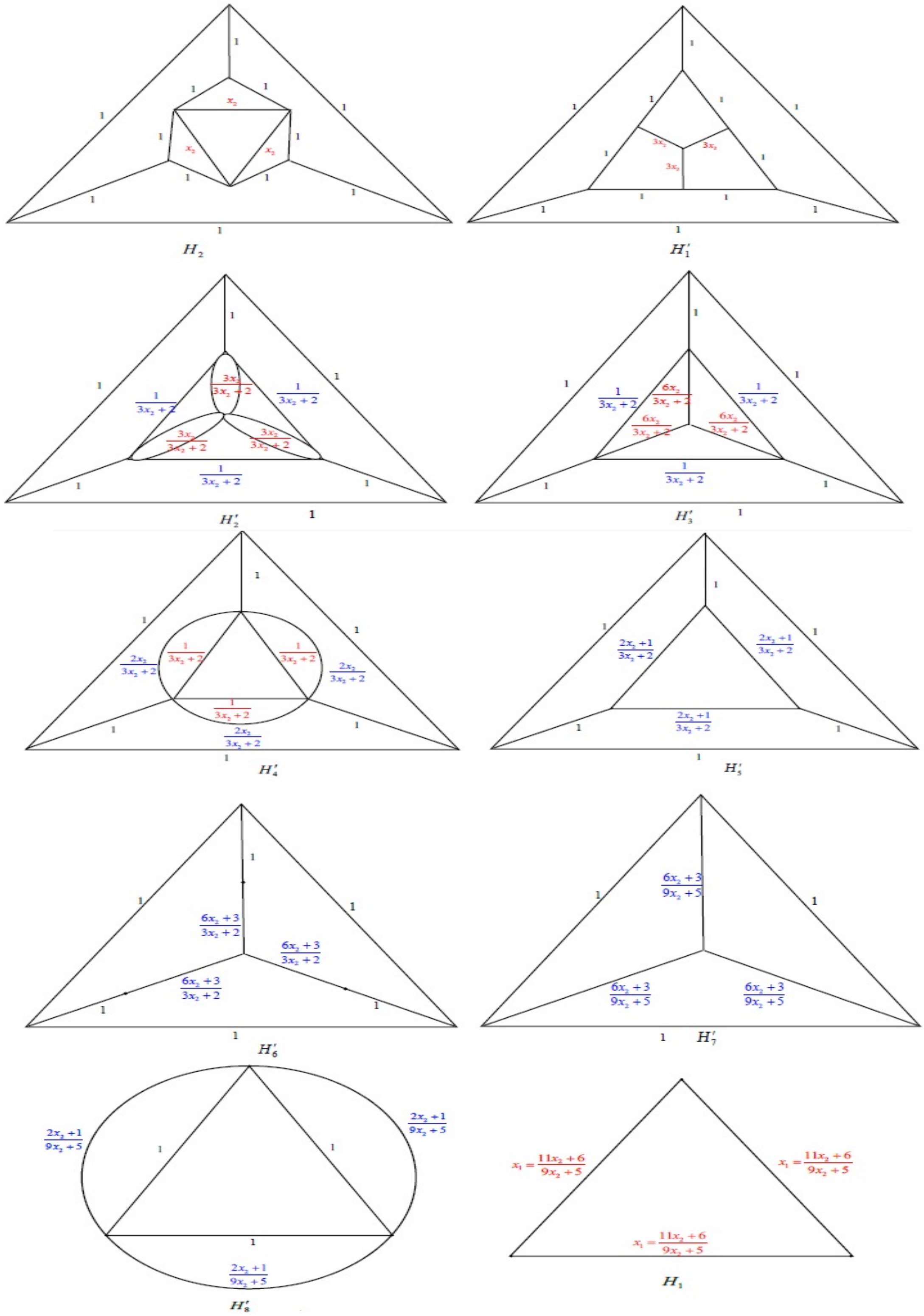

Figure 4 The transformations from H_2 to H_1

Proof We use the electrically equivalent transformation to transform H_i to H_{i-1}. Fig.4 illustrates the transformation process from H_2 to H_1.

Using the properties given in section 2, we have the following the transformations:

$$\tau(H_1') = 9\,\tau(H_2), \qquad \tau(H_2') = \left[\frac{1}{3x_2+2}\right]^3 \tau(H_1'),$$
$$\tau(H_3') = \tau(H_2'), \qquad \tau(H_4') = \frac{3x_2+2}{18x_2}\,\tau(H_3'),$$
$$\tau(H_5') = \tau(H_4'), \qquad \tau(H_6') = 9\left(\frac{2x_2+1}{3x_2+2}\right)\tau(H_5'),$$
$$\tau(H_7') = \left[\frac{3x_2+2}{9x_2+5}\right]^3 \tau(H_6'), \quad \tau(H_8') = \frac{9x_2+5}{3(6x_2+3)}\,\tau(H_6'),$$
$$\tau(H_1) = \tau(H_8') \qquad .$$

Combining these nine transformations, we have

$$\tau(H_2) = 2\ (9x_2+5)^2\,\tau(H_1). \tag{17}$$

Further,

$$\tau(H_n) = \prod_{i=2}^{n} 2\ (9x_i+5)^2\,\tau(H_1) = 3\ \times\ 2^{n-1}\,x_1^2\ \left[\prod_{i=2}^{n}(9x_i+5)\right]^2 \tag{18}$$

where $x_{i-1} = \frac{11\,x_i+6}{9\,x_i+5}$, $i = 2, 3, \cdots, n$. Its characteristic equation is $9u^2 - 6u - 6 = 0$, with roots $u_1 = \frac{1-\sqrt{7}}{3}$ and $u_2 = \frac{1+\sqrt{7}}{3}$.

Subtracting these two roots into both sides of $x_{i-1} = \frac{11\,x_i+6}{9\,x_i+5}$, we get

$$x_{i-1} - \frac{1-\sqrt{7}}{3} = \frac{11\,x_i+6}{9\,x_i+5} - \frac{1-\sqrt{7}}{3} = (8+3\sqrt{7})\ \frac{x_i\ -\ \frac{1-\sqrt{7}}{3}}{9\,x_i\ +5}, \tag{19}$$

$$x_{i-1} - \frac{1+\sqrt{7}}{3} = \frac{11\,x_i+6}{9\,x_i+5} - \frac{1+\sqrt{7}}{3} = (8-3\sqrt{5})\ \frac{x_i\ -\ \frac{1+\sqrt{7}}{3}}{9\,x_i\ +5}. \tag{20}$$

Let $y_i = \frac{x_i - \frac{1-\sqrt{7}}{3}}{x_i - \frac{1+\sqrt{7}}{3}}$. Then by Eqs.(19) and(20), we get $y_{i-1} = (127+48\sqrt{7})\,y_i$ and $y_{i-1} = (127+48\sqrt{7})^{n-i}\ y_i$.

Therefore,

$$x_{\,i} = \frac{(127+48\sqrt{7})^{n-i}\,(\frac{1+\sqrt{7}}{3})\,y_n - \frac{1-\sqrt{7}}{3}}{(127+48\sqrt{7})^{n-i}y_n - 1}.$$

Thus,

$$x_{\,i} = \frac{(127+48\sqrt{7})^{\,n-1}\ (\frac{1+\sqrt{7}}{3})\ y_n - \frac{1-\sqrt{7}}{3}}{(127+48\sqrt{7})^{\,n-1}y_n - 1}. \tag{21}$$

Using the expression $x_{n-1} = \frac{11\,x_n+6}{9x_n+5}$ and denoting the coefficients of $11\,x_n+6$ and $9\,x_n+5$

as h_n and k_n we have

$$\begin{aligned}
9\,x_n+5 &= h_0\,(11x_n+6)+k_0\,(9x_n+5),\\
9\,x_{n-1}+5 &= \frac{h_1(11x_n+6)+k_1(9x_n+5)}{h_0\,(11x_n+6)+k_0\,(9x_n+5)},\\
9\,x_{n-2}+5 &= \frac{h_2(11x_n+6)+k_2(9x_n+5)}{h_1\,(11x_n+6)+k_1\,(9x_n+5)},\\
&\vdots\\
9\,x_{n-i}+5 &= \frac{h_i(11x_n+6)+k_i(9x_n+5)}{h_{i-1}\,(11x_n+6)+k_{i-1}\,(9x_n+5)},
\end{aligned} \tag{22}$$

$$9\,x_{n-(i+1)}+5 = \frac{h_{i+1}\,(11x_n+6)+k_{i+1}\,(9x_n+5)}{h_i\,(11x_n+6)+k_i\,(9x_n+5)}, \tag{23}$$

$$\vdots$$

$$9\,x_2+5 = \frac{h_{n-2}\,(11x_n+6)+k_{n-2}\,(9x_n+5)}{h_{n-3}\,(11x_n+6)+k_{n-3}\,(9x_n+5)} \tag{24}$$

Thus, we obtain

$$\tau(H_n) = 3\times 2^{n-1}\;x_1^2\;[h_{n-2}\,(11x_n+6)+h_{n-2}\,(9x_n+5)]^2, \tag{25}$$

where $h_0=0$, $k_0=1$ and $h_1=9$, $k_1=5$. By the expression $x_{n-1}=\frac{11\,x_n+6}{9x_n+5}$ and Eqs.(23) and (24), we have

$$h_{i+1}=16h_i-h_{i-1}\;;\;\;k_{i+1}=16k_i-k_{i-1}. \tag{26}$$

The characteristic equation of Eq.(26) is $v^2-16v+1=0$ with roots $v_1=8+3\sqrt{7}$ and $v_2=8-3\sqrt{7}$. The general solutions of Eq. (26) are

$$h_i=a_1\;v_1^i\;+a_2\;v_2^i\;;\;\;k_i=b_1\;v_1^i\;+b_2\;v_2^i.$$

Using the initial conditions $h_0=0$, $k_0=1$ and $h_1=9$, $k_1=5$, yields

$$\begin{aligned}
h_i &= \frac{3\sqrt{7}}{14}\left(8+3\sqrt{7}\right)^i-\frac{3\sqrt{7}}{14}\left(8-3\sqrt{7}\right)^i\\
k_i &= \left(\frac{7-\sqrt{7}}{14}\right)\left(8+3\sqrt{7}\right)^i+\left(\frac{7+\sqrt{7}}{14}\right)\left(8-3\sqrt{7}\right)^i.
\end{aligned} \tag{27}$$

If $x_n=1$, it means that H_n is without any electrically equivalent transformation. Plugging Eq.(27) into Eq.(250, we have

$$\tau(H_n)=3\times 2^{n-1}\,x_1^2\;\left[\left(\frac{98+37\sqrt{7}}{14}\right)\left(8+3\sqrt{7}\right)^{n-2}+\left(\frac{98-37\sqrt{7}}{14}\right)\left(8-3\sqrt{7}\right)^{n-2}\right]^2 \tag{28}$$

if $n\geqslant 2$. When $n=1$, $\tau(H_1)=3$ which satisfies Eq.(28). Therefore, the number of spanning trees in the sequence of the graph H_n is given by

$$\tau(H_n) = 3\times 2^{n-1}\, x_1^2\ \left[\left(\frac{98+37\sqrt{7}}{14}\right)\ \left(8+3\sqrt{7}\right)^{n-2} + \left(\frac{98-37\sqrt{7}}{14}\right)\ \left(8-3\sqrt{7}\right)^{n-2}\right]^2 \quad (29)$$

if $n \geqslant 1$, where

$$x_1 = \frac{(127+48\sqrt{7})^{n-1}(13+5\sqrt{7}) + (1-\sqrt{7})}{(127+48\sqrt{7})^{n-1}(11+4\sqrt{7}) + 3}\ ,\ n \geqslant 1. \quad (30)$$

Inserting Eq.(30) into Eq.(29) we obtain the desired result. □

Consider the sequence of graphs $L_1,\ L_2,\cdots,\ L_n$ constructed as shown in Fig.5

According to the construction, the number of total vertices $|V(L_n)|$ and edges $|E(L_n)|$are $|V(L_n)| = 9n-6$,$|E(L_n)| = 18n-15$, $n = 1,2,\cdots$. It is clear that the average degree is approximately 4 for a large n.

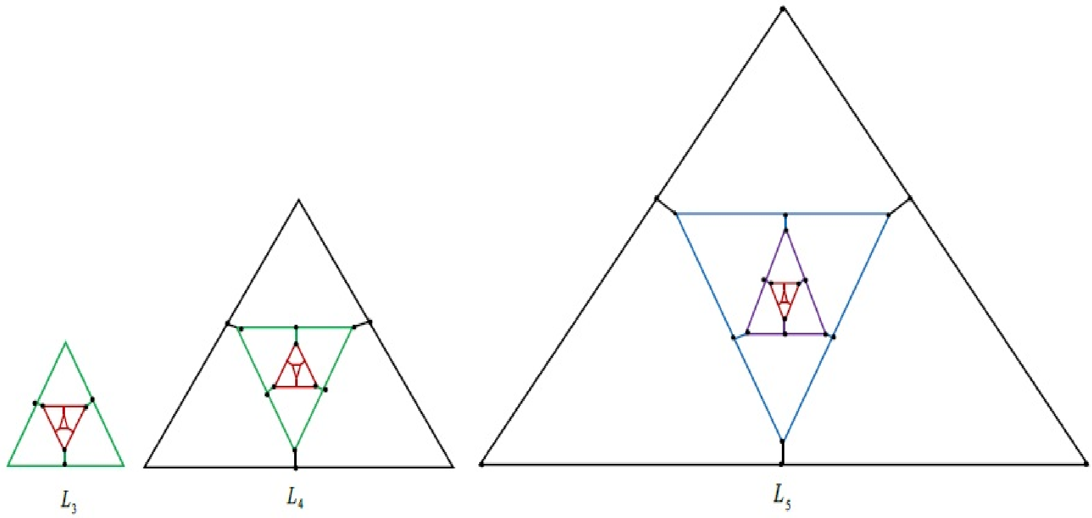

Figure 5 Some sequences of the graph L_n

Theorem 3.3 *For $n \geqslant 1$, the number of spanning trees in the sequence of the graph L_n is given by*

$$\frac{2^{-(n+1)}\left(\left(11+5\sqrt{5}\right)\ \left(14+6\sqrt{5}\right)^n + \left(14-6\sqrt{5}\right)^n\left(123+55\sqrt{5}\right) - 2\ \left(644+288\sqrt{5}\right)^n\right)^2}{3\ \left(2^n\ \left(47+21\sqrt{5}\right) - \left(3+\sqrt{5}\right)\ \left(47+21\sqrt{5}\right)^n\right)^2}.$$

Proof We use the electrically equivalent transformation to transform L_i to L_{i-1}. Fig.6 illustrates the transformation process from L_2 to L_1.

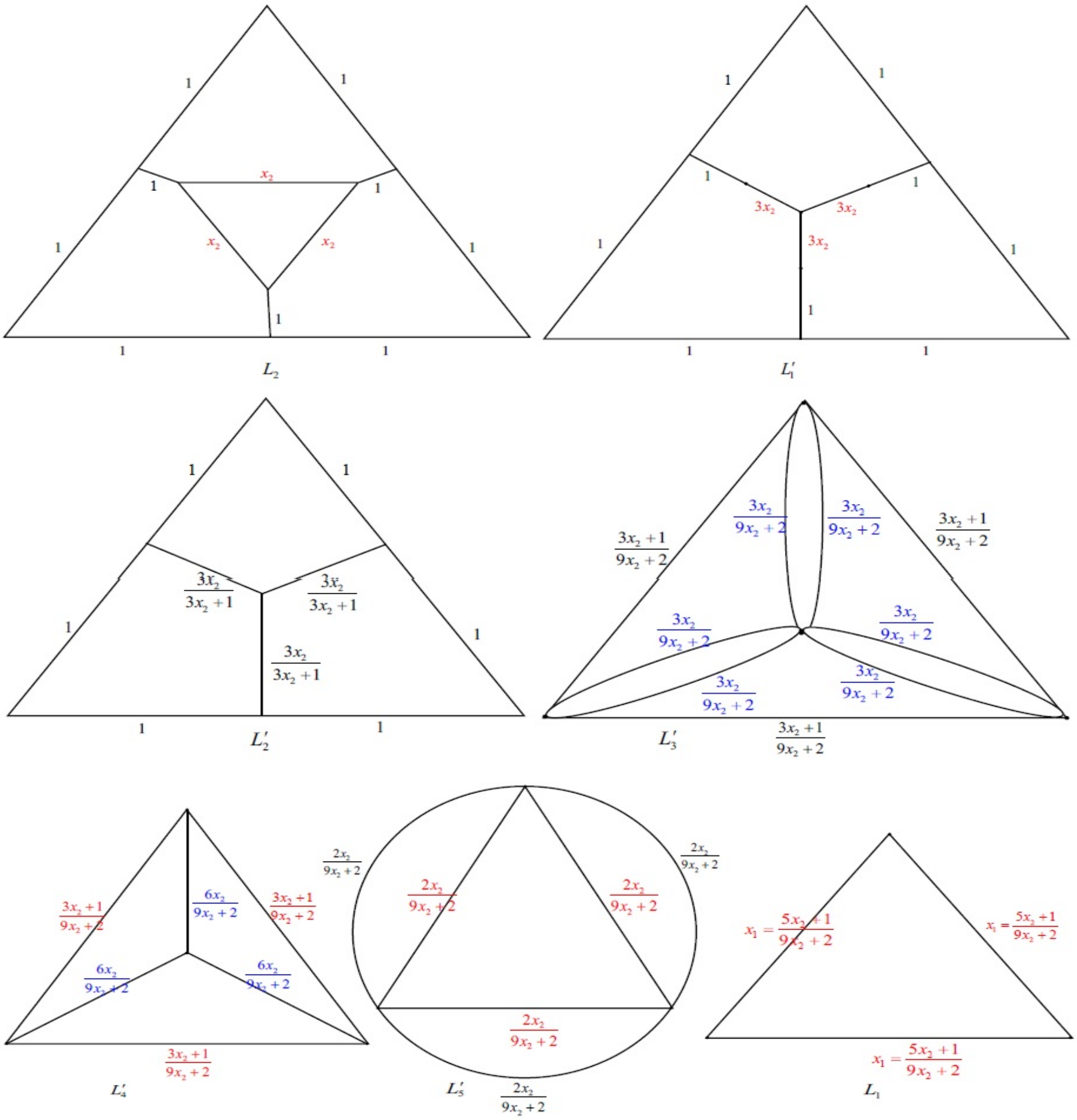

Figure 6 The transformations from L_2 to L_1

Using the properties given in section 2, we have the following transformations

$$\tau(L_1') = 9x_2\,\tau(L_2)\,, \qquad \tau(L_2') = [\frac{1}{3x_2+1}]^3\,\tau(L_1'),$$
$$\tau(L_3') = \left(\frac{3x_2+1}{9x_2+2}\right)^3 \tau(L_2'), \quad \tau(L_4') = \tau(L_3'),$$
$$\tau(L_5') = \left(\frac{9x_2+2}{18x_2}\right) \tau(L_4'), \quad \tau(L_1) = \tau(L_5').$$

Combining these six transformations, we have

$$\tau(L_2) = 2\,(9x_2+2)^2\,\tau(L_1). \tag{31}$$

Further

$$\tau(L_n)=\prod_{i=2}^{n}2\ (9x_i+2)^2\,\tau(L_1)=3\ \times\ 2^{n-1}\,x_1^2\ \left[\prod_{i=2}^{n}(9x_i+2)\right]^2, \tag{32}$$

where $x_{i-1}\ =\frac{5\,x_i+1}{9\,x_i+2}$, $i=2,3,\cdots,n$. Its characteristic equation is $9u^2-3u-1=0$ with roots $u_1=\frac{1-\sqrt{5}}{6}$ and $u_2=\frac{1+\sqrt{5}}{6}$.

Subtracting these two roots into both sides of $x_{i-1}\ =\frac{5\,x_i+1}{9\,x_i+2}$, we get

$$x_{i-1}\ -\frac{1-\sqrt{5}}{6}=\frac{5\,x_i+1}{9x_i+2}-\frac{1-\sqrt{5}}{6}=(7+3\sqrt{5})\ .\ \frac{x_i\ -(\frac{1-\sqrt{5}}{6})}{2(9x_i\ +2)} \tag{33}$$

$$x_{i-1}\ -\frac{1+\sqrt{5}}{6}=\frac{5\,x_i+1}{9x_i+2}-\frac{1+\sqrt{5}}{6}=(7-3\sqrt{5})\ .\ \frac{x_i\ -(\frac{1+\sqrt{5}}{6})}{2(9x_i\ +2)} \tag{34}$$

Let $y_i=\frac{x_i-\frac{1-\sqrt{5}}{6}}{x_i-\frac{1+\sqrt{5}}{6}}$. Then, by Eqs.(33) and (34), we get $y_{i-1}=(\frac{47+21\sqrt{5}}{2})\,y_i$ and $y_i=y_{i-1}=(\frac{47+21\sqrt{5}}{2})^{\,n-i}\ y_n$. Therefore,

$$x_{\,i}=\frac{(\frac{47+21\sqrt{5}}{2})^{n-i}\,(d\frac{1+\sqrt{5}}{6})\ y_n-\dfrac{1-\sqrt{5}}{6}}{(\dfrac{47+21\sqrt{5}}{2})^{n-i}\ y_n-1}.$$

Thus,

$$x_1=\frac{(\frac{47+21\sqrt{5}}{2})^{n-1}\,(\dfrac{1+\sqrt{5}}{6})\ y_n-\dfrac{1-\sqrt{5}}{6}}{(\dfrac{47+21\sqrt{5}}{2})^{n-1}\ y_n-1}. \tag{35}$$

Using the expression $x_{n-1}\ =\frac{5\,x_n+1}{9\,x_n+2}$ and denoting the coefficients of $5\,x_n+1$ and $9x_n+2$ as h_n and k_n, we have

$$\begin{aligned}
9\,x_n+2\ &=\ h_0\,(5x_n+1)+k_0\,(9x_n+2),\\
9x_{n-1}+2\ &=\ \frac{h_1(5x_n+1)+k_1(9x_n+2)}{h_0\,(5x_n+1)+k_0\,(9x_n+2)},\\
9x_{n-2}+2\ &=\ \frac{h_2(5x_n+1)+k_2(9x_n+2)}{h_1\,(5x_n+1)+k_1\,(9x_n+2)},\\
&\ \ \vdots
\end{aligned}$$

$$9x_{n-i}+2=\frac{h_i(5x_n+1)+k_i(9x_n+2)}{h_{i-1}\,(5x_n+1)+k_{i-1}\,(9x_n+2)}, \tag{36}$$

$$9x_{n-(i+1)}+2=\frac{h_{i+1}\,(5x_n+1)+k_{i+1}\,(9x_n+2)}{h_i\,(5x_n+1)+k_i\,(9x_n+2)}, \tag{37}$$

$$\vdots$$

$$9\,x_2+2=\frac{h_{n-2}\,(5x_n+1)+k_{n-2}\,(9x_n+2)}{h_{n-3}\,(5x_n+1)+k_{n-3}\,(9x_n+2)}. \tag{38}$$

Thus, we obtain

$$\tau(L_n) = 3 \times 2^{n-1} x_1^2 \, [h_{n-2} (5x_n + 1) + k_{n-2} (9x_n + 2)]^2, \tag{39}$$

where $h_0 = 0$, $k_0 = 1$ and $h_1 = 9$, $k_1 = 2$. By the expression $x_{n-1} = \frac{5x_n+1}{9x_n+2}$, Eqs.(36) and (37), we have

$$h_{i+1} = 7h_i - h_{i-1} \; ; \; k_{i+1} = 7k_i - k_{i-1} \tag{40}$$

The characteristic equation of Eq.(40) is $v^2 - 7v + 1 = 0$ with roots $v_1 = \frac{7+3\sqrt{5}}{2}$ and $v_2 = \frac{7-3\sqrt{5}}{2}$. The general solutions of Eq.(40) are

$$h_i = a_1 \, v_1^i + a_2 \, v_2^i \; ; \; k_i = b_1 \, v_1^i + b_2 \, v_2^i.$$

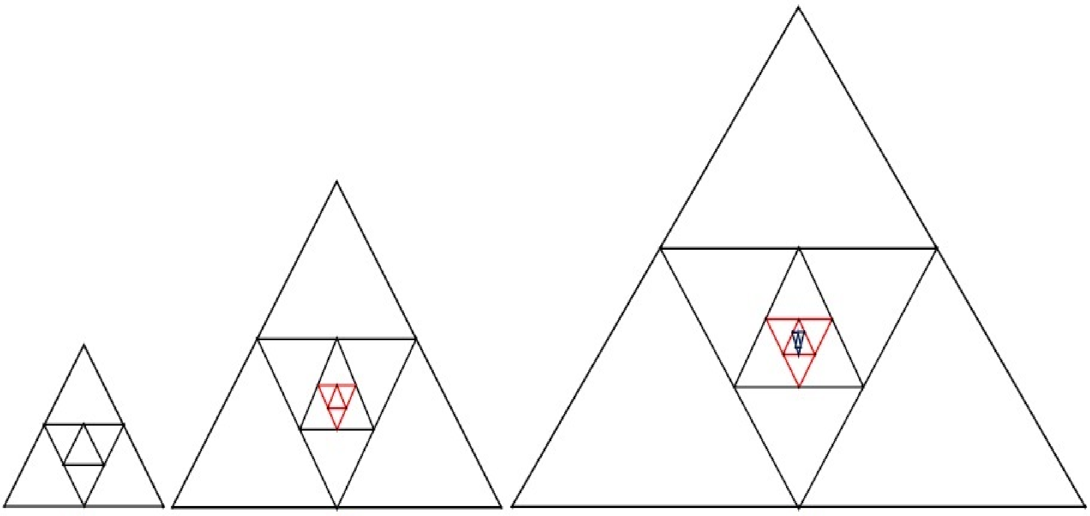

Figure 7 Some sequences of the graph M_n

Using the initial conditions $h_0 = 0$, $k_0 = 1$ and $h_1 = 9$, $k_1 = 2$, yields

$$\begin{aligned} h_i &= \frac{3\sqrt{5}}{5} (\frac{7+3\sqrt{5}}{2})^i - \frac{3\sqrt{5}}{5} (\frac{7-3\sqrt{5}}{2})^i; \\ k_i &= (\frac{5-\sqrt{5}}{10})(\frac{7+3\sqrt{5}}{2})^i + (\frac{5+\sqrt{5}}{10}) (\frac{7-3\sqrt{5}}{2})^i. \end{aligned} \tag{41}$$

If $x_n = 1$, it means that L_n is without any electrically equivalent transformation. Plugging Eq.(41) into Eq.(39), we have

$$\begin{aligned} \tau(L_n) &= 3 \times 2^{n-1} x_1^2 \, [(\frac{11+5\sqrt{5}}{2}) (\frac{7+3\sqrt{5}}{2})^{n-2} \\ &\quad + (\frac{11-5\sqrt{5}}{2}) (\frac{7-3\sqrt{5}}{2})^{n-2}]^2 \end{aligned} \tag{42}$$

if $n \geqslant 2$. When $n = 1$, $\tau(L_1) = 3$ which satisfies Eq.(42). Therefore, the number of spanning trees in the sequence of the graph is given by

$$\tau(L_n) = 3\times 2^{n-1}x_1^2\,[(\frac{11+5\sqrt{5}}{2})(\frac{7+3\sqrt{5}}{2})^{n-2} + (\frac{11-5\sqrt{5}}{2})(\frac{7-3\sqrt{5}}{2})^{n-2}]^2 \quad (43)$$

if $n \geqslant 1$, where

$$x_1 = \frac{2(\frac{47+21\sqrt{5}}{2})^{n-1}(2+\sqrt{5})-(1-\sqrt{5})}{3(\frac{47+21\sqrt{5}}{2})^{n-1}(3+\sqrt{5})-6},\ n\geqslant 1. \quad (44)$$

Inserting Eq.(44) into Eq.(43) we obtain the desired result. □

Consider the sequence of graphs $M_1,\ M_2,\cdots,\ M_n$ constructed as shown in Fig.7.

Theorem 3.4 *For $n \geqslant 1$, the number of spanning trees in the sequence of the graph M_n is given by*

$$\frac{4^{n-2}\ \left((-3+\sqrt{3})\,(1351+780\sqrt{3})^n+(1713+989\sqrt{3})\ (7-4\sqrt{3})^n\right)^2}{3\ \left(97+56\sqrt{3}-(2+\sqrt{3})\ (97+56\sqrt{3})^n\right)^2}.$$

Proof We use the electrically equivalent transformation to transform M_i to M_{i-1} and the transformation process from M_2 to M_1 is illustrated in Fig.8.

Using the properties given in Section 2, we have the following transformations

$$\tau(M_1') = 9x_2\,\tau(M_2), \qquad \tau(M_2') = (\frac{1}{9x_2+2})^3\tau(M_1'),$$
$$\tau(M_3') = \tau(M_2'), \qquad \tau(M_4') = (\frac{3x_2+2}{18x_2})\,\tau(M_3'),$$
$$\tau(M_5') = \tau(M_4'), \qquad \tau(M_6') = (\frac{18x_2+9}{3x_2+2})\,\tau(M_5'),$$
$$\tau(M_7') = (\frac{3x_2+2}{12x_2+7})\,\tau(M_6'), \qquad \tau(M_8') = \tau(M_7'),$$
$$\tau(M_9') = d\tfrac{1}{9}(\frac{12x_2+7}{3x_2+2})\,\tau(M_8'), \qquad \tau(M_1) = \tau(M_9').$$

Combining these ten transformations, we have

$$\tau(M_2) = 4\,(12x_2+7)^2\,\tau(M_1). \quad (45)$$

Further,

$$\tau(M_n) = \prod_{i=2}^{n} 4\ (12x_i+7)^2\,\tau(M_1) = 3\ \times\ 4^{n-1}\,x_1^2\ \left[\prod_{i=2}^{n}(12x_i+7)\right]^2, \quad (46)$$

where $x_{i-1} = \frac{7\,x_i+4}{12\,x_i+7}$, $i = 2,3,\cdots,n$. Its characteristic equation is $3u^2-1=0$ with roots $u_1 = \frac{-\sqrt{3}}{3}$ and $u_2 = \frac{\sqrt{3}}{3}$.

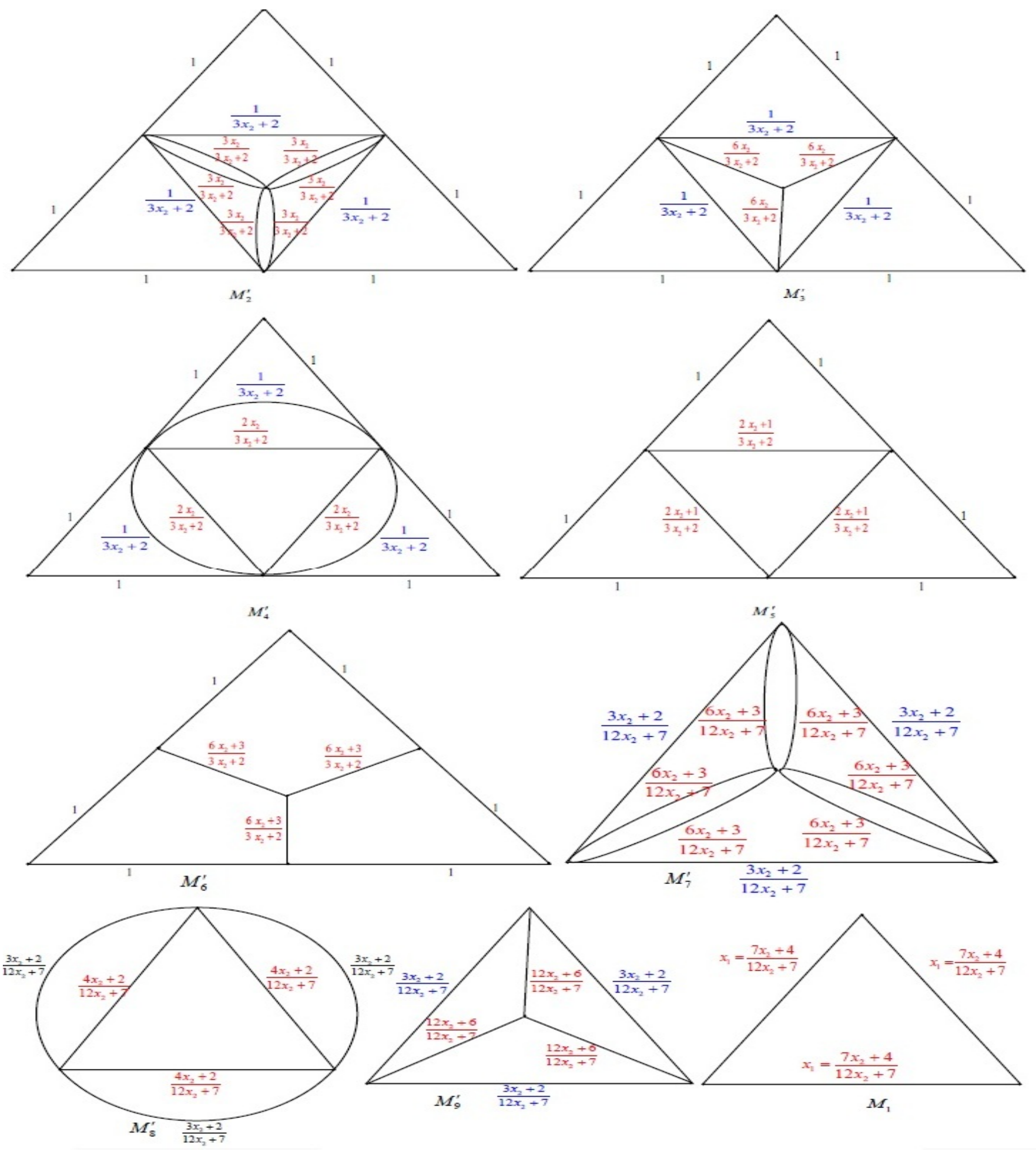

Figure 8 The transformations from M_2 to M_1

Subtracting these two roots into both sides of $x_{i-1} = \frac{7x_i+4}{12\,x_i+7}$, we get

$$x_{i-1} - \left(\frac{-\sqrt{3}}{3}\right) = \frac{7x_i+4}{12x_i+7} + \frac{\sqrt{3}}{3} = (7+4\sqrt{3}) \times \frac{x_i + \frac{\sqrt{3}}{3}}{12x_i + 7}, \tag{47}$$

$$x_{i-1} - \frac{\sqrt{3}}{3} = \frac{7x_i+4}{12x_i+7} - \frac{\sqrt{3}}{3} = (7-4\sqrt{3}) \times \frac{x_i - \frac{\sqrt{3}}{3}}{12x_i + 7}. \tag{48}$$

Let $y_i = \frac{x_i + \frac{\sqrt{3}}{3}}{x_i - \frac{\sqrt{3}}{3}}$. Then by Eqs.(47) and (48), we get $y_{i-1} = (97+56\sqrt{3})\,y_i$ and $y_i =$

$(97+56\sqrt{3})^{\,n-i}\; y_n$. Therefore,

$$x_{\,i}=\frac{(97+56\sqrt{3})^{n-i}\,(\dfrac{\sqrt{3}}{3})\;y_n+\dfrac{\sqrt{3}}{3}}{(97+56\sqrt{3})^{n-i}\;y_n-1}.$$

Thus,

$$x_{\,1}=\frac{(97+56\sqrt{3})^{n-1}\,(\frac{\sqrt{3}}{3})\;y_n+\frac{\sqrt{3}}{3}}{(97+56\sqrt{3})^{n-1}\;y_n-1}. \tag{49}$$

Using the expression $x_{n-1}\;=\frac{7\,x_n+4}{12\,x_n+7}$ and denoting the coefficients of $7x_n+4$ and $12x_n+7$ as h_n and k_n, we have

$$\begin{aligned}
12\,x_n+7&=h_0\,(7x_n+4)+k_0\,(12x_n+7),\\
12x_{n-1}+7&=\frac{h_1(7x_n+4)+k_1(12x_n+7)}{h_0\,(7x_n+4)+k_0\,(12x_n+7)},\\
12x_{n-2}+7&=\frac{h_2(7x_n+4)+k_2(12x_n+7)}{h_1\,(7x_n+4)+k_1\,(12x_n+7)},\\
&\vdots
\end{aligned}$$

$$12x_{n-i}+7=\frac{h_i(7x_n+4)+k_i(12x_n+7)}{h_{i-1}\,(7x_n+4)+k_{i-1}\,(12x_n+7)}, \tag{50}$$

$$12x_{n-(i+1)}+7=\frac{h_{i+1}\,(7x_n+4)+k_{i+1}\,(12x_n+7)}{h_i\,(7x_n+4)+k_i\,(12x_n+7)}, \tag{51}$$

$$\vdots$$

$$12\,x_2+7=\frac{h_{n-2}\,(7x_n+4)+k_{n-2}\,(12x_n+7)}{h_{n-3}\,(7x_n+4)+k_{n-3}\,(12x_n+7)}. \tag{52}$$

Thus, we obtain

$$\tau(M_n)=3\times 4^{n-1}\,x_1^2\;\left[h_{n-2}\;(7x_n+4)+k_{n-2}\;(12x_n+7)\right]^2, \tag{53}$$

where $h_0=0$, $k_0=1$ and $h_1=12$, $k_1=7$. By the expression$x_{n-1}\;=\frac{7\,x_n+4}{12\,x_n+7}$ and Eqs.(53) and (51), we have

$$h_{i+1}=14h_i-h_{i-1}\;;\;k_{i+1}=14k_i-k_{i-1}. \tag{54}$$

The characteristic equation of Eq.(54) is $v^2-14v+1=0$ with roots $v_1\;=7+4\sqrt{3}$ and $v_2=7-4\sqrt{3}$. The general solutions of Eq. (54) are

$$h_i=a_1\;v_1^i\;+a_2\;v_2^i\;;\;k_i=b_1\;v_1^i\;+b_2\;v_2^i\,.$$

Using the initial conditions $h_0=0$, $k_0=1$ and $h_1=12$, $k_1=7$, yields

$$h_i\;=\frac{\sqrt{3}}{2}\,\left(7+4\sqrt{3}\right)^i-\frac{\sqrt{3}}{2}\,\left(7-4\sqrt{3}\right)^i;\;k_i\;=\frac{1}{2}\left(7+4\sqrt{3}\right)^i+\frac{1}{2}(7-4\sqrt{3})^i. \tag{55}$$

If $x_n=1$, it means that M_n is without any electrically equivalent transformation. Plugging

Eq.(55) into Eq.(53), we have

$$\tau(M_n) = 3\times 4^{n-1} x_1^2 \left[\left(\frac{19+11\sqrt{3}}{2}\right)\left(7+4\sqrt{3}\right)^{n-2} + \left(\frac{19-11\sqrt{3}}{2}\right)\left(7-4\sqrt{3}\right)^{n-2}\right]^2 \quad (56)$$

if $n \geqslant 2$. When $n = 1$, $\tau(M_1) = 3$ which satisfies Eq.(56). Therefore, the number of spanning trees in the sequence of the graph is given by

$$\tau(M_n) = 3\times 4^{n-1} x_1^2 \left[\left(\frac{19+11\sqrt{3}}{2}\right)\left(7+4\sqrt{3}\right)^{n-2} + \left(\frac{19-11\sqrt{3}}{2}\right)\left(7-4\sqrt{3}\right)^{n-2}\right]^2 \quad (57)$$

if $n \geqslant 2$, where

$$x_1 = \frac{(97+56\sqrt{3})^{n-1}\,(\frac{3+2\sqrt{3}}{3}) + \frac{\sqrt{3}}{3}}{(97+56\sqrt{3})^{n-1}\,(2+\sqrt{3})-1}. \quad (58)$$

Inserting Eq.(58) into Eq.(57) we obtain the desired result. $\square$.

Consider the sequence of graphs $Q_1, Q_2, \cdots, Q_n$ constructed as shown in Fig.9.

According to the construction, the number of total vertices $|V(Q_n)|$ and edges $|E(Q_n)|$are $|V(Q_n)| = 6n-3$,$|E(Q_n)| = 15n-12$, $n = 1, 2, \cdots$. It is clear that the average degree is approximately 5for a large n.

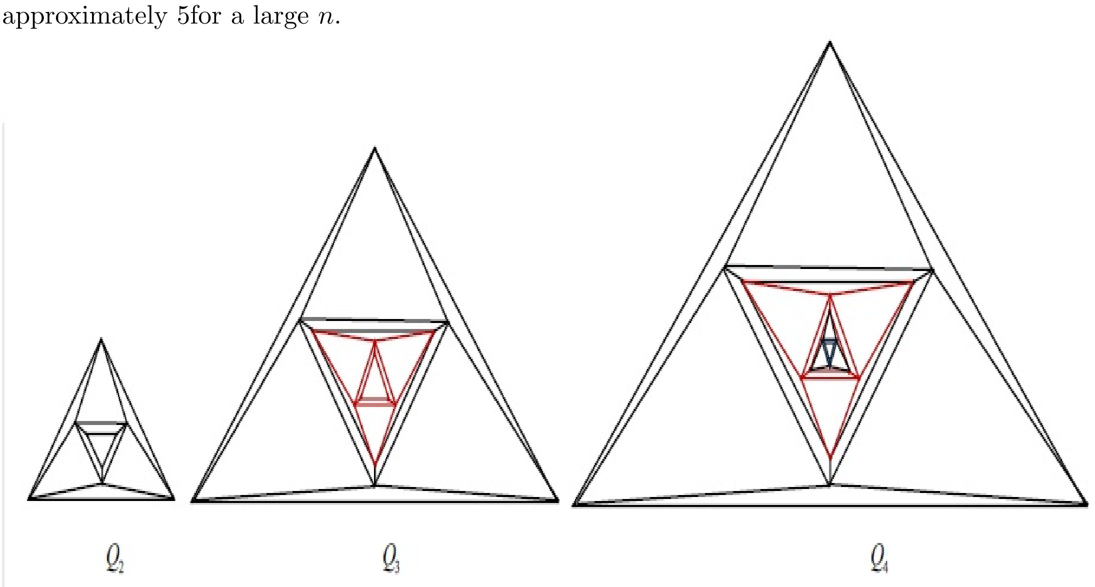

Figure 9 Some sequences of the graph Q_n

Theorem 3.5 *For $n \geqslant 1$, the number of spanning trees in the sequence of the graph Q_n is* *given by* $\dfrac{C}{D}$, *where*

$$\begin{aligned} C &= 2^{n-5}\left(\left(44-31\sqrt{2}\right)\left(17+12\sqrt{2}\right)^n+\left(44+31\sqrt{2}\right)\left(17-12\sqrt{2}\right)^n\right)^2 \\ &\quad\times\left(-7\left(239+169\sqrt{2}\right)\left(17+13\sqrt{2}\right)\left(577+408\sqrt{2}\right)^n\right)^2 \\ D &= 3\left(7\left(577+408\sqrt{2}\right)+(9+4\sqrt{2})\left(577+408\sqrt{2}\right)^n\right)^2 \end{aligned}$$

Proof We use the electrically equivalent transformation to transform Q_i to Q_{i-1}. In the Fig.$10-1$ and $10-2$, we illustrate the transformation process from Q_2 to Q_1.

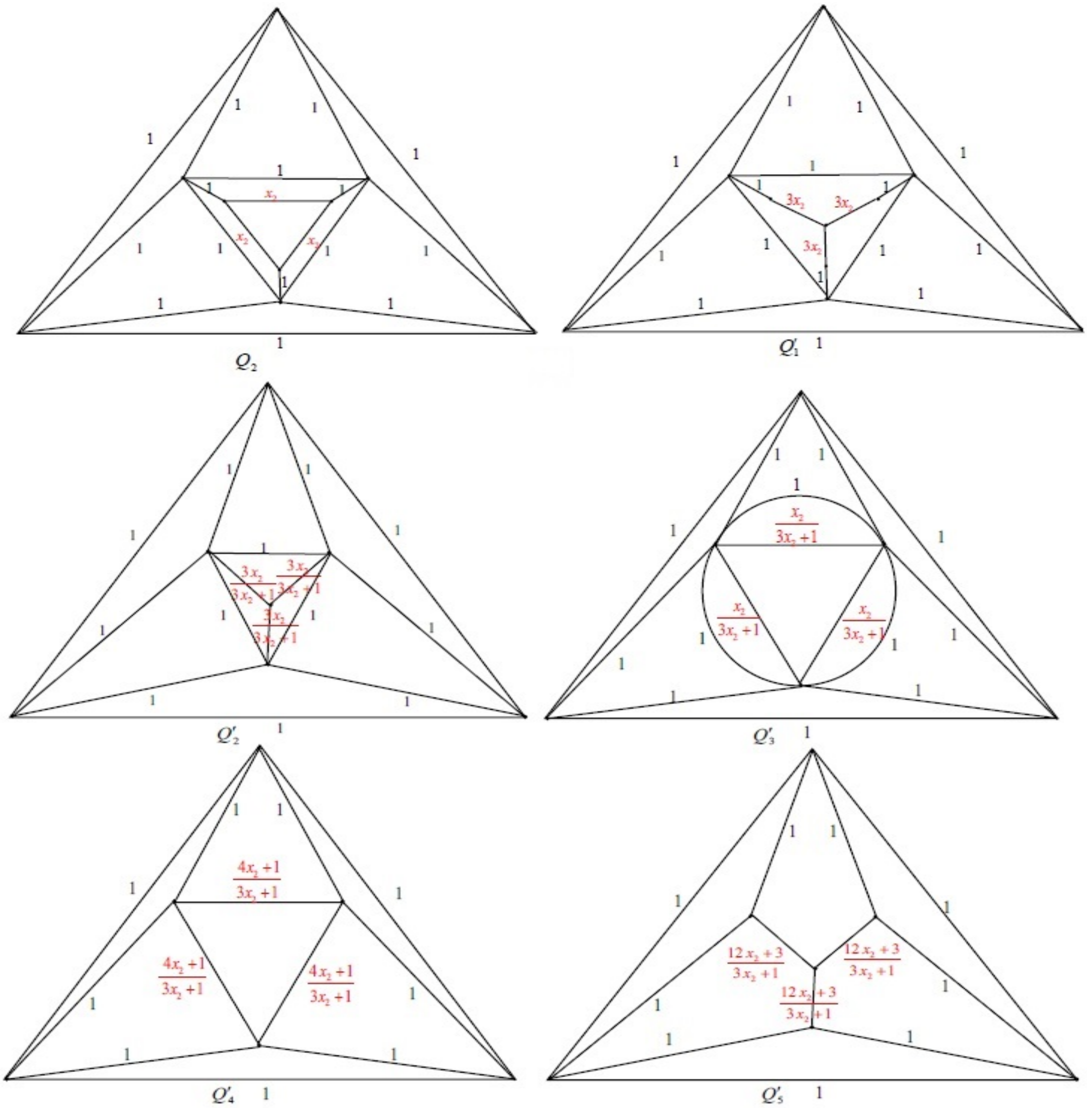

Figure 10-1 The transformations from Q_2 to Q_1

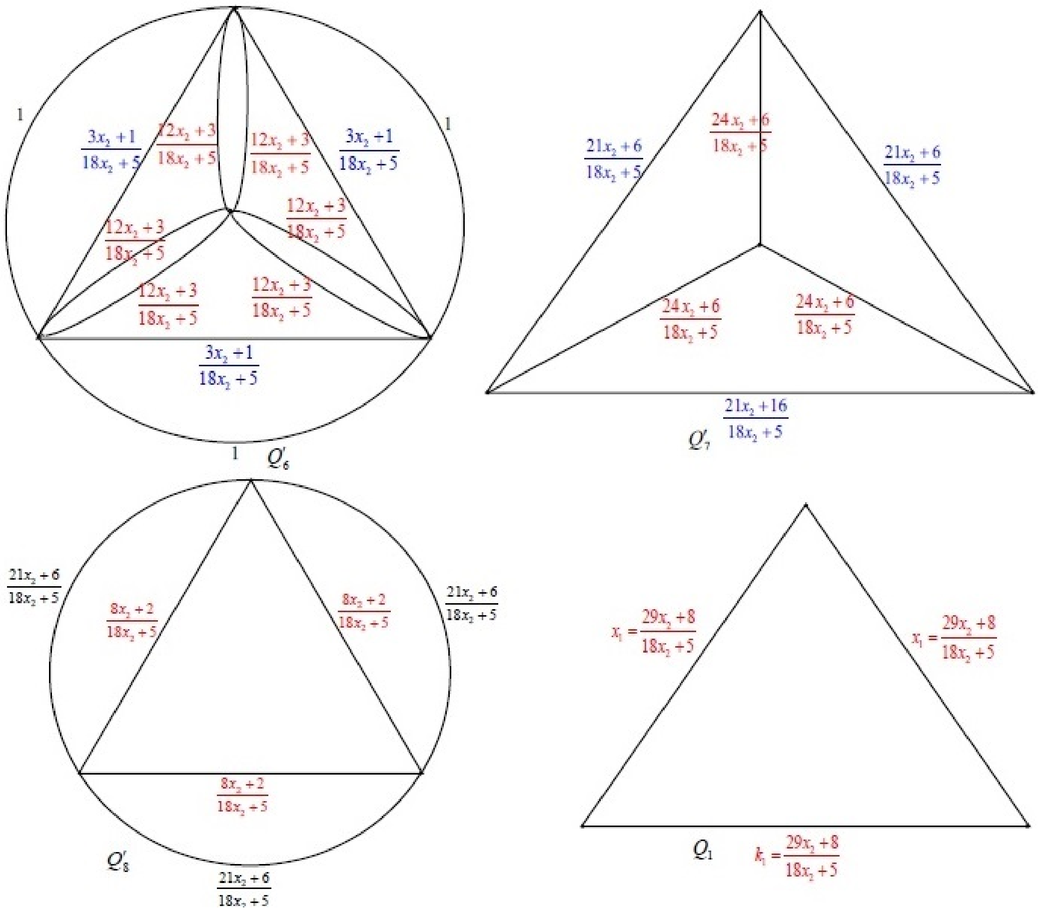

Figure 10-2 The transformations from Q_2 to Q_1

Using the properties given in Section 2, we have the following the transformations:

$$\tau(Q_1') = 9_2\,\tau(Q_2)\,, \qquad \tau(Q_2') = (\frac{1}{3x_2+1})^3\tau(Q_1'),$$

$$\tau(Q_3') = (\frac{3x_2+1}{9x_2})\tau(Q_2'), \qquad \tau(Q_4') = \tau(Q_3'),$$

$$\tau(Q_5') = 9\,(\frac{4x_2+1}{3x_2+1})\,\tau(Q_4'), \quad \tau(Q_6') = (\frac{3x_2+1}{18x_2+5})^3\,\tau(Q_5'),$$

$$\tau(Q_7') = \tau(Q_6'), \qquad \tau(Q_8') = \frac{1}{9}(\frac{18x_2+5}{8x_2+2})^3\,\tau(Q_7')$$

and $\tau(Q_1) = \tau(Q_8')$.

Combining these nine transformations, we have

$$\tau(Q_2) = 2\,(18x_2+5)^2\,\tau(Q_1). \tag{59}$$

Further,

$$\tau(Q_n) = \prod_{i=2}^{n} 2\ (18x_i+5)^2\, \tau(Q_1) = 3\ \times\ 2^{n-1}\, x_1^2\ \left[\prod_{i=2}^{n} (18x_i+5)\right]^2, \tag{60}$$

where $x_{i-1} = \frac{29\,x_i+8}{18x_i+5}$, $i = 2, 3, \cdots, n$. Its characteristic equation is $9u^2 - 12u - 4 = 0$ with roots $u_1 = \frac{2-2\sqrt{2}}{3}$ and $u_2 = \frac{2+2\sqrt{2}}{3}$.

Subtracting these two roots into both sides of $x_{i-1} = \frac{29x_i+8}{18\,x_i+5}$, we get

$$x_{i-1} - \frac{2-2\sqrt{5}}{3} = \frac{29\,x_i+8}{18x_i+5} - \frac{2-2\sqrt{2}}{3} = (17+12\sqrt{2})\ .\ \frac{x_i\ - (\frac{2-2\sqrt{2}}{3})}{18x_i\ +5}, \tag{61}$$

$$x_{i-1} - \frac{2+2\sqrt{2}}{3} = \frac{29\,x_i+8}{18x_i+5} - \frac{2+2\sqrt{2}}{3} = (17-12\sqrt{2})\ .\ \frac{x_i\ - (\frac{2+2\sqrt{2}}{3})}{18x_i\ +5}. \tag{62}$$

Let $y_i = \frac{x_i\ -\ \frac{2-2\sqrt{2}}{3}}{x_i\ -\ \frac{2+2\sqrt{2}}{3}}$. Then by Eqs.(61) and (62), we get $y_{i-1} = (577+408\sqrt{2})\, y_i$ and $y_i = (577+408\sqrt{2})^{\,n-i}\ y_n$.

Therefore,

$$x_{\,i} = \frac{(577+408\sqrt{2})^{n-i}\,(\frac{2+2\sqrt{2}}{3})\ y_n - \frac{2-2\sqrt{2}}{3}}{(577+408\sqrt{2})^{n-i}\ y_n - 1}.$$

Thus,

$$x_{\,i} = \frac{(577+408\sqrt{2})^{n-1}\,(\frac{2+2\sqrt{2}}{3})\ y_n - \frac{2-2\sqrt{2}}{3}}{(577+408\sqrt{2})^{n-1}\ y_n - 1}. \tag{63}$$

Using the expression $x_{n-1} = \frac{29\,x_n+8}{18x_n+5}$ and denoting the coefficients of $29x_n+8$ and $18x_n+5$ as h_n and k_n, we have

$$\begin{aligned}
18x_n+5 &= h_0\,(29x_n+8) + k_0\,(18x_n+5),\\
18x_{n-1}+5 &= \frac{h_1(29x_n+8) + k_1(18x_n+5)}{h_0\,(29x_n+8) + k_0\,(18x_n+5)},\\
18x_{n-2}+5 &= \frac{h_2(29x_n+8) + k_2(18x_n+5)}{h_1\,(29x_n+8) + k_1\,(18x_n+5)},\\
&\vdots
\end{aligned}$$

$$18x_{n-i}+5 = \frac{h_i(29x_n+8) + k_i(18x_n+5)}{h_{i-1}\,(29x_n+8) + k_{i-1}\,(18x_n+5)}, \tag{64}$$

$$18x_{n-(i+1)}+5 = \frac{h_{i+1}\,(29x_n+8) + k_{i+1}\,(18x_n+5)}{h_i\,(29x_n+8) + k_i\,(18x_n+5)}, \tag{65}$$

$$\vdots$$

$$18x_2+5 = \frac{h_{n-2}\,(29x_n+8) + k_{n-2}\,(18x_n+5)}{h_{n-3}\,(29x_n+8) + k_{n-3}\,(18x_n+5)}. \tag{66}$$

Thus, we obtain

$$\tau(Q_n) = 3 \times 2^{n-1}\, x_1^2\ [h_{n-2}\,(29x_n+8) + k_{n-2}\,(18x_n+5)]^{\,2}, \tag{67}$$

where $h_0 = 0$, $k_0 = 1$ and $h_1 = 18$, $k_1 = 5$. By the expression $x_{n-1} = \frac{29\,x_n+8}{18x_n+5}$ and Eqs.(65) and (66), we have

$$h_{i+1} = 34h_i - h_{i-1}\ ;\ k_{i+1} = 34k_i - k_{i-1}, \tag{68}$$

The characteristic equation of Eq.(68) is $v^2 - 34v + 1 = 0$ with roots $v_1 = 17 + 12\sqrt{2}$ and $v_2 = 17 - 12\sqrt{2}$. The general solutions of Eq. (68) are

$$h_i = \lambda a_1\ v_1^i\ + a_2\ v_2^i\ ;\ k_i = b_1 v_1^i\ + b_2 v_2^i.$$

Using the initial conditions $h_0 = 0$, $k_0 = 1$ and $h_1 = 18$, $k_1 = 5$, yields

$$\begin{aligned} h_i &= \frac{3\sqrt{2}}{8}\left(17+12\sqrt{2}\right)^i - \frac{3\sqrt{2}}{8}\left(17-12\sqrt{2}\right)^i; \\ k_i &= \left(\frac{2-\sqrt{2}}{4}\right)\left(17+12\sqrt{2}\right)^i + \left(\frac{2+\sqrt{2}}{4}\right)\left(17-12\sqrt{2}\right)^i. \end{aligned} \tag{69}$$

If $x_n = 1$, it means that Q_n is without any electrically equivalent transformation. Plugging Eq.(68) into Eq.(67), we have

$$\begin{aligned} \tau(Q_n) &= 3\times 2^{n-1}\,x_1^2\ \Bigg[\left(\frac{92+65\sqrt{2}}{8}\right)\left(17+12\sqrt{2}\right)^{n-2} \\ &\quad + \left(\frac{92-65\sqrt{2}}{8}\right)\left(17-12\sqrt{2}\right)^{n-2}\Bigg]^2 \end{aligned} \tag{70}$$

if $n \geqslant 2$. When $n = 1$, $\tau(Q_1) = 3$ which satisfies Eq.(69). Therefore, the number of spanning trees in the sequence of the graph is given by

$$\begin{aligned} \tau(Q_n) &= 3\times 2^{n-1}\,x_1^2\ \Bigg[\left(\frac{92+65\sqrt{2}}{8}\right)\left(17+12\sqrt{2}\right)^{n-2} \\ &\quad + \left(\frac{92-65\sqrt{2}}{8}\right)\left(17-12\sqrt{2}\right)^{n-2}\Bigg]^2, \end{aligned} \tag{71}$$

if $n \geqslant 1$, where

$$x_1 = \frac{(577+408\sqrt{2})^{n-1}\,(\dfrac{34+26\sqrt{2}}{21})\ +(\dfrac{2-2\sqrt{2}}{3})}{(577+408\sqrt{2})^{n-1}\,(\dfrac{9+4\sqrt{2}}{7})+1}. \tag{72}$$

Inserting Eq.(72) into Eq.(71) we obtain the desired result. □

Consider the sequence of graphs R_1, R_2, $\cdots$, R_n constructed as shown in Fig.11.

According to the construction, the number of total vertices $|V(R_n)|$ and edges $|E(R_n)|$

are $|V(R_n)| = 9n-6, |E(R_n)| = 18n-15,\ n = 1, 2, \cdots$. It is clear that the average degree is approximately 4 for a large n.

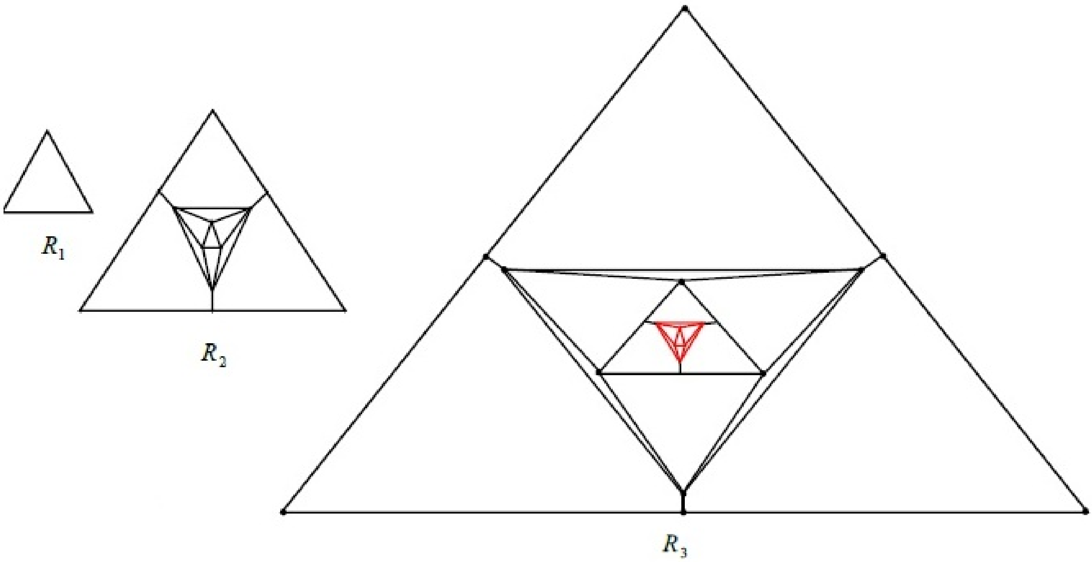

Figure 11 Some sequences of the graph R_n

Theorem 3.6 *For $n \geqslant 1$, the number of spanning trees in the sequence of the graph R_n is given by $\dfrac{A}{B}$, where*

$$\begin{aligned} A &= (59+\sqrt{3477}^{2n}(-111\times 2^{n+1}(47519+806\sqrt{3477})+17(3479+59\sqrt{3477})^n \\ &\quad \times(-511119+8467\sqrt{3477})+629(3479-59\sqrt{3477})^n(169194297+2869349\sqrt{3477})^2) \end{aligned}$$

$$B = (16119372\,(-629\times 2^n(3479+59\sqrt{477})+(2417+35\sqrt{3477})(3479+59\sqrt{3477}])^n)^2).$$

Proof We use the electrically equivalent transformation to transform R_i to R_{i-1}. In Figs.12 − 1, 12 − 2 and 12 − 3, we illustrate the transformation process from R_2 to R_1.

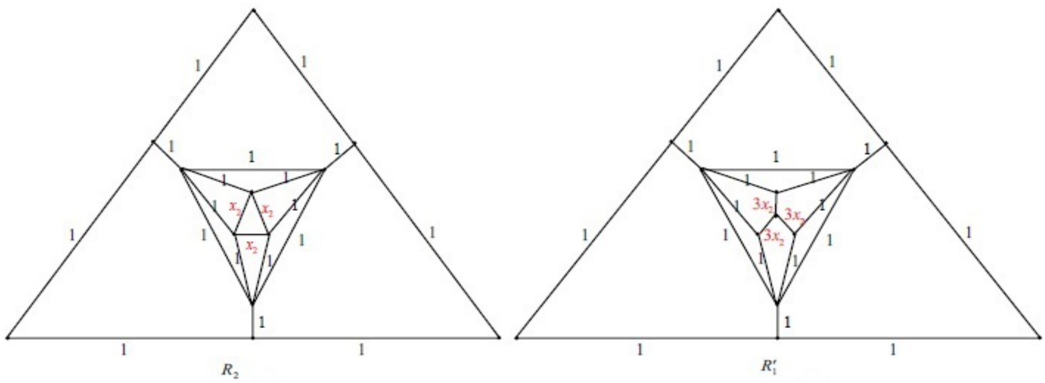

Figure 12-1 The transformations from R_2 to R_1

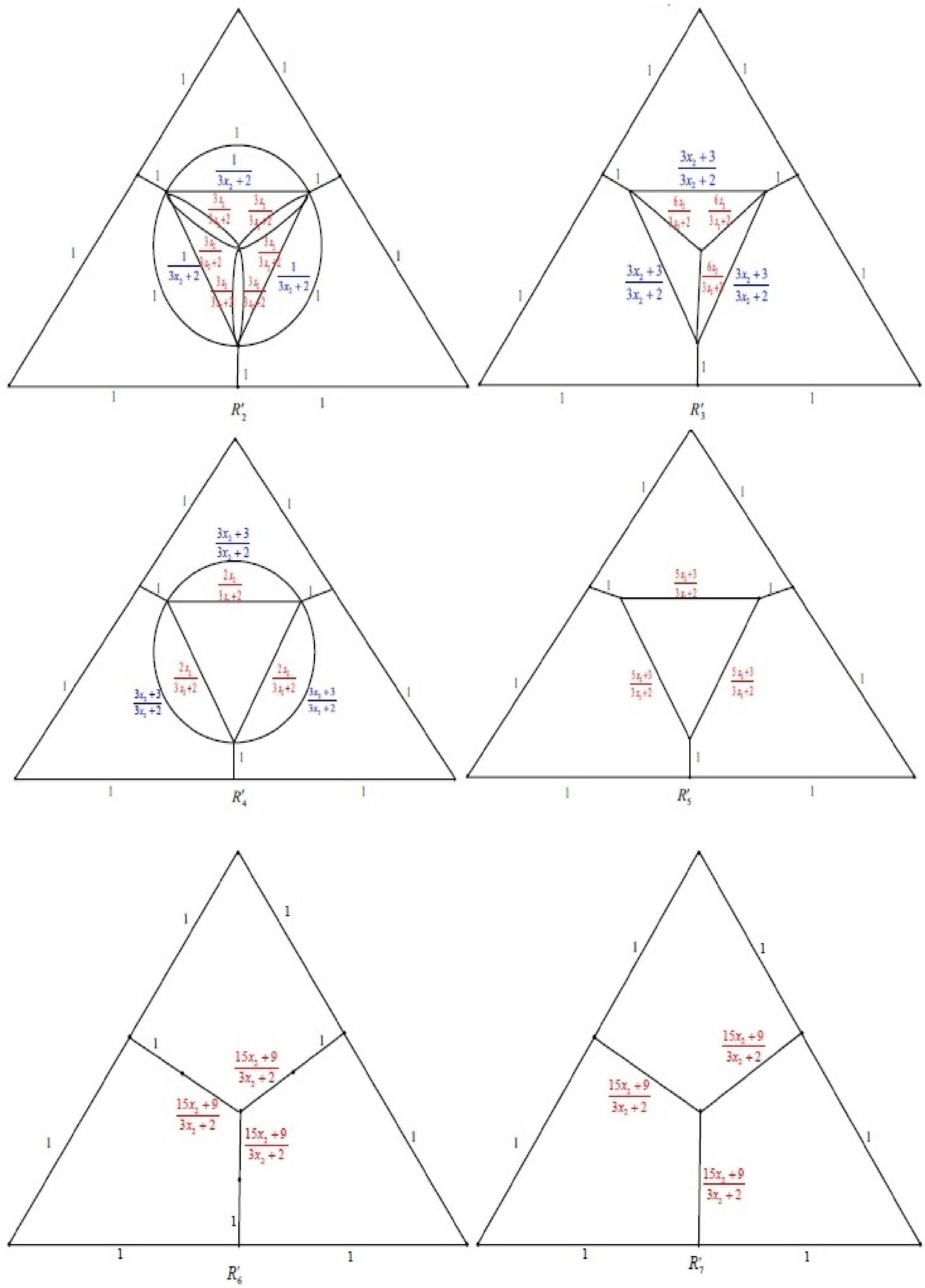

Figure 12-2 The transformations from R_2 to R_1

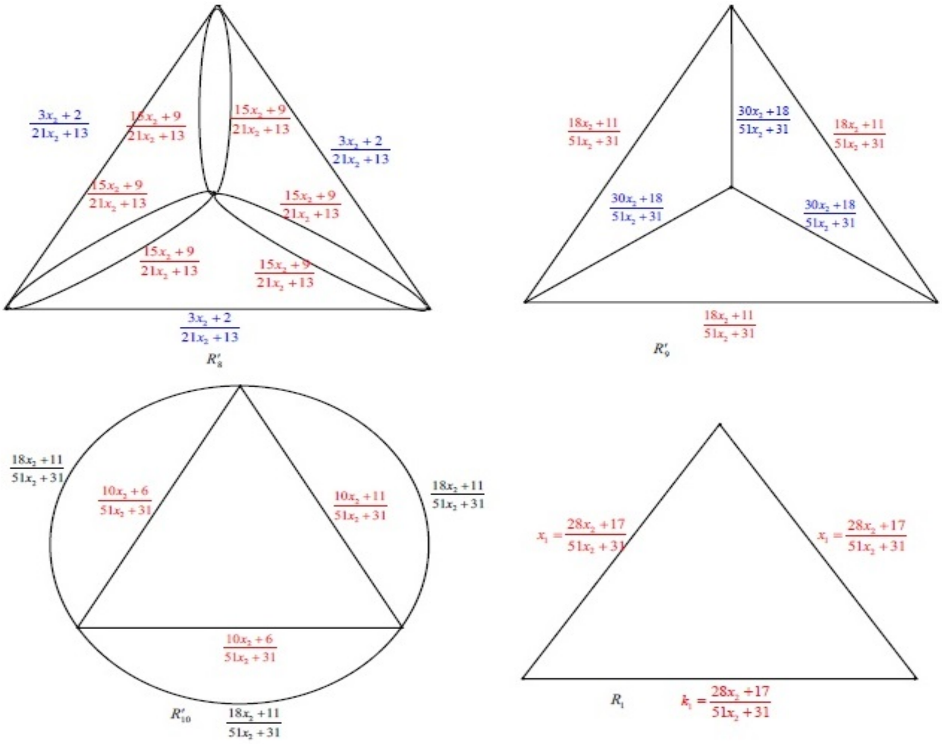

Figure 12-2 The transformations from R_2 to R_1

Using the properties given in section 2, we have the following the transformations:

$$\tau(R_1') = 9x_2\,\tau(R_2)\,, \qquad \tau(R_2') = (\frac{1}{3x_2+2})^3\tau(R_1'),$$

$$\tau(R_3') = \tau(R_2'), \qquad \tau(R_4') = \frac{3x_2+2}{18x_2}\tau(R_3'),$$

$$\tau(R_5') = \tau(R_4'), \qquad \tau(R_6') = 9(\frac{5x_2+3}{3x_2+2})\tau(R_5'),$$

$$\tau(R_7') = (\frac{3x_2+2}{18x_2+11})^3\tau(R_6'), \qquad \tau(R_8') = (\frac{18x_2+11}{51x_2+31})^3\,\tau(R_7'),$$

$$\tau(R_9') = \tau(R_8'), \qquad \tau(R_{10}') = \frac{51x_2+31}{90x_2+54}\,\tau(R_9')$$

and $\tau(R_1) = \tau(R_{10}')$. Combining these eleven transformations, we have

$$\tau(R_2) = 4\,(51x_2+31)^2\,\tau(R_1). \tag{73}$$

Further,

$$\tau(R_n) = \prod_{i=2}^{n} 4\,(51x_i+31)^2\,\tau(R_1) = 3\;\times\;4^{n-1}\,x_1^2\,\left[\prod_{i=2}^{n}(51k_i+31)\right]^2, \tag{74}$$

where $x_{i-1} = \frac{28\,x_i+17}{51x_i+31}$, $i = 2, 3, \cdots, n$. Its characteristic equation is $51\,u^2 + 3u - 17 = 0$ with roots $u_1 = \frac{-3-\sqrt{3477}}{102}$ and $u_2 = \frac{-3+\sqrt{3477}}{102}$.

Subtracting these two roots into both sides of $x_{i-1} = \frac{28x_i+17}{51\,x_i+31}$, we get

$$x_{i-1} + \frac{3+\sqrt{3477}}{102} = \frac{28\,x_i + 17}{51x_i + 31} + \frac{3+\sqrt{3477}}{102} = (59+\sqrt{3477})\ .\ \frac{x_i\ + (\dfrac{3+\sqrt{3477}}{2})}{2(51x_i\ + 31)}, \qquad (75)$$

$$x_{i-1} - \frac{3-\sqrt{3477}}{102} = \frac{28\,x_i + 17}{51x_i + 31} + \frac{3-\sqrt{3477}}{102} = (59-\sqrt{3477})\ .\ \frac{x_i\ + (\frac{3-\sqrt{3477}}{2})}{2(51x_i\ + 31)}. \qquad (76)$$

Let $y_i = \frac{x_i + \frac{3+\sqrt{3477}}{102}}{x_i + \frac{3-\sqrt{3477}}{102}}$. Then by Eqs.(75) and (76), we get $y_{i-1} = (\frac{3479+59\sqrt{3477}}{2})\, y_i$ and $y_i = (\frac{3479+59\sqrt{3477}}{2})^{\,n-i}\ y_n$.

Therefore

$$x_{\,i} = \frac{(\frac{-3+\sqrt{3477}}{102})(\dfrac{3479+59\sqrt{3477}}{2})^{n-i}\ y_n + \frac{3+\sqrt{3477}}{102}}{(\dfrac{3479+59\sqrt{3477}}{2})^{n-i}\ y_n - 1}.$$

Thus

$$x_{\,i} = \frac{(\dfrac{-3+\sqrt{3477}}{102})(\dfrac{3479+59\sqrt{3477}}{2})^{n-1}\ y_n + \dfrac{3+\sqrt{3477}}{102}}{(\dfrac{3479+59\sqrt{3477}}{2})^{n-1}\ y_n - 1}. \qquad (77)$$

Using the expression $x_{n-1} = \frac{28\,x_n+17}{51x_n+31}$ and denoting the coefficients of $28x_n + 17$ and $51x_n + 31$ as h_n and k_n, we have

$$\begin{aligned}
51x_n + 31 &= h_0\,(28x_n + 17) + k_0\,(51x_n + 31),\\
51x_{n-1} + 31 &= \frac{h_1(28x_n + 17) + k_1(51x_n + 31)}{h_0\,(28x_n + 17) + k_0\,(18x_n + 31)},\\
51x_{n-2} + 31 &= \frac{h_2(28x_n + 17) + k_2(51x_n + 31)}{h_1\,(28x_n + 17) + k_1\,(51x_n + 31)},\\
&\ \vdots
\end{aligned}$$

$$51x_{n-i} + 31 = \frac{h_i(28x_n + 17) + k_i(51x_n + 31)}{h_{i-1}\,(28x_n + 17) + k_{i-1}\,(51x_n + 31)}, \qquad (78)$$

$$51x_{n-(i+1)} + 31 = \frac{h_{i+1}\,(28x_n + 17) + k_{i+1}\,(51x_n + 31)}{h_i\,(28x_n + 17) + k_i\,(51x_n + 31)}, \qquad (79)$$

$$\vdots$$

$$51x_2 + 31 = \frac{h_{n-2}\,(28x_n + 17) + k_{n-2}\,(51x_n + 31)}{h_{n-3}\,(28x_n + 17) + k_{n-3}\,(51x_n + 31)}. \qquad (80)$$

Thus, we obtain

$$\tau(R_n) = 3 \times 4^{n-1}\,x_1^2\,\left[h_{n-2}\,(28x_n + 17) + k_{n-2}\,(51x_n + 31)\right]^2, \qquad (81)$$

where $h_0 = 0$, $k_0 = 1$ and $h_1 = 51$, $k_1 = 31$. By the expression $x_{n-1} = \frac{28\,x_n+17}{51x_n+31}$ and Eqs.(78)

and (79), we have

$$h_{i+1} = 59h_i - h_{i-1}\ ;\ k_{i+1} = 59k_i - k_{i-1}. \tag{82}$$

The characteristic equation of Eq.(82) is $v^2 - 59v + 1 = 0$ with roots $v_1 = \frac{59+\sqrt{3477}}{2}$ and $v_2 = \frac{59-\sqrt{3477}}{2}$. The general solutions of Eq.(82) are

$$h_i = a_1\ v_1^i\ + a_2\ v_2^i\ ;\ k_i = b_1 v_1^i\ + b_2 v_2^i.$$

Using the initial conditions $h_0 = 0$, $k_0 = 1$ and $h_1 = 51$, $k_1 = 31$, yields

$$\begin{aligned} h_i &= \frac{17\sqrt{3477}}{1159}\,(\frac{59+\sqrt{3477}}{2})^i - \frac{17\sqrt{3477}}{1159}\,(\frac{59-\sqrt{3477}}{2})^i; \\ k_i &= (\frac{1159+\sqrt{3477}}{2218})(\frac{59+\sqrt{3477}}{2})^i \\ &\quad +(\frac{1159-\sqrt{3477}}{2218})(\frac{59-\sqrt{3477}}{2})^i. \end{aligned} \tag{83}$$

If $x_n = 1$, it means that R_n is without any electrically equivalent transformation. Plugging Eq.(83) into Eq.(81), we have

$$\begin{aligned} \tau(R_n) &= 3\times 4^{n-1}\,x_1^2\,[(\frac{47519+806\sqrt{3477}}{1159})\,(\frac{59+\sqrt{3477}}{2})^{n-2} \\ &\quad +(\frac{47519-806\sqrt{3477}}{1159})\,(\frac{59-\sqrt{3477}}{2})^{n-2}]^2 \end{aligned} \tag{84}$$

if $n \geqslant 2$. When $n = 1$, $\tau(R_1) = 3$ which satisfies Eq.(84). Therefore, the number of spanning trees in the sequence of the graph is given by

$$\begin{aligned} \tau(R_n) &= 3\times 4^{n-1}\,x_1^2\,[(\frac{47519+806\sqrt{3477}}{1159})\,(\frac{59+\sqrt{3477}}{2})^{n-2} \\ &\quad +(\frac{47519-806\sqrt{3477}}{1159})\,(\frac{59-\sqrt{3477}}{2})^{n-2}]^2 \end{aligned} \tag{85}$$

if $n \geqslant 2$, where

$$x_1 = \frac{(\frac{3479+59\sqrt{3477}}{2})^{n-1}\,(\frac{99+2\sqrt{3477}}{111})\ +(\frac{3+\sqrt{3477}}{102})}{(\frac{3479+59\sqrt{3477}}{2})^{n-1}\ (\frac{2417+35\sqrt{3477}}{1258}) - 1}. \tag{86}$$

Inserting Eq.(86) into Eq.(85) we obtain the desired result. □

§4. Numerical Results

The following two tables illustrate some the values of the number of spanning trees in the graphs G_n, H_n, L_n, M_n, Q_n and R_n.

n	$\tau(G_n)$	$\tau(H_n)$	$\tau(L_n)$
1	3	3	3
2	10800	1734	216
3	29128368	881292	20172
4	78529953792	447690264	1895064
5	211716289555200	227423130672	178054848
6	570785860162301952	115529159623776	16729574496

Table 1. Some values of $\tau(G)_n$, $\tau(H_n)$ and $\tau(L_n)$.

n	$\tau(M_n)$	$\tau(Q_n)$	$\tau(R_n)$
1	3	3	3
2	1452	8214	24300
3	1123632	18960588	338098368
4	871902912	43761009624	4704976258752
5	676578632448	101000334380592	65474444206252800
6	525011068136448	233108596706389344	91114229029 4589960192

Table 2. Some values of $\tau(M_n)$, $\tau(O_n)$ and $\tau(R_n)$

§5. **Spanning Tree Entropy**

After having explicit Formulas for the number of spanning trees of the sequence of the six graphs G_n , H_n , L_n , M_n , Q_n and R_n, we can calculate its spanning tree entropy Z which is a finite number and a very interesting quantity characterizing the network structure, defined in [16,17] to be

$$Z(G) = \lim_{n\to\infty} \frac{\ln \tau(G)}{|V(G)|}. \tag{87}$$

for a graph G. We have known that

$$\begin{aligned}
Z(G_n) &= \frac{1}{6}\left(\ln[4] + 2\ln\left[13 + 2\sqrt{42}\right]\right) = 1.316587045,\\
Z(T_n) &= \frac{1}{6}\left(\ln[2] - 2\ln\left[127 + 48\sqrt{7}\right] + 2\ln\left[2024 + 765\sqrt{7}\right]\right)\\
&= 1.038410991,\\
Z(L_n) &= \frac{1}{6}\left(\ln[8] - 2\ln\left[47 + 21\sqrt{5}\right] + 2\ln\left[161 + 72\sqrt{5}\right]\right)\\
&= 0.757140296,
\end{aligned}$$

$$\begin{aligned}
Z(M_n) &= \frac{1}{6}\left(\ln\,[4] - 2\ln\left[97+56\sqrt{3}\right] + 2\ln\left[1351+780\sqrt{3}\right]\right)\\
&= 1.0109020991,\\
Z(Q_n) &= \frac{1}{6}\left(\ln\,[2] + 2\ln\left[17+12\sqrt{2}\right]\right) = 1.0290689313,\\
Z(R_n) &= \frac{2}{9}\left(\ln\left[59+\sqrt{3477}\right]\right) = 1.060088273.
\end{aligned}$$

Now we compare the value of entropy in our graphs with other graphs. It is clear that the entropy of the graph G_n is larger than the entropy of the graph Q_n of the same average degree 5. Also the entropy of the graph M_n of average degree 4 is larger than the entropy of the graph H_n, R_n and $L_n$of the same average degree 4. In addition the entropies of the graphs M_n and H_n are larger than the entropy of the fractal scale free lattice [18] which has the entropy1.040and has the same average degree 4, while the entropies of the graphs L_n and R_n are smaller than fractal scale free lattice. Also the entropies of the graphs M_n , H_n , L_n and R_n are smaller than the entropy of the two dimensional Sierpinski gasket [19] which has the entropy1.166 of the same average degree4. Moreover, the entropies of the graphs H_n , L_n and R_n are smaller than entropy of the 3-prism graph [20] which has the entropy1.0445 but the graph M_n has entropy larger than 3-prism graph.

§6. Conclusions

In this paper, we calculate the number of spanning trees in the sequences of some graphs generated by triangle graph using electrically equivalent transformations. The feature of this technique lies in the parry of strenuous computation of Laplacian spectra that is prerequisite for a generic method for determining spanning trees. In addition, our results have shown that the entropy is related to the average degree of the graph.

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CR-Sub-Manifolds of (ϵ, δ)-Trans-Sasakian Manifolds Admitting Generalized Symmetric Metric Connection

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Abstract: A new connection on (ϵ, δ)-trans-Sasakian manifolds is introduced in this paper, which is the generalization both of the semi-symmetric and quarter-symmetric connection. We discuss the properties of CR-Sub-manifold of an (ϵ, δ)-trans-Sasakian manifolds with respect to a generalized symmetric metric connection and also presented the integrability conditions of distributions on CR-Sub-manifolds in this paper.

Key Words: CR-Sub-manifold, (ϵ, δ)-trans-Sasakian manifolds, ϵ-Kenmotsu manifolds, integrability condition.

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§1. Introduction

The metric connection with a torsion different from zero was introduced by Hayden [3] on a Riemannian manifold. In [2], Golab mentioned that the quarter symmetric connections, being more generalized form of semi-symmetric connections on a differentiable manifold. Tripathi [10] introduced and studied many types of connections which includes the semi-symmetric and quarter symmetric connections.

CR-Sub-manifolds of Sasakian manifold were studied by Kobayashi [6] and Hasan Shahid et al. [9]. Moreover, Kenmotsu [5] studied new class of almost contact Riemannian manifolds, known as Kenmotsu manifolds. CR-Sub-manifolds of such manifolds was studied by Papaghuic [7]. More general, one has the notion of α-Sasakian structure and β-Kenmotsu structure (See [14]). Motivating by these results, in this paper we studied the concept of (ϵ, δ)-trans-Sasakian manifolds.

A linear connection on a Riemannian manifold M is suggested to be a generalized sym-

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metric connection if its torsion tensor T is presented as follows:

$$T(U,V) = \alpha(u(V)U - u(U)V) + \beta(u(V)\phi U - u(U)\phi Y), \tag{1.1}$$

for all vector fields U and V on M, where α and β are smooth functions on M, ϕ is of tensor type (1,1) and u is regarded as a 1-form connected with the vector field. The connection mentioned here is a generalized metric one when a Riemannian metric g in M is available as $\overline{\nabla} g = 0$ or else it is non-metric.

In (1.1), if $\alpha = 0$, $\beta \neq 0$; $\alpha \neq 0$, $\beta = 0$, then the generalized symmetric connection is called β-quarter-symmetric connection and α-semi-symmetric connection respectively. Therefore, generalizing semi-symmetric and quarter-symmetric connection gives the generalized symmetric metric connection.

In this paper, we define a new connection on (ϵ, δ)-trans-Sasakian manifolds which is the generalization of semi-symmetric and quarter-symmetric connection. Nagaraja et al. [8] introduced (ϵ, δ)- trans-Sasakian manifold, which generalizes both ϵ-Sasakian and ϵ-Kenmotsu manifolds. In Section 2, the preliminaries of (ϵ, δ)-trans-Sasakian manifolds discussed. Section 3, illustrates generalized symmetric connection on an (ϵ, δ)-trans-Sasakian manifolds. In Section 4, we study the properties of CR-Sub-manifold of an (ϵ, δ)-trans-Sasakian manifolds with respect to a generalized symmetric metric connection. We also presented the integrability conditions of distributions on CR-Sub-manifolds.

§2. **Preliminaries**

Let M be a differentiable manifold of dimension n endowed with a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a 1-form η and metric g, which satisfies

$$\phi^2 X_1 = -X_1 + \eta(X_1)\xi, \;\; \phi\xi = 0, \;\; \eta(\phi X_1) = 0, \;\;\; \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) - \epsilon\eta(X_1)\eta(X_2), \;\; \eta(X_1) = \epsilon g(X, \xi) \tag{2.2}$$

$$\begin{aligned}(\nabla_{X_1}\phi)X_2 = \alpha(g(X_1, X_2)\xi - \epsilon\eta(X_2)X_1) \\ + \beta(g(\phi X_1, X_2)\xi - \delta\eta(X_2)\phi(X_1))\end{aligned} \tag{2.3}$$

$$\nabla_{X_1}\xi = -\epsilon\alpha\phi X_1 - \beta\delta\phi^2 X_1, \;\;\; rank \;\; \phi = n-1, \tag{2.4}$$

for any $X_1, Y_1 \in TM$, where ∇ denotes the Levi-Civita connection with respect to the (ϵ, δ)-trans-Sasakian manifolds metric g. Such manifold (M, ϕ, ξ, η, g) is called (ϵ, δ)-trans-Sasakian manifolds. In addition, if η is closed on an (ϵ, δ)-trans-Sasakian manifolds then we have

$$(\nabla_{X_1}\eta)X_2 = -\alpha g(\phi X_1, X_2) + \epsilon\delta\beta g(\phi X_1, \phi X_2) \tag{2.5}$$

for any vector field X_1 and X_2.

The Gauss and Weingarten formulae give by

$$\nabla_{X_1} X_2 = \nabla'_{X_1} X_2 + h(X_1, X_2), \quad \forall X_1, X_2 \in \Gamma(TM'), \tag{2.6}$$

$$\nabla_{X_1} N = -A_N X_1 + \nabla^{\perp}_{X_1} N, \ \ \forall N \in \Gamma(T^{\perp} M'), \tag{2.7}$$

where $(\nabla_{X_1} X_2, A_N X_1)$ and $(h(X_1, X_2), \nabla^{\perp}_{X_1} N)$ belong to $\Gamma(TM')$ and $\Gamma(T^{\perp} M')$, respectively.

§3. (ϵ, δ)-Trans-Sasakian Manifold with Generalized Symmetric Metric Connection

We have $\overline{\nabla}$ as a linear connection and ∇ as a Levi-Civita connection of (ϵ, δ)-trans-Sasakian manifold M, in such a way that

$$\overline{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + H(X_1, X_2), \tag{3.1}$$

for all the vector field X_1 and X_2. Since $\overline{\nabla}$ is a generalized symmetric metric connection of ∇, where H is $(1, 2)$ tensor type.

$$H(X_1, X_2) = \frac{1}{2}[T(X_1, X_2) + T'(X_1, X_2) + T'(X_2, X_1)] \tag{3.2}$$

$$g(T'(X_1, X_2), W) = g(T(W, X_1), X_2). \tag{3.3}$$

Therefore, from (1.1) and (3.3), we have:

$$T'(X_1, X_2) = \alpha(\eta(X_1)X_2 - g(X_1, X_2)\xi) + \beta(\eta(X_1)\phi X_2 - g(\phi X_1, X_2)\xi) \tag{3.4}$$

now taking (1.1), (3.2) and (3.4), we get:

$$H(X_1, X_2) = \alpha(\eta(X_2)X_1 - g(X_1, X_2)\xi) + \beta(\eta(X_2)\phi X_1 - g(\phi X_1, X_2)\xi) \tag{3.5}$$

Corollary 3.1 *For an (ϵ, δ)-trans-Sasakian manifold, the generalized symmetric metric connection $\overline{\nabla}$ of type (α, β) is given by*

$$\overline{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + \alpha(\eta(X_2)X_1 - g(X_1, X_2)\xi) + \beta(\eta(X_2)\phi X_1 - g(\phi X_1, X_2)\xi). \tag{3.6}$$

If $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$, the generalized metric connection is declined to a semi-symmetric metric and a quarter-symmetric metric one as given below

$$\overline{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + \eta(X_2)X_1 - g(X_1, X_2)\xi \tag{3.7}$$

$$\overline{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + \eta(X_2)\phi X_1 - g(\phi X_1, X_2)\xi. \tag{3.8}$$

Using (2.3), (2.5) and (3.6), we have the following lemma.

Lemma 3.1 *If M is (ϵ,δ)-trans-Sasakian manifold with the generalized symmetric metric connection then the following relations holds*

$$(\overline{\nabla}_{X_1}\phi)X_2 = [(\alpha-\beta)g(X_1,X_2) + (\alpha+\beta)g(\phi X_1,X_2) + \beta\eta(X_1)\eta(X_2)(\epsilon-1)]\xi - \alpha\epsilon\eta(X_2)X_1 - \eta(X_2)\phi X_1(\alpha+\delta\beta), \quad (3.9)$$

$$\overline{\nabla}_{X_1}\xi = -\alpha\epsilon\phi X_1 + (\alpha+\delta\beta)(X_1 - \eta(X_1)\xi), \quad (3.10)$$

$$(\overline{\nabla}_{X_1}\eta)X_2 = (\alpha+\delta\beta)g(\phi X_1,\phi X_2) - \alpha\epsilon g(\phi X_1,X_2). \quad (3.11)$$

for every $X_1, X_2 \in \Gamma(TM)$.

Proof We know that $(\overline{\nabla}_{X_1}\phi)X_2 = \overline{\nabla}_{X_1}\phi X_2 - \phi(\overline{\nabla}_{X_1}X_2)$. Replacing X_2 with ϕX_2 in (3.6) we have

$$\overline{\nabla}_{X_1}\phi X_2 = (\alpha-\beta)g(X_1,X_2) + (\alpha+\beta)g(\phi X_1,X_2)\xi + \beta\epsilon\eta(X_1)\eta(X_2)\xi - \alpha\epsilon\eta(X_2)X_1 - \beta\delta\eta(X_2)\phi X_1 + \phi(\nabla_{X_1}X_2) \quad (3.12)$$

Substituting (3.12) in $(\overline{\nabla}_{X_1}\phi)X_2$, we obtain (3.9), and put $X_2 = \xi$ in (3.9) we get (3.10). Similarly, taking $(\overline{\nabla}_{X_1}\eta)X_2 = g(X_2, \overline{\nabla}_{X_1}\xi)$, we get (3.11). Hence, the proof is completes. □

§4. CR-Sub-Manfolds of (ϵ,δ)-Trans-Sasakian Manifold with Generalized Symmetric Metric Connection

An n-dimensional Riemannian manifold M of an (ϵ,δ)-trans-Sasakian manifold M' is called a CR-sub-manifold if ξ tangent to M and there exists on M a differentiable distribution $D : x \to D_x \subset T_x(M)$ such that

(i) D is invariant under ϕ, that is, $\phi D \subset D$;

(ii) The orthogonal complement distribution $D^\perp : x \to D_x^\perp \subset T_x(M)$ of the distribution D on M is totally real, that is, $\phi D^\perp \subset T^\perp M$.

Here, the distribution D is called horizontal distribution. The pair $(D, D^\perp)$ is called ξ-horizontal if $\xi \in \Gamma(D)$. The CR-sub-manifold is also called ξ-horizontal if $\xi \in \Gamma(D)$.

The orthogonal component $\phi D^\perp$ in $T^\perp M$ is given by

$$TM = D \oplus D^\perp, \quad T^\perp M = \phi D^\perp \oplus \mu,$$

where $\phi\mu = \mu$.

Let M be a CR-sub-manifold of (ϵ,δ)-trans-Sasakian manifold M' with generalized symmetric metric connection $\overline{\nabla}$. For any $X \in \Gamma(TM')$ and $X \in \Gamma(T^\perp M')$, we have

$$X_1 = PX_1 + QX_1, PX_1 \in \Gamma(D), \quad QX_1 \in \Gamma(D^\perp), \quad (4.1)$$

$$\phi N = BN + CN, \quad BN \in \Gamma(D^\perp), CN \in \Gamma(\mu). \quad (4.2)$$

The Gauss and Weingarten formulae with respect to $\overline{\nabla}$ are as follows

$$\overline{\nabla}_{X_1} X_2 = \overline{\nabla}'_{X_1} X_2 + \overline{h}(X_1, X_2) \tag{4.3}$$

$$\overline{\nabla}_{X_1} N = -\overline{A}_N X_1 + \overline{\nabla}^{\perp}_{X_1} N \tag{4.4}$$

for any $X_1, X_2 \in \Gamma(TM^{\perp})$. Now, the above equation becomes

$$\begin{aligned} P\overline{\nabla}'_{X_1} X_2 = P\nabla'_{X_1} X_2 + \alpha\eta(X_2)PX - \alpha g(X_1, X_2)P\xi + \beta\eta(X_2\phi PX \\ - \beta(g(\phi X, X_2)P\xi)) \end{aligned} \tag{4.5}$$

$$\overline{h}(X_1, X_2) = h(X_1, X_2) + \beta(\eta(X_2)\phi QX_1 \tag{4.6}$$

$$Q\overline{\nabla}'_{X_1} X_2 = \nabla'_{X_1} X_2 + \alpha(\eta(X_2)QX_1 - \alpha g(X_1, X_2)Q\xi - \beta g(\phi X_1, X_2)Q\xi) \tag{4.7}$$

for any $X_1, X_2 \in \Gamma(TM')$.

The Gauss and Weingarten formulae with respect to generalized symmetric metric connection is of the form (See [1]):

$$\overline{\nabla}_{X_1} X_2 = \overline{\nabla}'_{X_1} X_2 + h(X_1, X_2) + \beta\eta(X_2)\phi QX_1 \tag{4.8}$$

$$\overline{\nabla}_{X_1} N = -A_N X_1 + \nabla^{\perp} N + \alpha\eta(N)X_1 + \beta\eta(N)\phi X_1 - \beta g(\phi X_1, N)\xi. \tag{4.9}$$

Theorem 4.1 *Let M be a CR-submanifold of (ϵ,δ)-trans-Sasakian manifold M' with generalized symmetric metric connection. Then*

$$h(X_1, \phi PX_2) + \nabla^{\perp}_{X_1} \phi QX_2 = Ch(X_1, X_2) - (\alpha + \delta\beta)\eta(X_2)\phi QX_1 + \phi Q\overline{\nabla}' X_1 X_2, \tag{4.10}$$

$$\begin{aligned} P\overline{\nabla}' X_1 \phi PX_2 - PA_{\phi}QX_2X_1 - \beta g(\phi X_1, \phi QX_2)P\xi = (\alpha - \beta)g(X_1, X_2)P\xi \\ + (\alpha + \beta)g(\phi X_1, X_2)P\xi + (\epsilon - 1)\beta\eta(X_1)\eta(X_2)P\xi - \alpha\epsilon\eta(X_2)P(X_1) \\ - (\alpha + \delta\beta)\eta(X_2)\phi PX_1 + \phi P\overline{\nabla}' X_1 X_2 + \beta\eta(X_2)\eta(QX_1)P\xi, \end{aligned} \tag{4.11}$$

$$\begin{aligned} Q\overline{\nabla}' \phi PX_2 - QA_{\phi QX_2} X_1 - \beta g(\phi X_1, \phi QX_2)Q\xi = (\alpha - \beta)g(X_1, X_2)Q\xi \\ + (\alpha + \beta)g(\phi X_1, X_2)Q\xi + (\epsilon - 1)\beta\eta(X_1)\eta(X_2)Q\xi - \alpha\epsilon\eta(X_2)Q(X_1) \\ + Bh(X_1, X_2) + \beta\eta(X_2)QX_1 + \beta\eta(X_2)\eta(QX_1)Q\xi. \end{aligned} \tag{4.12}$$

for any $X_1, X_2 \in \Gamma(TM)$.

Proof We know that $\overline{\nabla}_{X_1} \phi X_2 = (\overline{\nabla}_{X_1} \phi)X_2 + \phi(\overline{\nabla}_{X_1} X_2)$. Consider LHS of the above equation and using (3.12) we have

$$\overline{\nabla}_{X_1} \phi X_2 = \overline{\nabla}_{X_1} \phi PX_2 + \overline{\nabla}_{X_1} \phi QX_2 \tag{4.13}$$

Now using (4.8) for the tangential part and (4.9) for the normal part of the above equation we get

$$\overline{\nabla}_{X_1} \phi X_2 = \overline{\nabla}'_{X_1} \phi PX_2 + h(X_1, \phi PX_2) - A_{\phi QX_2} X_1 + \nabla^{\perp}_{X_1} \phi QX_2 - \beta g(\phi Q, \phi QX_2)\xi \tag{4.14}$$

Again by using (3.12) we obtain

$$\overline{\nabla}_{X_1}\phi X_2 = P\overline{\nabla}'_{X_1}\phi PX_2 + h(X_1,\phi PX_2) + \nabla^{\perp}_{X_1}\phi QX_2 - \beta g(\phi X_1,\phi QX_2)P\xi - \beta g(\phi X_1,\phi QX_2)Q\xi + Q\overline{\nabla}'_{X_1}\phi QX_2 - PA_{\phi QX_2} - QA_{\phi QX_2}. \tag{4.15}$$

Now consider RHS of $\overline{\nabla}_{X_1}\phi X_2 = (\overline{\nabla}_{X_1}\phi)X_2 + \phi(\overline{\nabla}_{X_1}X_2)$ and using (3.9) we obtain

$$(\overline{\nabla}_{X_1}\phi)X_2 + \phi(\overline{\nabla}_{X_1}X_2) = [(\alpha-\beta)g(X_1,X_2) + (\alpha+\beta)g(\phi X_1,X_2) + \beta\eta(X_1)\eta(X_2)(\epsilon-1)]\xi - \alpha\epsilon\eta(X_2)X_1 - \eta(X_2)\phi X_1(\alpha+\delta\beta) + \phi[\overline{\nabla}'_{X_1}X_2 + h(X_1,X_2) + \beta\eta(X_2)\phi QX_1] \tag{4.16}$$

Now using (4.7) and (4.8) in the above equation we obtain

$$\begin{aligned}(\overline{\nabla}_{X_1}\phi)X_2 + \phi(\overline{\nabla}_{X_1}X_2) = {}&(\alpha-\beta)g(X_1,X_2)P\xi + (\alpha-\beta)g(X_1,X_2)Q\xi\\ &+ (\alpha+\beta)g(\phi X_1,X_2)P\xi + (\alpha+\beta)g(\phi X_1,X_2)Q\xi\\ &+ (\epsilon-1)\beta\eta(X_1)\eta(X_2)P\xi + (\epsilon-1)\beta\eta(X_1)\eta(X_2)Q\xi\\ &- \alpha\epsilon\eta(X_2)X_1P\xi - \alpha\epsilon\eta(X_2)X_1Q\xi - (\alpha+\delta\beta)\eta(X_2)\phi PX_1\\ &- (\alpha+\delta\beta)\eta(X_2)\phi QX_1 + \phi P\overline{\nabla}'_{X_1}X_2 + \phi Q\overline{\nabla}'_{X_1}X_2 + Bh(X_1,X_2)\\ &+ Ch(X_1,X_2) + \beta\eta(X_2)QX_1 + \beta\eta(X_2)\eta(QX_1)\xi\end{aligned}$$

Now on comparing LHS and RHS the normal,horizontal and vertical components we obtain (4.9), (4.10) and (4.11). The proof is completes. □

Theorem 4.2 *Let M be a ξ -vertical CR-submanifold of a (ϵ,δ)-trans-Sasakian manifold M' with generalized symmetric metric connection. Then*

$$\phi[X_2,X_3] = A_{\phi X_3}X_2 - A_{\phi X_2}X_3 + 2(\alpha+\beta)g(X_3,\phi X_2)\xi. \tag{4.17}$$

for any $X_2, X_3 \in (\Gamma D^{\perp})$.

Proof Consider

$$g(\phi([X_2,X_3]),V) = g(\phi(\overline{\nabla}'_{X_2}X_3 - \overline{\nabla}'_{X_3}X_2),V),$$

Using (4.7) we take

$$g(\phi([X_2,X_3]),V) = -g(\overline{\nabla}'_{X_3}\phi X_2,V) + g((\overline{\nabla}'_{X_3}\phi)X_2,V) + g(\overline{\nabla}'_{X_2}\phi X_3,V) - g((\overline{\nabla}'_{X_2}\phi)X_3,V) + \beta\eta(X_3)g(QX_2,V).$$

Now using (4.8) and also (3.9) in the above equation we have

$$\begin{aligned} g(\phi([X_2,X_3]),V) &= g(A_{\phi X_2}X_3,V) - g(\nabla^{\perp}_{X_3}\phi X_2,V) + (\alpha+\beta)g(\phi X_3,X_2)\epsilon\eta(V) \\ &\quad - g(A_{\phi X_3}X_2,V) + g((\nabla^{\perp}_{X_2}\phi)X_3,V) - (\alpha+\beta)g(\phi X_2,X_3)\epsilon\eta(V) \\ &\quad + \alpha\epsilon\eta(X_3)g(X_2,V) + \eta X_3 g(X_2,V) + \eta(X_3)g(\phi X_2,V)(\alpha+\delta\beta) \\ &= -g(\nabla^{\perp}_{X_3}\phi X_2,V) - g(A_{\phi X_3}X_2,V) + g(A_{\phi X_2}X_3,V) \\ &\quad + g(\nabla^{\perp}_{X_2}\phi X_3,V) + 2\epsilon(\alpha+\beta)g(\phi X_3,X_2)\eta(V). \end{aligned}$$

This completes the proof. □

Hence this theorem is verifying the following corollary.

Corollary 4.1 *Let M be a ξ vertical CR-submanifold of (ϵ,δ)-trans-Sasakian manifold M' with generalized symmetric metric connection. Then the distribution $D^{\perp}$ is integrable iff*

$$A_{\phi X_2}X_3 - A_{\phi X_3}X_2 = 2(\alpha+\beta)g(\phi X_2,X_3)\xi \tag{4.19}$$

for any $X_2, X_3 \in \Gamma(D^{\perp})$.

Corollary 4.2 *Let M be a ξ vertical CR-submanifold of (ϵ,δ)-trans-Sasakian manifold M' with generalized symmetric metric connection. Then the distribution $D^{\perp}$ is integrable iff*

$$A_{\phi X_2}X_3 = A_{\phi X_3}X_2$$

for any $X_2, X_3 \in \Gamma(D^{\perp})$.

Corollary 4.3 *Let M be a ξ vertical CR-submanifold of (ϵ,δ)-trans-Sasakian manifold M' with generalized symmetric metric connection. Then the distribution $D^{\perp}$ is integrable iff*

$$A_{\phi X_2}X_3 - A_{\phi X_3}X_2 = g(\phi X_2,X_3)\xi$$

for any $X_2, X_3 \in \Gamma(D^{\perp})$.

Theorem 4.3 *Let M be a CR-submanifold of (ϵ,δ)-trans-Sasakian manifold M' with generalized symmetric metric connection. Then we have*

$$\nabla_{X_1}\xi = (\alpha+\delta\beta)PX_1 - \alpha\epsilon BX_1, \tag{4.20}$$
$$h(X_1,\xi) = -\alpha\epsilon CX_1 - (\alpha+\delta\beta)(QX_1 + \eta(X_1)\xi), \tag{4.21}$$
$$\nabla_{X_2}\xi = (\alpha+\delta\beta)PX_2 - \alpha\epsilon BX_2, \tag{4.22}$$
$$h(X_2,\xi) = -\alpha\epsilon CX_2 - (\alpha+\delta\beta)(QX_2 + \eta(X_2)\xi), \tag{4.23}$$
$$\nabla_{\xi}\xi = (\alpha+\delta\beta)P\xi, \tag{4.24}$$
$$h(\xi,\xi) = -(\alpha+\delta\beta)QX_2. \tag{4.25}$$

for any $X_2, X_3 \in \Gamma(D^{\perp})$.

Proof The above theorem is proved from (3.10) by using (3.12), (4.1) and (4.2) considering

$$\begin{aligned}\nabla_{X_1}\xi + h(X_1, \xi) = &-\alpha\epsilon BX_1 - \alpha\epsilon CX_1 + (\alpha + \delta\beta)PX_1 + (\alpha + \delta\beta)QX_1 \\ &- (\alpha + \delta\beta)\eta(X_1)\xi \end{aligned} \tag{4.26}$$

On equating LHS and RHS we have (4.18) and (4.20). Now by replacing X_1 to X_2 in (4.25) and on equating LHS and RHS we get (4.21) and (4.22). Again by replacing X_1 to ξ in (4.25) and on equatingLHS and RHS we get (4.23) and (4.24). The proof is completed. □

Theorem 4.4 *Let M be a ξ horizontal CR-submanifold of (ϵ,δ)-trans-Sasakian manifold M' with generalized symmetric metric connection. Then the distribution D is integrable iff*

$$h(\phi X_1, X_2) = h(X_1, \phi X_2) \tag{4.27}$$

for any $X_1, X_2 \in \Gamma(D)$.

Proof Assuming M to be ξ horizontal we have from (4.9)

$$h(X_1, \phi PX_2) = Ch(X_1, X_2) - (\alpha + \delta\beta)\eta(X_2)\phi QX_1 + \phi Q\overline{\nabla}' X_1 X_2$$

for all $X_1, X_2 \in D$.

Since $[X_1, X_2] \in D$, we have D is integrable iff

$$h(\phi X_1, X_2) = h(X_1, \phi X_2) \tag{4.28}$$

This completes the proof. □

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A Proof of Reciprocity Theorem by Use of Loop Integrals

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Abstract: In this paper, we give a proof of the reciprocity theorem of Ramanujan using loop integrals.

Key Words: Reciprocity theorem, loop integrals, residue calculus.

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§1. Introduction

In his lost notebook [12], Ramanujan recorded the following beautiful reciprocity theorem

$$\rho(a,b)-\rho(b,a)=\left(\frac{1}{b}-\frac{1}{a}\right)\frac{(aq/b,bq/a,q)_\infty}{(-aq,-bq)_\infty}, \tag{1}$$

where

$$\rho(a,b)=\left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^n b^{-n}}{(-aq)_n}$$

and $a,\ b$ are complex numbers other than 0 and $-q^{-n}$. Throughout this paper, we assume $|q|<1$ and employ the customary notations

$$(a)_\infty := (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n),$$

$$(a)_n := (a;q)_n := \frac{(a)_\infty}{(aq^n)_\infty}, \qquad n \text{ is an integer.}$$

The first proof of (1) was given by Andrews [2] by employing his four-variable identity and the well-known Jacobi's triple product identity which is a special case of (1). Somashekara and Fathima [13] used Ramanujan's ${}_1\psi_1$ summation formula and Heine's transformation formula to establish an equivalent version of (1). Bhargava, Somashekara and Fathima [5] provided another proof of (1). Kim, Somashekara and Fathima [10] gave a proof of (1) using only q-binomial theorem. Guruprasad and Pradeep [8] also have devised a proof of (1) using q-binomial theorem.

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Adiga and Anitha [1] established a proof of (1) by the method of analytic continuation. Berndt, Chan, Yeap and Yee [4] found three different proofs of (1). Kang [9] constructed a proof of (1) along the lines of Venkatachaliengar's proof of Ramanujan's ${}_1\psi_1$ summation formula. In [14], Somashekara and Narasimha Murthy gave a proof of (1) using Abel's lemma and Jacobi's triple product identity. Recently, Somashekara, Narasimha Murthy and Shalini [16] have proved an equivalent form of the general identity of Andrews [2] using the parameter augumentation method and employed the same to derive (1). Further, Somashekara and Narasimha Murthy [15] gave a finite form of (1). For more details one may refer the book [3] by Andrews and Berndt .

In 1988, K.Mimachi [11] gave a proof of Ramanujan's ${}_1\psi_1$ summation using loop integrals. Motivated by this, we give a proof of Ramanujan's Reciprocity theorem [12] using loop integrals.

§2. Proof of the Reciprocity Theorem

Making the substitution $a = -cz/q$, $b = c/q$ in (1) we obtain

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(cz)_{n+1}} z^n + \sum_{n=1}^{\infty} \frac{q^{n(n-1)/2}}{(-c)_n} z^{-n} = \frac{(-qz)_\infty(-1/z)_\infty(q)_\infty}{(-c)_\infty(cz)_\infty}. \tag{2}$$

Using the Heine's transformation [7, eq(III.2), p.359],

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(q)_n(c)_n} z^n = \frac{(c/b)_\infty(bz)_\infty}{(c)_\infty(z)_\infty} \sum_{n=0}^{\infty} \frac{(abz/c)_n(b)_n}{(q)_n(bz)_n} \left(\frac{c}{b}\right)^n \tag{3}$$

in (2) we obtain

$$\sum_{n=-\infty}^{\infty} (-q/c)_n (cz)^n = \frac{(-qz)_\infty(-1/z)_\infty(q)_\infty}{(-c)_\infty(cz)_\infty}, \quad 0 < |z| < \frac{1}{|c|}, \tag{4}$$

where $|q| < 1$, $z \neq 0$, $c \neq -1, -q^{-n}$, $n \in \mathbb{Z}^+$.

Set $C_n := \rho_n e^{i\phi}$ where $\rho_n := \frac{1}{2}|c|\left(|q|^n + |q|^{n+1}\right)$, $0 \le \phi \le 2\pi$. Then the series in (4) is defined in $C_n \setminus \{0\}$. Define

$$\begin{aligned} f(t) &:= \frac{(-1/t)_\infty(-tq)_\infty}{(c/t)_\infty(1-tz)}, \\ F(t) &:= \frac{(q)_\infty(-1/t)_\infty(-tq)_\infty}{(-c)_\infty(c/t)_\infty(1-tz)} \\ &= \frac{(q)_\infty}{(-c)_\infty} f(t), \\ I(C) &:= \frac{1}{2\pi i}\int_C f(t)dt \quad \text{and} \qquad (5) \\ \underset{t=y}{Res}\ \phi(t) &:= \text{“the residue of } \phi(t) \text{ at } t = y\text{”}. \end{aligned}$$

The function F(t) has simple poles at $t = cq^j$, $j = 0, 1, 2, \cdots$ and $t = 1/z$. The infinite point ∞ and the origin 0 are essential singularities.

We now show that

$$\sum_{n=-\infty}^{\infty}(-q/c)_n(cz)^n - \frac{(q)_\infty(-qz)_\infty(-1/z)_\infty}{(-c)_\infty(cz)_\infty} = \sum_{j=0}^{\infty}\operatorname*{Res}_{t=cq^j}F(t) + \operatorname*{Res}_{t=1/z}F(t). \tag{6}$$

Consider

$$\begin{aligned}\sum_{n=0}^{\infty}(-q/c)_n(cz)^n &= \sum_{n=0}^{\infty}\frac{(-q/c)_\infty}{(-q^{n+1}/c)_\infty}(cz)^n \\ &= (-q/c)_\infty\sum_{n=0}^{\infty}\frac{1}{(-q^{n+1}/c)_\infty}(cz)^n.\end{aligned}$$

On using the special case [7, eq(1.3.15), p.10],

$$\frac{1}{(z)_\infty} = \sum_{n=0}^{\infty}\frac{z^n}{(q)_n}$$

of the well known q-binomial theorem [6], we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}(-q/c)_n(cz)^n &= (-q/c)_\infty\sum_{n=0}^{\infty}(cz)^n\sum_{j=0}^{\infty}\frac{(-q^{n+1}/c)^j}{(q)_j} \\ &= (-q/c)_\infty\sum_{j=0}^{\infty}\frac{(-q/c)^j}{(q)_j}\frac{1}{1-czq^j} = \sum_{j=0}^{\infty}\operatorname*{Res}_{t=cq^j}F(t).\end{aligned} \tag{7}$$

Now, consider

$$\begin{aligned}\operatorname*{Res}_{t=1/z}F(t) &= \lim_{t\to 1/z}(t-1/z)F(t) \\ &= -\frac{(q)_\infty(-qz)_\infty(-1/z)_\infty}{(-c)_\infty(cz)_\infty}.\end{aligned} \tag{8}$$

Since

$$\sum_{n=-\infty}^{-1}(-q/c)_n(cz)^n = \sum_{n=1}^{\infty}\frac{1}{(-q/cq^n)_n}\frac{1}{(cz)^n}$$

is analytic in $C_n \setminus \{0\}$ as $0 < |z| < 1/|c|$ and from (7) and (8), (6) follows.

Now, it remains to prove that for $|q| < 1$ and $0 < |z| < 1/|c|$,

$$\sum_{j=0}^{\infty}\operatorname*{Res}_{t=cq^j}F(t) + \operatorname*{Res}_{t=1/z}F(t) = 0.$$

The simple poles $t = cq^j$, $j = 0, 1, 2, \cdots, n$ and $1/z$ lie in the deleted neighbourhood $C_n \setminus \{0\}$. So, the function is analytic in the region $C_n \setminus \{0\}$ except inside the circles $\gamma_0, \gamma_1, \cdots, \gamma_n, \alpha$ where γ_j is the circle centred at cq^j with very small radius, $0 \le j \le n$ and α is the circle centred at $1/z$ with very small radius.

Thus, by Cauchy's theorem, we have

$$\int_{C_n\setminus\{0\}-\sum_{j=0}^{n}\gamma_j-\alpha} f(t)dt=0.$$

Using the properties of integrals, we have

$$\int_{C_n\setminus\{0\}} f(t)dt=\int_{\sum_{j=0}^{n}\gamma_j} f(t)dt+\int_{\alpha} f(t)dt.$$

Hence

$$\frac{1}{2\pi i}\int_{C_n\setminus\{0\}} f(t)dt=\sum_{j=0}^{n}\underset{t=cq^j}{Res}\, f(t)+\underset{t=1/z}{Res}\, f(t).$$

This yields

$$\sum_{j=0}^{n}\underset{t=cq^j}{Res}\, f(t)+\underset{t=1/z}{Res}\, f(t)=I(C_n\setminus\{0\}),$$

on using (5). Therefore, the problem reduces to proving $I(C_n\setminus\{0\})\longrightarrow 0$ as $n\longrightarrow\infty$. For $n=1,2,3,\cdots$, we have from the definition

$$f(|q|^n t)=\frac{(-1/|q|^n t)_\infty(-|q|^n tq)_\infty}{(c/|q|^n t)_\infty(1-|q|^n tz)}.$$

Hence, for $0\le\phi\le 2\pi$,

$$\begin{aligned}|f(\rho_n e^{i\phi})| &= |f(\rho_0|q|^n e^{i\phi})| \\ &= \left|\frac{1}{c}\right|^n\left|\left(\frac{-\rho_0|q|^n e^{i\phi}q}{q^n}\right)_n\left(\frac{-q^n}{|q|^n\rho_0 e^{i\phi}}\right)_\infty(-|q|^n\rho_0 e^{i\phi}q)_\infty\right| \\ &\quad\times\left|\left(\frac{|q|^n\rho_0 e^{i\phi}q}{cq^n}\right)_n\left(\frac{cq^n}{|q|^n\rho_0 e^{i\phi}}\right)_\infty(1-|q|^n tz)\right|^{-1}. \end{aligned}\tag{9}$$

For each factor in the right hand side, we have the following estimates

$$\begin{aligned}\left|\left(\frac{-\rho_0|q|^n e^{i\phi}q}{q^n}\right)_n\right| &= \left|\prod_{j=0}^{n-1}\left(1+\frac{\rho_0|q|^n e^{i\phi}qq^j}{q^n}\right)\right| \\ &\le \prod_{j=0}^{n-1}\left(1+|\rho_0 q^{j+1}|\right)\le\prod_{j=0}^{\infty}\left(1+|\rho_0 q^{j+1}|\right). \end{aligned}\tag{10}$$

$$\left|\left(\frac{-q^n}{|q|^n\rho_0 e^{i\phi}}\right)_\infty\right|=\left|\prod_{j=0}^{\infty}\left(1+\frac{q^n q^j}{|q|^n\rho_0 e^{i\phi}}\right)\right|\le\prod_{j=0}^{\infty}\left(1+\left|\frac{q^j}{\rho_0}\right|\right).\tag{11}$$

$$\begin{aligned}\left|(-|q|^n\rho_0 e^{i\phi}q)_\infty\right| &= \left|\prod_{j=0}^{\infty}\left(1+|q|^n\rho_0 e^{i\phi}qq^j\right)\right| \\ &\leq \prod_{j=0}^{\infty}\left(1+|q^{j+1+n}\rho_0|\right) \leq \prod_{j=0}^{\infty}\left(1+|\rho_0 q^{j+1}|\right). \end{aligned} \tag{12}$$

$$\begin{aligned}\left|\left(\frac{|q|^n\rho_0 e^{i\phi}q}{cq^n}\right)_n\right| &= \left|\prod_{j=0}^{n-1}\left(1-\frac{|q|^n\rho_0 e^{i\phi}qq^j}{cq^n}\right)\right| \\ &\geq \prod_{j=0}^{n-1}\left(1-\left|\frac{\rho_0 q^{j+1}}{c}\right|\right) \geq \prod_{j=0}^{\infty}\left(1-\left|\frac{\rho_0 q^{j+1}}{c}\right|\right) > 0. \end{aligned} \tag{13}$$

$$\left|\left(\frac{cq^n}{|q|^n\rho_0 e^{i\phi}}\right)_\infty\right| = \left|\prod_{j=0}^{\infty}\left(1-\frac{cq^nq^j}{|q|^n\rho_0 e^{i\phi}}\right)\right| \geq \prod_{j=0}^{\infty}\left(1-\frac{cq^j}{\rho_0}\right) > 0. \tag{14}$$

$$\left|(1-|q|^n\rho_0 e^{i\phi}z)\right| \geq 1-|q^n\rho_0 z| \geq 1-|\rho_0 z| > 0. \tag{15}$$

On using (10)-(15) in (9), it follows that there exists a positive number M such that

$$|f(\rho_n e^{i\phi})| \leq M\left|\frac{1}{c}\right|^n, \ (0 \leq \phi \leq 2\pi).$$

Hence,

$$\begin{aligned}|I(C_n\setminus\{0\})| &= \left|\frac{1}{2\pi i}\int_{C_n\setminus\{0\}} f(t)dt\right| \leq \frac{\rho_n}{2\pi}\int_0^{2\pi}|f(\rho_n e^{i\phi})||d\phi| \\ &\leq \frac{\rho_n}{2\pi}\ \underset{0\leq\phi\leq 2\pi}{Max}|f(\rho_n e^{i\phi})|2\pi \leq \rho_0|q|^n M\left|\frac{1}{c}\right|^n. \end{aligned}$$

Consequently, for $|c| > 1$, $I(C_n\setminus\{0\}) \longrightarrow 0$ as $n \longrightarrow \infty$.

Hence under the condition $|q| < 1,\ 0 < |z| < 1/|c|$ where $c \neq -1, -q^{-n}, n \in \mathbb{Z}^+$, we have (4), completing the proof. □

§3. Acknowlodgement

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On Skew-Quotient of Randić and Sum-Connectivity Energy of Digraphs

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Abstract: In this paper we introduce the concept of skew-quotient of Randić and sum-connectivity energy of directed graphs. We then obtain upper and lower bounds for skew-quotient of Randić and sum-connectivity energy of digraphs. Then, we compute the skew-quotient of Randić and sum-connectivity energy of some graphs such as star digraph, complete bipartite digraph, the $S_m \wedge P_2$ digraph and a crown digraph.

Key Words: Skew-quotient, Randić and sum-connectivity energy, adjacency matrix, digraph.

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§1. Introduction

In [3], Puttaswamy and Bhavya C. A have introduced the quotient of Randić and sum-connectivity energy of a simple graph G as follows. Let a and b be two nonnegative real numbers with $a \neq 0$. The quotient of Randić and sum-connectivity adjacency matrix of G is the $n \times n$ matrix $A_{qrs} = (a_{ij})$, where

$$a_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{\sqrt{\frac{a(d_i+d_j)}{b(d_i d_j)}}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent}, \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

The quotient of Randić and sum-connectivity energy of G is defined to be the sum of absolute values of eigenvalues of the quotient of Randić and sum-connectivity adjacency matrix of G.

In 2010, Adiga, Balakrishnan and Wasin So [1] have introduced the skew energy of a digraph as follows. Let D be a digraph of order n with vertex set $V(D) = \{v_1, v_2, \cdots, v_n\}$ and arc set $\Gamma(D) \subset V(D) \times V(D)$ where $(v_i, v_i) \notin \Gamma(D)$ for all i and $(v_i, v_j) \in \Gamma(D)$ implies

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$(v_j, v_i) \notin \Gamma(D)$. The skew-adjacency matrix of D is the $n \times n$ matrix $S(D) = (s_{ij})$ where $s_{ij} = 1$ whenever $(v_i, v_j) \in \Gamma(D)$, $s_{ij} = -1$ whenever $(v_j, v_i) \in \Gamma(D)$ and $s_{ij} = 0$ otherwise. Hence $S(D)$ is a skew symmetric matrix of order n and all its eigenvalues are of the form $i\lambda$ where $i = \sqrt{-1}$ and λ is a real number. The skew energy of G is the sum of the absolute values of eigenvalues of $S(D)$.

Motivated by these works, we introduce the concept of skew-quotient of Randić and sum-connectivity energy of a digraph as follows. Let a and b be two nonnegative real numbers with $a \neq 0$ and D be a digraph of order n with vertex set $V(D) = \{v_1, v_2, \cdots, v_n\}$ and arc set $\Gamma(D) \subset V(D) \times V(D)$ where $(v_i, v_i) \notin \Gamma(D)$ for all i and $(v_i, v_j) \in \Gamma(D)$ implies $(v_j, v_i) \notin \Gamma(D)$. Then the skew-quotient of Randić and sum-connectivity adjacency matrix of D is the $n \times n$ matrix $A_{sqrs} = (a_{ij})$ where

$$a_{ij} = \begin{cases} \frac{1}{\sqrt{\frac{a(d_i+d_j)}{b(d_i d_j)}}}, & \text{if } (v_i, v_j) \in \Gamma(D), \\ -\frac{1}{\sqrt{\frac{a(d_i+d_j)}{b(d_i d_j)}}}, & \text{if } (v_j, v_i) \in \Gamma(D), \\ 0, & \text{otherwise.} \end{cases}$$

Then, the skew-quotient of Randić and sum-connectivity energy $E_{sqrs}(D)$ of D is defined to be the sum of the absolute values of eigenvalues of A_{sqrs}.

In Section 2 of this paper we obtain the upper and lower bounds for skew-quotient of Randić and sum-connectivity energy of digraphs. In Section 3 we compute the skew-quotient of Randić and sum-connectivity energy of some directed graphs such as complete bipartite digraph, star digraph, the $(S_m \wedge P_2)$ digraph and a crown digraph.

§2. Upper and Lower Bounds for Skew-Quotient of Randić and Sum-Connectivity Energy

Theorem 2.1 *Let D be a simple digraph of order n and a, b be as defined above. Then*

$$E_{sqrs}(D) \leq \sqrt{2n \sum_{j \sim k} \frac{b(d_j d_k)}{a(d_j + d_k)}}.$$

Proof Let $i\lambda_1, i\lambda_2, i\lambda_3, \cdots, i\lambda_n$, be the eigenvalues of A_{sqrs}, where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \cdots \geq \lambda_n$. Since

$$\sum_{j=1}^{n} (i\lambda_j)^2 = tr(A_{sqrs}^2) = -\sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk}^2 = -2 \sum_{j \sim k} \frac{b(d_j d_k)}{a(d_j + d_k)},$$

we have

$$\sum_{j=1}^{n} |\lambda_j|^2 = 2 \sum_{j \sim k} \frac{b(d_j d_k)}{a(d_j + d_k)}. \tag{1}$$

Applying the Cauchy-Schwartz inequality

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \left(\sum_{j=1}^{n} a_j^2\right) \cdot \left(\sum_{j=1}^{n} b_j^2\right)$$

with $a_j = 1$, $b_j =\mid \lambda_j \mid$, we obtain

$$E_{sqrs}(D) = \sum_{j=1}^{n} |\lambda_j| = \sqrt{(\sum_{j=1}^{n} |\lambda_j|)^2} \leq \sqrt{n \sum_{j=1}^{n} |\lambda_j|^2} = \sqrt{2n \sum_{j \sim k} \frac{b(d_j d_k)}{a(d_j + d_k)}}.$$

This completes the proof. □

Theorem 2.2 *Let D be a simple digraph of order n with and a, b be as defined above. Then*

$$E_{sqrs}(D) \geq \sqrt{2 \sum_{j \sim k} \frac{b(d_j d_k)}{a(d_j + d_k)} + n(n-1) p^{\frac{2}{n}}}, \tag{2}$$

where

$$p = |det A_{sqrs}| = \prod_{j=1}^{n} |\lambda_j|.$$

Proof A calculation shows that

$$(E_{sqrs}(D))^2 = \left(\sum_{j=1}^{n} |\lambda_j|\right)^2 = \sum_{j=1}^{n} |\lambda_j|^2 + \sum_{1 \leq j \neq k \leq n} |\lambda_j||\lambda_k|.$$

By arithmetic − geometric mean inequality, we get

$$\begin{aligned}
\sum_{1 \leq j \neq k \leq n} |\lambda_j||\lambda_k| &= |\lambda_1|(|\lambda_2| + |\lambda_3| + \cdots + |\lambda_n|) \\
&\quad + |\lambda_2|(|\lambda_1| + |\lambda_3| + \cdots + |\lambda_n|) + \cdots \\
&\quad + |\lambda_n|(|\lambda_1| + |\lambda_2| + \cdots + |\lambda_{n-1}|) \\
&\geq n(n-1)(|\lambda_1||\lambda_2| \cdots |\lambda_n|)^{\frac{1}{n}} (|\lambda_1|^{n-1}|\lambda_2|^{n-1} \cdots |\lambda_n|^{n-1})^{\frac{1}{n(n-1)}} \\
&= n(n-1)(\prod_{j=1}^{n} |\lambda_j|)^{\frac{1}{n}} (\prod_{j=1}^{n} |\lambda_j|)^{\frac{1}{n}} \\
&= n(n-1) \left(\prod_{j=1}^{n} |\lambda_j|\right)^{\frac{2}{n}}.
\end{aligned}$$

Thus

$$(E_{sqrs}(D))^2 \geq \sum_{j=1}^{n} |\lambda_j|^2 + n(n-1) \left(\prod_{j=1}^{n} |\lambda_j|\right)^{\frac{2}{n}}.$$

From the equation (1), we get

$$(E_{sqrs}(D))^2 \geq 2\sum_{j\sim k}\frac{b(d_jd_k)}{a(d_j+d_k)}+n(n-1)p^{\frac{2}{n}},$$

which gives the inequality (2). □

§3. Skew-Quotient of Randić and Sum-Connectivity Energies of Some Families of Graphs

We begin with some basic definitions and notations.

Definition 3.1[4] *A graph G is said to be complete if every pair of its distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n.*

Definition 3.2[4] *A bigraph or bipartite graph G is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every line of G joins V_1 with V_2. (V_1, V_2) is a bipartition of G. If G contains every line joining V_1 and V_2, then G is a complete bigraph. If V_1 and V_2 have m and n points, we write $G = K_{m,n}$. A star is a complete bigraph $K_{1,n}$.*

Definition 3.3[2] *The Crown graph S_n^0 for an integer $n \geq 3$ is the graph with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and edge set $\{u_iv_j; 1 \leq i, j \leq n, i \neq j\}$. S_n^0 is therefore S_n^0 coincides with complete bipartite graph $K_{n,n}$ with the horizontal edges removed.*

Definition 3.4[5] *The conjunction $S_m \wedge P_2$ of $S_m = \overline{K}_m + K_1$ and P_2 is the graph having the vertex set $V(S_m) \times V(P_2)$ and edge set $\{(v_i, v_j)(v_k, v_l)|v_iv_k \in E(S_m)$ and $v_jv_l \in E(P_2)$ and $1 \leq i, k \leq m+1, 1 \leq j, l \leq 2\}$.*

Now, we compute skew- quotient of Randić and sum-connectivity energies of some directed graphs such as complete bipartite digraph, star digraph, the $(S_m \wedge P_2)$ digraph and a crown digraph.

Theorem 3.5 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of $K_{m,n}$ complete bipartite digraph be respectively given by $V(D) = \{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n\}$ and $\Gamma(D) = \{(u_i, v_j) \,|\, 1 \leq i \leq m, 1 \leq j \leq n\}$. Then the skew-quotient of Randić and sum-connectivity energy of the complete bipartite digraph is*

$$2\sqrt{\frac{(mn)^2}{a(m+n)}}.$$

Proof The skew-quotient of Randić and sum-connectivity matrix of complete bipartite digraph is given by

$$A_{sqrs} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \gamma & \gamma & \cdots & \gamma \\ 0 & 0 & \cdots & 0 & \gamma & \gamma & \cdots & \gamma \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \gamma & \gamma & \cdots & \gamma \\ -\gamma & -\gamma & \cdots & -\gamma & 0 & 0 & \cdots & 0 \\ -\gamma & -\gamma & \cdots & -\gamma & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma & -\gamma & \cdots & -\gamma & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $\gamma = \frac{1}{\sqrt{\frac{a(m+n)}{b(mn)}}}$. Then, its characteristic polynomial is

$$\begin{aligned} |\lambda I - A_{sqrs}| &= \begin{vmatrix} \lambda & 0 & \cdots & 0 & -\gamma & -\gamma & \cdots & -\gamma \\ 0 & \lambda & \cdots & 0 & -\gamma & -\gamma & \cdots & -\gamma \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & -\gamma & -\gamma & \cdots & -\gamma \\ \gamma & \gamma & \cdots & \gamma & \lambda & 0 & \cdots & 0 \\ \gamma & \gamma & \cdots & \gamma & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma & \gamma & \cdots & \gamma & 0 & 0 & \cdots & \lambda \end{vmatrix} \\ &= \begin{vmatrix} \lambda I_m & -\frac{1}{\sqrt{\frac{a(m+n)}{b(mn)}}} J^T \\ \frac{1}{\sqrt{\frac{a(m+n)}{b(mn)}}} J & \lambda I_n \end{vmatrix}, \end{aligned}$$

where J is an $n \times m$ matrix with all the entries are equal to 1. Hence the characteristic equation is given by

$$\begin{vmatrix} \lambda I_m & -\frac{1}{\sqrt{\frac{a(m+n)}{b(mn)}}} J^T \\ \frac{1}{\sqrt{\frac{a(m+n)}{b(mn)}}} J & \lambda I_n \end{vmatrix} = 0,$$

which can be written also as

$$|\lambda I_m| \left| \lambda I_n - \left(\frac{1}{\sqrt{\frac{a(m+n)}{b(mn)}}} J \right) \frac{I_m}{\lambda} \left(-\frac{1}{\sqrt{\frac{a(m+n)}{b(mn)}}} J^T \right) \right| = 0.$$

On simplification, we obtain

$$\frac{\lambda^{m-n}}{\frac{a(m+n)}{b(mn)}}\left|(\frac{a(m+n)}{b(mn)})\lambda^2 I_n + JJ^T\right| \quad = \quad 0,$$

which can be written as

$$\frac{\lambda^{m-n}}{\frac{a(m+n)}{b(mn)}} P_{JJ^T}(-(\frac{a(m+n)}{b(mn)})\lambda^2) \quad = \quad 0,$$

where $P_{JJ^T}(\lambda)$ is the characteristic polynomial of the matrix ${}_mJ_n$. Thus, we have

$$\frac{\lambda^{m-n}}{\frac{a(m+n)}{b(mn)}}\left(\frac{a(m+n)}{b(mn)}\lambda^2 + mn\right)\left(\frac{a(m+n)}{b(mn)}\lambda^2\right)^{n-1} = 0,$$

which is the same as

$$\lambda^{m+n-2}\left(\lambda^2 + \frac{mn}{\frac{a(m+n)}{b(mn)}}\right) = 0.$$

Hence,

$$Spec\,(D) = \begin{pmatrix} 0 & i\sqrt{\frac{mn}{\frac{a(m+n)}{b(mn)}}} & -i\sqrt{\frac{mn}{\frac{a(m+n)}{b(mn)}}} \\ m+n-2 & 1 & 1 \end{pmatrix}$$

and the skew-quotient of Randić and sum-connectivity energy of $K_{m,n}$ is

$$E_{sg}(D) = 2\sqrt{\frac{b(mn)^2}{a(m+n)}}. \qquad \square$$

Theorem 3.6 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of S_n star digraph be respectively given by $V(D) = \{v_1, v_2, \cdots, v_n\}$ and $\Gamma(D) = \{(v_1, v_j)\,|\, 2 \le j \le n\}$. Then, the skew-quotient of Randić and sum-connectivity energy of D is*

$$2\sqrt{\frac{b(n-1)^2}{an}}.$$

Proof The skew-quotient of Randić and sum-connectivity matrix of the star digraph D is given by

$$A_{sqrs} = \begin{pmatrix} 0 & \frac{1}{\sqrt{\frac{an}{b(n-1)}}} & \frac{1}{\sqrt{\frac{an}{b(n-1)}}} & \cdots & \frac{1}{\sqrt{\frac{an}{b(n-1)}}} & \frac{1}{\sqrt{\frac{an}{b(n-1)}}} \\ -\frac{1}{\sqrt{\frac{an}{b(n-1)}}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{\frac{an}{b(n-1)}}} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{1}{\frac{an}{b(n-1)}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\frac{an}{b(n-1)}} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Hence, the characteristic polynomial is given by

$$
\begin{aligned}
|\lambda I - A_{sqrs}| &= \begin{vmatrix} \lambda & -\frac{1}{\sqrt{\frac{an}{b(n-1)}}} & -\frac{1}{\sqrt{\frac{an}{b(n-1)}}} & \cdots & -\frac{1}{\sqrt{\frac{an}{b(n-1)}}} \\ \frac{1}{\sqrt{\frac{an}{b(n-1)}}} & \lambda & 0 & \cdots & 0 \\ \frac{1}{\sqrt{\frac{an}{b(n-1)}}} & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{\frac{an}{b(n-1)}}} & 0 & 0 & \cdots & \lambda \end{vmatrix} \\
&= \left(\frac{1}{\sqrt{\frac{an}{b(n-1)}}}\right)^n \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ 1 & \mu & 0 & \cdots & 0 & 0 \\ 1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & \mu & 0 \\ 1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix},
\end{aligned}
$$

where $\mu = \lambda\sqrt{\frac{an}{b(n-1)}}$. Then, we get that

$$
|\lambda I - A_{sqrs}| = \phi_n(\mu)\left(\frac{1}{\sqrt{\frac{an}{b(n-1)}}}\right)^n,
$$

where

$$
\phi_n(\mu) = \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ 1 & \mu & 0 & \cdots & 0 & 0 \\ 1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & \mu & 0 \\ 1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix}.
$$

Using the properties of the determinants, we obtain

$$
\phi_n(\mu) = (\mu^{n-2} + \mu\phi_{n-1}(\mu))
$$

after some simplifications. By iterating, we obtain

$$
\phi_n(\mu) = \mu^{n-2}(\mu^2 + n - 1).
$$

Therefore

$$|\lambda I - A_{sqrs}| = \left(\frac{1}{\sqrt{\frac{an}{b(n-1)}}}\right)^n \left[\left((\frac{an}{b(n-1)})\lambda^2 + (n-1)\right)(\lambda\sqrt{\frac{an}{b(n-1)}})^{n-2}\right].$$

Thus the characteristic equation is given by

$$\lambda^{n-2}\left(\lambda^2 + \frac{n-1}{\frac{an}{b(n-1)}}\right) = 0.$$

Hence,

$$Spec\,(D) = \begin{pmatrix} 0 & i\sqrt{\frac{n-1}{\frac{an}{b(n-1)}}} & -i\sqrt{\frac{n-1}{\frac{an}{b(n-1)}}} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Therefore, the skew-quotient of Randić and sum-connectivity energy of D is

$$E_{sqrs}(D) = 2\sqrt{\frac{b(n-1)^2}{an}}.$$ □

Theorem 3.7 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of S_n^0 crown digraph be respectively given by $V(D) = \{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and $\Gamma(D) = \{(u_i, v_j) \,|\, 1 \le i \le n, 1 \le j \le n, i \ne j\}$. Then, the skew-quotient of Randić and sum-connectivity energy of the crown digraph is*

$$\frac{4\sqrt{b}(n-1)^2}{\sqrt{2a(n-1)}}.$$

Proof The skew-quotient of Randić and sum-connectivity matrix of crown digraph is given by

$$A_{sqrs} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \gamma & \cdots & \gamma \\ 0 & 0 & \cdots & 0 & \gamma & 0 & \cdots & \gamma \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \gamma & \gamma & \cdots & 0 \\ 0 & -\gamma & \cdots & -\gamma & 0 & 0 & \cdots & 0 \\ -\gamma & 0 & \cdots & -\gamma & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma & -\gamma & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $\gamma = \frac{1}{\sqrt{\frac{2a(n-1)}{b(n-1)^2}}}$. Then, its characteristic polynomial is

$$|\lambda I - A_{sqrs}| = \begin{vmatrix} \lambda I_n & -\frac{1}{\sqrt{\frac{2a(n-1)}{b(n-1)^2}}}K^T \\ \frac{1}{\sqrt{\frac{2a(n-1)}{b(n-1)^2}}}K & \lambda I_n \end{vmatrix},$$

where K is an $n \times n$ matrix. Hence the characteristic equation is given by

$$\begin{vmatrix} \lambda I_n & -\frac{1}{\sqrt{\frac{2a(n-1)}{b(n-1)^2}}} K^T \\ \frac{1}{\sqrt{\frac{2a(n-1)}{b(n-1)^2}}} K & \lambda I_n. \end{vmatrix} = 0.$$

This is same as

$$|\lambda I_n| \left| \lambda I_n - \left(\frac{K}{\sqrt{\frac{2a(n-1)}{b(n-1)^2}}} \right) \frac{I_n}{\lambda} \left(-\frac{K^T}{\sqrt{\frac{2a(n-1)}{b(n-1)^2}}} \right) \right| = 0,$$

which can be written as

$$\frac{1}{(\frac{2a(n-1)}{b(n-1)^2})^n} P_{KK^T} \left(-\frac{2a(n-1)}{b(n-1)^2} \lambda^2 \right) = 0,$$

where $P_{KK^T(\lambda)}$ is the characteristic polynomial of the matrix KK^T. Thus, we have

$$\frac{1}{\left(\frac{2a(n-1)}{b(n-1)^2}\right)^n} \left[\frac{2a(n-1)}{b(n-1)^2} \lambda^2 + (n-1)^2 \right] \left[\frac{2a(n-1)}{b(n-1)^2} \lambda^2 + 1 \right]^{n-1} = 0,$$

which is same as

$$\left(\lambda^2 + \frac{(n-1)^2}{\frac{2a(n-1)}{b(n-1)^2}} \right) \left(\lambda^2 + \frac{1}{\frac{2a(n-1)}{b(n-1)^2}} \right)^{n-1} = 0.$$

Therefore

$$Spec\,(D) = \begin{pmatrix} i\zeta & -i\zeta & i\tau & -i\tau \\ 1 & 1 & n-1 & n-1 \end{pmatrix},$$

where,

$$\zeta = \sqrt{\frac{(n-1)^2}{\frac{2a(n-1)}{b(n-1)^2}}}, \quad \text{and} \quad \tau = \frac{1}{\sqrt{\frac{2a(n-1)}{b(n-1)^2}}}.$$

Hence, the skew-quotient of Randić and sum-connectivity energy of crown digraph is

$$E_{sqrs}(D) = \frac{4\sqrt{b}(n-1)^2}{\sqrt{2a(n-1)}},$$

as desired. □

Theorem 3.8 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of $S_m \wedge P_2$ digraph be respectively given by $V(D) = \{v_1, v_2, \cdots, v_{2m+2}\}$ and $\Gamma(D) = \{(v_1, v_j), (v_{m+2}, v_k) \,|\, 2 \le k \le m+1, m+3 \le j \le 2m+2\}$. Then, the skew-quotient of Randić and sum-connectivity energy of D is*

$$4\sqrt{\frac{b(n-1)^2}{an}}.$$

Proof The skew-quotient of Randić and sum-connectivity matrix of $S_m \wedge P_2$ digraph is given by

$$A_{sqrs} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \gamma & \cdots & \gamma \\ 0 & 0 & \cdots & 0 & -\gamma & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -\gamma & 0 & \cdots & 0 \\ 0 & \gamma & \cdots & \gamma & 0 & 0 & \cdots & 0 \\ -\gamma & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{2n\times 2n},$$

where,

$$m+1=n \ \text{ and } \ \gamma = \frac{1}{\sqrt{\frac{an}{b(n-1)}}}.$$

Then, its characteristic polynomial is given by

$$|\lambda I - A_{sqrs}| = \begin{vmatrix} \lambda & 0 & \cdots & 0 & 0 & -\gamma & \cdots & -\gamma \\ 0 & \lambda & \cdots & 0 & \gamma & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & \gamma & 0 & \cdots & 0 \\ 0 & -\gamma & \cdots & -\gamma & \lambda & 0 & \cdots & 0 \\ \gamma & 0 & \cdots & 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{2n\times 2n}.$$

Hence, the characteristic equation is given by

$$\left(\frac{1}{\sqrt{\frac{an}{b(n-1)}}}\right)^{2n} \begin{vmatrix} \Lambda & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 \\ 0 & \Lambda & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda & 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n} = 0,$$

where $\Lambda = \sqrt{\frac{an}{b(n-1)}}\lambda$.

Now, let

$$\phi_{2n}(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \Lambda & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n}$$

$$= (-1)^{2n+2n}\Lambda \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \Lambda & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{(2n-1)\times(2n-1)}$$

$$+(-1)^{2n+1} \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}$$

and let

$$\Psi_{2n-1}(\Lambda) = (-1)^{2n+1}\begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}.$$

By properties of the determinants, we obtain

$$\Psi_{2n-1}(\Lambda) = \Lambda^{n-2}\Theta_n(\Lambda)$$

after some simplifications, where

$$\Theta_n(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 1 \\ 0 & \Lambda & 0 & \cdots & 1 \\ 0 & 0 & \Lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{n\times n}.$$

Then,

$$\phi_{2n}(\Lambda) = \Lambda^{n-2}\Theta_n(\Lambda) + \Lambda\phi_{2n-1}(\Lambda).$$

Now, proceeding as above, we obtain

$$\begin{aligned}\phi_{2n-1}(\Lambda) &= (-1)^{(2n-1)+1}\Psi_{2n-2}(\Lambda) + (-1)^{(2n-1)+(2n-1)}\Lambda\phi_{2n-2}(\Lambda) \\ &= \Lambda^{n-3}\Theta_n(\Lambda) + \Lambda\phi_{2n-2}(\Lambda)\end{aligned}$$

and similarly, proceeding like this, we obtain

$$\phi_{2n}(\Lambda) = (n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{(n-1)}\xi_{n+1}(\Lambda)$$

at the $(n-1)^{th}$ step, where

$$\xi_{n+1}(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 \\ 0 & \Lambda & 0 & \cdots & 1 \\ 0 & 0 & \Lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{(n+1)\times(n+1)},$$

i.e.,

$$\begin{aligned} \phi_{2n}(\Lambda) &= (n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n-1}\Lambda\Theta_n(\Lambda) \\ &= (n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n}\Theta_n(\Lambda) \\ &= ((n-1)\Lambda^{n-2} + \Lambda^{n})\Theta_n(\Lambda). \end{aligned}$$

By properties of the determinants again, we obtain

$$\Theta_n(\Lambda) = (n-1)\Lambda^{n-2} + \Lambda^{n}.$$

Therefore

$$\phi_{2n}(\Lambda) = ((n-1)\Lambda^{n-2} + \Lambda^{n})^2.$$

Hence, the characteristic equation becomes

$$\left(\frac{1}{\sqrt{\frac{an}{b(n-1)}}}\right)^{2n} \phi_{2n}(\Lambda) = 0,$$

which is same as

$$\left(\frac{1}{\sqrt{\frac{an}{b(n-1)}}}\right)^{2n} ((n-1)\Lambda^{n-2} + \Lambda^{n})^2 = 0,$$

which reduces to

$$\lambda^{2n-4}\left((n-1) + \frac{an}{b(n-1)}\lambda^2\right)^2 = 0.$$

Therefore

$$Spec\,(D) = \begin{pmatrix} 0 & i\sqrt{\frac{n-1}{\frac{an}{b(n-1)}}} & -i\sqrt{\frac{n-1}{\frac{an}{b(n-1)}}} \\ 2n-4 & 2 & 2 \end{pmatrix}.$$

Hence the skew-quotient of Randić and sum-connectivity energy of $S_m \wedge P_2$ digraph is

$$E_{sqrs}(D) = 4\sqrt{\frac{b(n-1)^2}{an}}. \qquad \square$$

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On the Modular Graphic Family of a Graph

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Abstract: For a finite, connected simple graph G of order n with degree sequence, $s = (d_i : 1 \leq i \leq n, d_i = deg(v_i)$ and, $d_j \geq d_{j+1}, 1 \leq j \leq n-1)$ the *modular sequences*, $s(mod\ k)$, $1 \leq k \leq n$ are introduced. Those modular sequences which are graphic constitute the *modular graphic family* of a graph. Numerous introductory results are presented.

Key Words: Degree sequence, graphic sequence, modular sequence, modular graph family.

AMS(2010): 05C15, 05C38, 05C62, 05C75, 05C85.

§1. Introduction

It is assumed that the reader is familiar with most of the classical concepts in graph theory. Unless stated otherwise, only finite, connected simple graphs will be considered. For more on general notation and concepts in graphs see [1,4,8].

Recall that the vertices of a graph G of order n can be labeled $v_1, v_2, v_3, \cdots, v_n$ such that a sequence $s = (d_i : 1 \leq i \leq n, d_i = deg(v_i)$ and, $d_j \geq d_{j+1}, 1 \leq j \leq n-1)$, can be defined. This sequence is called the *degree sequence* of G. We say that the sequence is of order n. Conversely, a finite sequence of non-increasing, non-negative integers, $s = (d_1, \geq d_2, \geq d_3, \geq \cdots, \geq d_n)$ is said to be *simple graphic* (graphic or graphical for brevity) if there exists a finite, simple graph G (not necessarily connected) with degree sequence corresponding to s. It is said that G is a graphical realization of s. The notion of graphic integer sequences has been studied extensively. Characterizations of graphic integer sequences are found in [2,3,5].

Let $s = (d_1, d_2, d_3, \cdots, d_n)$ be the degree sequence of graph G. The *modular sequences* of G are, $s(mod\ k) = (d_1(mod\ k), d_2(mod\ k), d_3(mod\ k), \ldots, d_n(mod\ k))$, $1 \leq k \leq n$. For a given k a modular sequence is abbreviated as, $s(mod\ k) = (d_1, d_2, d_3, \ldots, d_n)(mod\ k)$. Two integer sequences both of order n are said to be distinct if after say, arranging both as non-increasing sequences there exists at least one entry say, the i^{th} entry in each sequence which are not equal. The set of distinct modular sequences which are graphic are called the *graphic family* of G and is denoted by, $\mathfrak{M}_{od}(G)$. Clearly, $|\mathfrak{M}_{od}(G)| \leq n$. Also, $deg_G(v_i) = d_i \leq d_i(mod\ k)$. Note that the realizations of graphic modular sequences need not be connected.

It is agreed that if the set of integers $X = \{b_1, b_2, b_3, \cdots, b_m\}$ is added (or inserted) to an

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integer sequence s it is denoted by, $s \cup X$.

The motivation for this study is firstly, that it is in principle acceptable to do mathematics for the sake of mathematics. Secondly, studying graph theoretic properties and parameters as it may relate between the realizations of modular sequences and the graph G could potentially find application in mathematical chemistry. Thirdly, it could find application to the family of molecular structures which can derive when say, a virus Type A mutates to virus Type B. In the context of the emphasized virology research on the virus, SARS-CoV-2 (causing Covid-19) this notion could be relevant. Other applications can also be conceptualized.

§2. Main Results

2.1 Preliminary Results

It is known that $n(mod\ 1) = 0$, $\forall n$. Hence, the modular sequence $(\underbrace{0, 0, 0, \cdots, 0}_{n\ entries})$ which corresponds to the degree sequence of the null graph (or edgeless graph) of order n is always in $\mathfrak{M}_{od}(G)$ despite the fact that G is a connected simple graph. Furthermore, since $\Delta(G) \leq n-1$ the modular sequence $(d_1, d_2, d_3, \cdots, d_n)(mod\ n) = (d_1, d_2, d_3, \cdots, d_n)$ for graphs of order n.

Proposition 2.1 *Any connected simple graph G of order $n \geq 2$ has, $2 \leq |\mathfrak{M}_{od}(G)| \leq n$.*

Proof For any null graph $\mathfrak{N}_n$ it follows that, $(\underbrace{0, 0, 0, \ldots, 0}_{n\ entries})(mod\ k) = (\underbrace{0, 0, 0, \cdots, 0}_{n\ entries})$, $1 \leq k \leq n$ thus, $|\mathfrak{M}_{od}(\mathfrak{N}_n)| = 1$, $\forall n$. However, for any graph G of order $n \geq 2$ and $\delta(G) \geq 1$ the result is trivial because both $(\underbrace{0, 0, 0, \cdots, 0}_{n\ entries})$ and the degree sequence of G are graphic. □

In fact, besides repetition of graphic modular sequences some graphs have modular sequences which are not graphic. For such graphs, $2 \leq |\mathfrak{M}_{od}(G)| < n$. For example, a triangle with each vertex on C_3 joined to a distinct pendent vertex is of order 6. The degree sequence of such a graph is $(3, 3, 3, 1, 1, 1)$ and $(3, 3, 3, 1, 1, 1)(mod\ 3) = (0, 0, 0, 1, 1, 1)$. Clearly, $(1, 1, 1, 0, 0, 0)$ is not graphical.

Two necessary conditions for a non-negative integer sequence to be graphic are

(i) $d_i \leq n-1$ and

(ii) $\sum\limits_{i=1}^{n} d_i$ is even.

Modular sequences as defined for the degree sequence of a graph always satisfy (i) but not always satisfy (ii). A fundamental avenue for research is, to characterize graphs which yield modular sequences which all satisfy condition (i) and (ii). The next step will be to validate which of the aforesaid modular sequences also satisfy the sufficient conditions to be graphic. We present a family of graphs from [6] which have an unexpected property. Let the family be denoted by $\mathcal{F}$. For $k \in \mathbb{N}$, $k \geq 2$ a graph $G \in \mathcal{F}$ is defined as follows.

Definition 2.2 *Let the complete graph be on vertices $v_1, v_2, v_3, \cdots, v_k$, $k \geq 2$. For each v_i,*

$i = 2, 3, 4, \cdots, k$ add a distinct pendent vertex u_i. Add the edges u_iv_j, $i = 2, 3, 4, \cdots, (k-1)$, $j = i+1, i+2, i+3, \cdots, k$ to obtain $G \in \mathcal{F}$.

Lemma 2.3 *If a sequence of positive integers, after possible arrangement, is a sequence of consecutive decreasing integers say, $s = (t, t-1, t-2, t-3, \cdots, 1)$ then the modular sequences, $s(mod\ k)$, $k = 1, 2, 3, \cdots, t$ are distinct.*

Proof Let $s_1 = (1)$. Clearly, $s_1(mod\ 1) = (0)$ which is distinct. Let $s_2 = (2, 1)$ then, $s(mod\ 1) = (0, 0)$ and $s_2(mod\ 2) = (0, 1)$. Also for $s_3 = (3, 2, 1)$ we have, $s_3(mod\ 1) = (0, 0, 0)$, $s_3(mod\ 2) = (1, 0, 1)$ and $s_3(mod\ 3) = (0, 2, 1)$. The results holds for $1 \leq t \leq 3$. Assume the results holds for $1 \leq t \leq \ell$.

Consider $t = \ell + 1$. Let $s' = (\ell+1, \ell, \ell-1, \ell-2, \cdots, 1)$. Clearly, $s' = s \cup \{\ell + 1\}$. It follows immediately that since $s_i(mod\ i) \neq s_j(mod\ j)$ for all distinct pairs, $1 \leq i, j \leq \ell$ then, $s_i(mod\ i) \cup \{(\ell+1)(mod\ i)\} \neq s_j(mod\ j) \cup \{(\ell+1)(mod\ j)\}$. Hence, the result follows through induction. □

Corollary 2.4 *If the maximum sub-sequence of distinct positive integers in a non-negative integer sequence s of order n, after possible re-arrangement, is the sequence of consecutive decreasing integers say, $(t, t-1, t-2, t-3, \cdots, 1)$, $t \leq n$ then the modular sequences, $s(mod\ k)$, $k = 1, 2, 3, \cdots, n$ are distinct.*

Proof The result is a direct consequence of Lemma 2.3. □

Observe that if a sequence $s = (a_1, a_2, a_3, \cdots, a_n)$ is graphic then, $s' = (a_1, a_2, a_3, \cdots, a_n, 0, 0, 0, \cdots, 0)$ is also graphic. We recall the useful result which is independently due to [3, 5]. It is called the Havel-Hakimi theorem. Also see [7].

Theorem 2.5([3, 5]) *Let $s = (a_1, a_2, a_3, \cdots, a_n)$ be a sequence of non-increasing, non-negative integers. Then s is graphic if and only if the sequence $s' = (a_2 - 1, a_3 - 1, \cdots, a_{a_1+1} - 1, a_{a_1+2}, \cdots, a_n)$ is graphic.*

Theorem 2.6 *If $G \in \mathcal{F}$ then, $|\mathfrak{M}_{od}(G)| = n$.*

Proof Observe that $G \in \mathcal{F}$ is a connected simple graph. It must be shown that the n modular sequences are distinct and graphic. Clearly, for $k \in \mathbb{N}$ the graph G has order $n = 2k - 1$. Also, $\Delta(G) = 2(k-1) = t$. The degree sequence is $s = (t, t-1, t-2, \ldots, t-k, t-k, t-(k+1), t-(k+2), \cdots, 1)$. It follows from Lemma 2.3 (or Corollary 2.4) that the modular sequences are all distinct. Hence, $n = 2k - 1$ such distinct modular sequences are yielded.

We must show that each modular sequence is graphic. It is known that all modular sequences, satisfy condition (i). Also it follows easily that if the number of odd integers in the sub-sequence $(t, t-1, t-2, \cdots, t-k)$ is odd then the number of odd integers in the sub-sequence $(t-k, t-(k+1), t-(k+2), \cdots, 1)$ is odd and conversely. Hence, same is found in any modular sequence $s(mod\ r)$, $1 \leq r \leq n$. Therefore all modular sequences satisfy condition (ii).

We are left to show sufficiency. Applying the Havel-Hakimi theorem to the sequence $s = (t, t-1, t-2, \cdots, t-k, t-k, t-(k+1), t-(k+2), \cdots, 1)$ give the derived sequence, $s' = (t-1, t-$

$2, t-3, \ldots, t-k-1, t-k-1, t-(k+2), t-(k+3), \cdots, 0)$. For the graphic property the zero entry may be deleted. Note that $s'' = (t-1, t-2, t-3, \cdots, t-k-1, t-k-1, t-(k+2), t-(k+3), \cdots, 1)$ is the degree sequence of $G' \in \mathcal{F}$ of order $n-2$. Furthermore, all modular sequences of G' are distinct and graphic, and the union of each with $\{t(mod\ k)\}$, $k = (n-1), n$ respectively does not change the graphic property and adds two more distinct modular sequences. The aforesaid is true because $(n, n-1)(mod\ n) = (0, n-1)$ and $(n, n-1)(mod\ (n-1)) = (1, 0)$. Hence, it follows through immediate induction that the results holds in general. □

The graph defined in Definition 2.2 was first constructed to prove a result related to degree tolerant coloring [6]. Following Theorem 2.5 it is proposed to refer to this family of graphs as the *Havel-Hakimi graphs*, the main topic of next section. It is observed that if both non-negative sequences s_1, s_2 are graphic then, $s_1 \cup s_2$ is graphic. It follows because the degree sequence s of the disjoint union $G_1 \cup G_2$ which is graphic by definition is, $s = s_1 \bigcup s_2$.

2.2 Disjoint Union of Graphs

Let graphs G and H have modular families $\mathfrak{M}_{od}(G)$, $\mathfrak{M}_{od}(G)$, respectively. When we consider the disjoint union $G \bigcup H$ the definition of modular sequences is relaxed to permit the derivative Cartesian product, $\mathfrak{M}_{od}(G) \times \mathfrak{M}_{od}(H) = s_i \bigcup s_j$, $s_i \in \mathfrak{M}_{od}(G)$, $s_j \in \mathfrak{M}_{od}(H)$.

Proposition 2.7 *For $G_1, G_2 \in \mathcal{F}$ of order n and m respectively, it follows that*

$$|\mathfrak{M}_{od}(G_1 \bigcup G_2)| \geq nm.$$

Proof From the n and m distinct and graphical modular sequences a total of nm sequences can be *cup'ed*. From each such $(n+m)$-sequence a total of $\binom{n+m}{n}$ sequences of the form, (n-sequence) $\bigcup$ (m-sequence), can be constructed. At least nm have been shown to be distinct and graphic by Theorem 2.5. Hence,

$$|\mathfrak{M}_{od}(G_1 \bigcup G_2)| \geq nm$$ □

Proposition 2.7 implies a generalization for disconnected graphs.

Corollary 2.8 *Let G be a simple graph with ℓ components and each component has order m_i, $i = 1, 2, 3, \cdots, \ell$. Then*

$$|\mathfrak{M}_{od}(G)| \geq \prod_{i=1}^{\ell} |\mathfrak{M}_{od}(G_i)|.$$

2.2 Complement Graph

This subsection presents a result of interest. The result appears to be trivial but it suggests a deeper problem which we pose in the next section.

Theorem 2.9 *Consider a connected simple graph G and $\overline{G}$ with degree sequences s_1, s_2, respectively. For the pairs of modular sequences, $s_1(mod\ k)$, $s_2(mod\ k)$, $k = 1, 2, 3, \cdots, n$ it follows* *that,* $s_1(mod\ k) = s_2(mod\ k)$ *if and only if* G *is self-complementary.*

Proof Let G be self-complementary. It is obvious that $s_1(mod\ k) = s_2(mod\ k)$, $k = 1, 2, 3, \cdots, n$. Now assume G is not self-complementary. Let s_1, s_2 be the degree sequences of respectively, G and $\overline{G}$. Also, assume that, $s_1(mod\ k) = s_2(mod\ k)$, $k = 1, 2, 3, \cdots, n$. Since G is not self-complementary there exists at least one $v \in V(G)$ such that $deg_G(v) \neq deg_{\overline{G}}(v)$. It implies that amongst the modular sequences at least, $s_1(mod\ n) \neq s_2(mod\ n)$. This contradiction suffices to settle the result. □

2.3 Complete Graphs $\mathfrak{M}_{od}(K_n)$, $k \geq 2$

A complete graph K_n is the graph of smallest order which permits a $(n-1)$-regular graph. Studying the complete graph with regards to the graph parameter, $\mathfrak{M}_{od}(K_n)$, $k \geq 2$ serves as a basis to study same for k-regular graphs in general.

Theorem 2.10 *For a complete graph,* K_n*,* $n \geq 2$*:*

$$|\mathfrak{M}_{od}(K_n)| = \begin{cases} \frac{n}{2} + 1, & \text{if } n \text{ is even;} \\ t - \lceil \frac{t}{2} \rceil + 1, t = \frac{n+1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof The proof is divided into two cases.

Case 1. n is even.

It follows immediately that $s = (\underbrace{n-1, n-1, n-1, \cdots, n-1}_{n-entries})(mod\ \frac{n}{2})$ results in the modular sequence, $(\underbrace{\frac{n}{2} - 1, \frac{n}{2} - 1, \frac{n}{2} - 1, \cdots, \frac{n}{2} - 1}_{n-entries})$.

Similarly through immediate induction, $s_i = (\underbrace{n-1, n-1, n-1, \cdots, n-1}_{n-entries})(mod\ (\frac{n}{2} + i)) =$
$(\underbrace{\frac{n}{2} - (i+1), \frac{n}{2} - (i+1), \cdots, \frac{n}{2} - (i+1)}_{n-entries})$, $i = 0, 1, 2, \ldots, \frac{n}{2} - 1$. Together with the degree sequence, $s(mod\ n) = (\underbrace{n-1, n-1, n-1, \cdots, n-1}_{n-entries})$. All distinct modular sequences have been obtained. The reason is that any $s(mod\ t)$, $t < \frac{n}{2}$ is a repetition as some modular sequence.

We are left to show all distinct modular sequences are graphic. It is easy to verify that all modular sequences satisfy both necessary conditions. finally applying the Havel-Hakimi theorem recursively to each modular sequence results in a sequence of one's which is graphic. Hence, by the Havel-Hakimi theorem all modular sequences are graphic. Therefore, $|\mathfrak{M}_{od}(K_n)| = \frac{n}{2} + 1$, if n is even.

Case 2. n is odd.

It follows through similar reasoning as Case 1 with the following exception. All distinct modular sequences with odd entries are non-graphic because such sequence prescribes an odd number of odd degrees. It is easy to verify that $\lceil \frac{n+1}{2} \rceil - 1$ such sequences exist. Hence, $|\mathfrak{M}_{od}(K_n)| = t - \lceil \frac{t}{2} \rceil + 1$, $t = \frac{n+1}{2}$, if n is odd. □

Theorem 2.10 leads to a useful corollary.

Corollary 2.11 *For a connected simple k-regular graph G, $n \geq k+1$, it follows that*

$$|\mathfrak{M}_{od}(G)| = \begin{cases} \frac{k+3}{2}, & \text{if } k+1 \text{ is even;} \\ t - \lceil \frac{t}{2} \rceil + 1, t = \frac{k+2}{2}, & \text{if } k+1 \text{ and } n \text{ are odd;} \\ \frac{k+2}{2}, & \text{if } k+1 \text{ is odd and } n \text{ is even} \end{cases}$$

Proof Since the degree sequence s of G has the modular sequences, $s(mod\ k_1)$, either $1 \leq k_1 \leq \frac{k}{2}$, k even, or $1 \leq k_1 \leq \frac{k-1}{2}$, k odd, and $s(mod\ k_2)$, $k+2 \leq k_2 \leq n$ as repetitions, the result is a direct consequence of Theorem 2.10. □

§3. Conclusion

The paper serves as an introduction to the notion of modular sequences of graphs. The scope for further research is evidently, enormous. Problems which could be worthy to research are listed below.

Problem 3.1 *Besides the Havel-Hakimi graphs which other graphs have, $|\mathfrak{M}_{od}(G)| = n$?*

Problem 3.2 *If possible characterize graphs which have, $|\mathfrak{M}_{od}(G)| < n$.*

Problem 3.3 *Can Proposition 2.7 be improved to $|\mathfrak{M}_{od}(G_1 \bigcup G_2)| = nm$?*

The cycle C_4 has distinct modular sequences $(0,0,0,0)$, $(2,2,2,2)$. The complement, $\overline{C_4} = P_2 \bigcup P_2$ cannot be realized by any of the modular sequences. However, the butterfly graph, G has distinct modular sequences, $(0,0,0,0,0)$, $(1,2,2,2,2)$, $(0,2,2,2,2)$, $(4,2,2,2,2)$. Note that $(4,2,2,2,2)(mod\ 4) = (0,2,2,2,2)$ which realizes $\overline{G} = C_4 \bigcup K_1$. See

www.graphclasses.org/smallgraphs.html

for details.

Problem 3.4 *If possible, characterize the graphs other than self-complementary graphs, which has a modular sequence which realization is the complement graph.*

Note that K_3 (or C_3), has exactly one modular graphic sequence i.e $(2,2,2)$ with the realization which is Hamiltonian. For K_n, $n \geq 4$ the $\mathfrak{M}_{od}(K_n)$ has at least two modular sequences of which the realizations are Hamiltonian. The two typical modular sequences are, $(\underbrace{2,2,2,\cdots,2}_{n-entries})$ and $(\underbrace{n-1,n-1,n-1,\ldots,n-1}_{n-entries})$. The modular sequence $(\underbrace{2,2,2,\cdots,2}_{n-entries})$ has an Eulerian realization whilst $(\underbrace{n-1,\cdots,n-1}_{n-entries})$ is an Eulerian realization if and only if n is odd.

Problem 3.5 *Find all Hamiltonian and Eulerian realizations in* $\mathfrak{M}_{od}(K_n)$, $n \geq 4$.

Finding results for graph operations and finding relations between other graph parameters of G and those for graphic modular sequence realizations would be of great interest.

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Algorithm for M modulo N Graceful Labeling of Ladder and Subdivision of Ladder Graphs

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Abstract: The ladder and subdivision of ladder graphs are an essential part of network and circuit theory, and the modulo operation helps to solve such type of problems. For that purpose, we consider the ladder and the exactly 1-subdivision of ladder and also proved both of them are M modulo N graceful labeling. In this paper, we also propose to develop a *C++* algorithm of M modulo N graceful labeling of the given graphs.

Key Words: Ladder, exactly 1-subdivision of ladder, one modulo N graceful labeling, M modulo N graceful labeling, labeling algorithm.

AMS(2010): 05C78, 05C85.

§1. Introduction

A graph G of size q is said to be graceful if an injective assignment of labels from the set $\{ 0, 1, \cdots, q \}$ to the vertices of G such that when each edge of G has been assigned a label defined by the absolute difference of its end-vertices, the resulting edge labels are distinct. A graph G is said to be M modulo N graceful labeling (where N is a positive integer and M = 1 to N) if there is a function f from the vertex set of G to { 0, M, N, N+M, 2N, $\cdots$, N(q-1), N(q-1) + M } in such a way that f is 1-1 and f induces a bijection f^* from edge set of G to { M, N + M, 2N + M,$\cdots$, N(q-1) + M } where $f^*(u,v) = |f(u) - f(v)|$ for all $u, v \in V(G)$. A graph G satisfied M modulo N graceful labeling is known as M modulo N graceful graph.

Rosa (1967) introduced graceful labeling and proved many graphs are graceful. Maryvonne Maheo (1980) introduced strongly graceful labeling and showed that a graph G is strongly graceful also $G + K_2$ is strongly graceful. Kathiresan (1992) proved that Subdivision of ladder S(Ln) graph are graceful. Ramya and Meenakshi (2017) computed graceful labeling , Harmonious labeling, Zumkeller labeling of Ladder graph, banana tree and Firecracker graph. Odd gracefulness was introduced by Gnanajothi (1991). Moussa and Badr (2016) showed that the ladder graph and subdivision of ladder graph Ln with m-pendant is odd graceful. Sekar [2002]

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introduced concept of one modulo three graceful labeling. Ramachandran, and Sekar (2014) introduced one modulo N graceful labeling and showed that supersubdivisions of ladders are graceful. Also they proved that arbitrary supersubdivision of disconnected path and cycle $P_n \bigcup C_r$ and the graph $P_n^+ - v_K^{(1)}$ are one modulo N graceful for every positive integer N. Further they showed that the arbitrary supersubdivisions of paths, disconnected paths, cycles and stars are one modulo N graceful for all positive integers N. Velmurugan and Ramachandran (2019) introduced M modulo N graceful labeling and proved that path and star are M modulo N graceful graph. In this paper, we prove that M modulo N graceful labeling of ladder and the exactly 1-subdivision of ladder graphs with its C++ algorithm.

§2. Main Result

Definition 2.1 *For an integer $n \geq 1$, a ladder L_n is defined by $L_n = P_n \times K_2$, where P_n is a path with n vertices and $\times$ denotes the Cartesian product. Clearly, L_n has $2n$ vertices and $3n-2$ edges.*

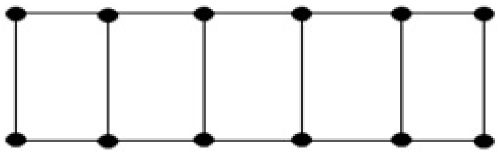

Figure 1 Ladder graph $L_6 = P_6 \times K_2$

Definition 2.2 *Let G be a ladder graph L_n. A graph H is said to be an exactly 1-subdivision of ladder G if H is obtained by subdividing every edge of G exactly once. Such a subdivision H is denoted by $S(G)$.*

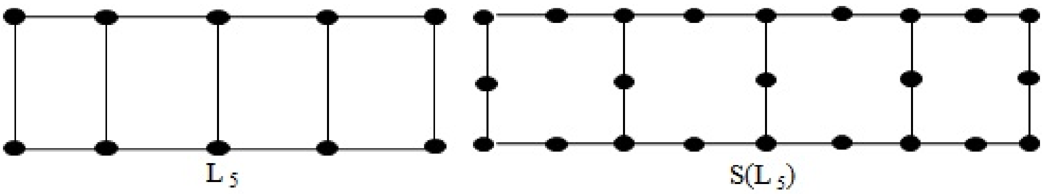

Figure 2 Subdivision of the ladder graph $S(L_5)$

Theorem 2.3 *A ladder graph $L_n = P_n \times K_2$ is* M *modulo N graceful labeling, where N is a positive integer and* M $= 1$ *to N.*

Proof Let the $2n$ vertices of a ladder graph L_n be $\{ u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n \}$ with $3n-2$ edges $\{ e_1, e_2, \cdots, e_n, e_{n+1}, \cdots, e_{3n-2} \}$.

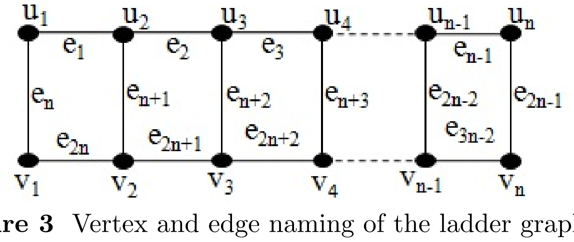

Figure 3 Vertex and edge naming of the ladder graph L_n

Case 1. n is odd.

In this case, an M modulo N graceful labeling of vertices in ladder graph L_n is given by

$f(u_{2i+1}) = [n-1+i]N$, $i = 0$ to $(n-1)/2$;

$f(u_{2(i+1)}) = [2n-3-i]N + \mathrm{M}$, $i = 0$ to $(n-3)/2$;

$f(v_{2i+1}) = [3(n-1)-i]N + \mathrm{M}$, $i = 0$ to $(n-1)/2$;

$f(v_{2(i+1)}) = Ni$, $i = 0$ to $(n-3)/2$

and an M modulo N graceful labeling of edges in ladder graph L_n is deduced as follows:

$f^*(e_{2i+1}) = [n-2-2i]N + \mathrm{M}$, $i = 0$ to $(n-3)/2$;

$f^*(e_{2(i+1)}) = [n-3-2i]N + \mathrm{M}$, $i = 0$ to $(n-3)/2$;

$f^*(e_{2n+2i}) = [3(n-1)-2i]N + \mathrm{M}$, $i = 0$ to $(n-3)/2$;

$f^*(e_{2n+2i+1}) = [3n-2i-4]N + \mathrm{M}$, $i = 0$ to $(n-3)/2$;

$f^*(e_{n+2i}) = [2(n-1-i)]N + \mathrm{M}$, $i = 0$ to $(n-1)/2$;

$f^*(e_{n+2i+1}) = [2n-3-2i]N + \mathrm{M}$, $i = 0$ to $(n-3)/2$.

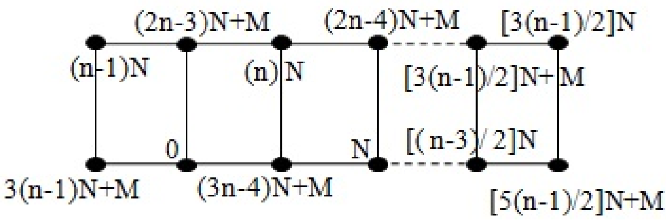

Figure 4 M modulo N graceful labeling of ladder graph L_n with odd n

Case 2. n is even.

In this case, an M modulo N graceful labeling of vertices in ladder graph L_n is given by

$f(u_{2i-1}) = [n+i-2]N$, $i = 1$ to $n/2$;

$f(u_{2i}) = [2n-i-2]N + \mathrm{M}$, $i = 1$ to $n/2$;

$f(v_{2i-1}) = [3n-i-2]N + \mathrm{M}$, $i = 1$ to $n/2$;

$f(v_{2i}) = N(i-1)$, $i = 1$ to $n/2$

and an M modulo N graceful labeling of edges in ladder graph L_n is deduced as follows:

$f^*(e_{2i-1}) = [n-2i]N + \mathrm{M}$, $i = 1$ to $n/2$;

$f^*(e_{2i}) = [n-2i-1]N + \mathrm{M}$, $i = 1$ to $(n-2)/2$;

$f^*(e_{2n+2i-2}) = [3n-2i-1]N + \mathrm{M}$, $i = 1$ to $n/2$;

$f^*(e_{2n+2i-1}) = [3n-2i-2]N + \mathrm{M}$, $i = 1$ to $(n-2)/2$;

$f^*(e_{n+2i-2}) = [2(n-i)]N + \mathrm{M}$, $i = 1$ to $n/2$;

$f^*(e_{n+2i-1}) = [2n-2i-1]N + \mathrm{M}$, $i = 1$ to $n/2$.

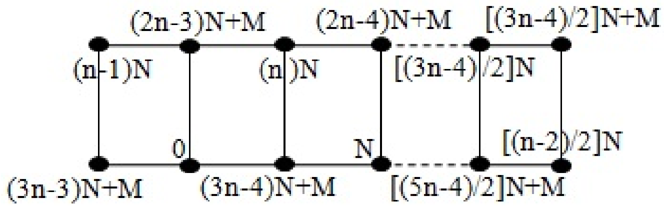

Figure 5 M modulo N graceful labeling of ladder graph L_n with even n

This completes the proof. □

Example 2.4 A 7 modulo 10 graceful labeling of ladder graph L_{11} is shown in Figure 6.

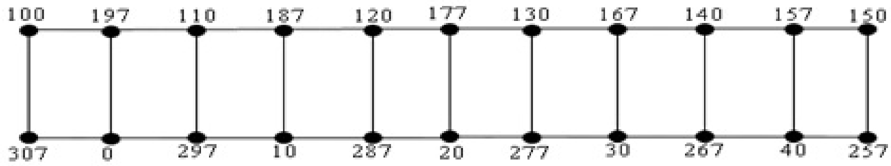

Figure 6

Example 2.5 A 3 modulo 5 graceful labeling of ladder graph L_6 is shown in Figure 7.

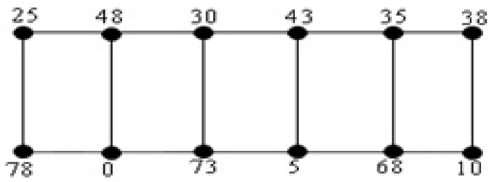

Figure 7

Corollary 2.6 *An algorithm on* M *modulo N graceful labeling of a ladder graph is presented in the following.*

```
♯include<iostream.h>
♯include<conio.h>
void main()
{
clrscr();
int i,j,n,M,N,Y;
cout≪"M modulo N graceful labeling of Ladder graph";
cout≪"Enter the N value N = ";
cin≫N;
cout≪"Enter the n value n = ";
cin≫n;
cout≪"Want to find particular M if yes enter 1 :";
cin≫Y;
```

```
if(Y==1)
{
cout≪”Enter the M value M = ”;
cin≫M;
goto X;
}
for(M=1;M<=N;M++)
{
X:
if(n%2!=0)
{
cout ≪ endl ≪ M ≪ ” modulo ” ≪ N ≪ ” graceful labeling of Vertices:”;
for(i=0;i<=(n-1)/2;i++)
{
cout≪” f(u”≪2*i+1≪”)=”≪(n-1+i)*N;
if(i<=(n-3)/2)
{
cout≪” f(u”≪2*(i+1)≪”)=”≪(2*n-3-i)*N+M;
} }
for(i=0;i<=(n-1)/2;i++)
{
cout≪” f(v”≪2*i+1≪”)=”≪(3*(n-1)-i)*N+M;
if(i<=(n-3)/2)
{
cout≪” f(v”≪2*(i+1)≪”)=”≪N*i;
} }
cout≪endl≪M≪” modulo ”≪N≪” graceful labeling of edges:”;
for(i=0;i<=(n-3)/2;i++)
{
cout≪” f*(e”≪2*i+1≪”)=”≪(n-2-(2*i))*N+M;
cout≪” f*(e”≪2*(i+1)≪”)=”≪(n-3-2*i)*N+M;
}
for(i=0;i<=(n-1)/2;i++)
{
cout≪” f*(e”≪n+2*i≪”)=”≪(2*(n-1-i))*N+M;
if(i<=(n-3)/2)
{
cout≪” f*(e”≪n+2*i+1≪”)=”≪(2*n-2*i-3)*N+M;
} }
for(i=0;i<=(n-3)/2;i++)
{
cout≪” f*(e”≪2*n+2*i≪”)=”≪(3*(n-1)-2*i)*N+M;
```

```
cout≪" f*(e"≪2*n+2*i+1≪")="≪(3*n-2*i-4)*N+M;
} }
else
{
cout≪endl≪M≪" modulo "≪N≪" graceful labeling of vertices:";
for(i=1;i<=(n)/2;i++)
{
cout≪" f(u"≪2*i-1≪")="≪(n-2+i)*N;
cout≪" f(u"≪2*i≪")="≪(2*n-2-i)*N+M;
}
for(i=1;i<=n/2;i++)
{
cout≪" f(v"≪2*i-1≪")="≪(3*n-2-i)*N+M;
cout≪" f(v"≪2*i≪")="≪N*(i-1);
}
cout≪endl≪M≪" modulo "≪N≪" graceful labeling of edges:";
for(i=1;i<=n/2;i++)
{
cout≪" f*(e"≪2*i-1≪")="≪(n-2*i)*N+M;
if(i<=(n-2)/2)
{
cout≪" f*(e"≪2*i≪")="≪(n-1-2*i)*N+M;
} }
for(i=1;i<=n/2;i++)
{
cout≪" f*(e"≪n+2*i-2≪")="≪(2*(n-i))*N+M;
cout≪" f*(e"≪n+2*i-1≪")="≪(2*n-2*i-1)*N+M;
}
for(i=1;i<=n/2;i++)
{
cout≪" f*(e"≪2*n+2*i-2≪")="≪(3*n-1-2*i)*N+M;
if(i<=(n-2)/2)
{
cout≪" f*(e"≪2*n+2*i-1≪")="≪(3*n-2*i-2)*N+M;
} } }
if(Y==1)
{
goto R;
} }
R:
getch();
}
```

Theorem 2.7 *An exactly* 1*-subdivision of ladder* $S(L_n)$ *is* M *modulo* N *graceful labeling for all positive integer* N *and* $\mathrm{M} = 1$ *to* N.

Proof Let $S(L_n)$, where every edge of L_n is subdividing exactly once. Notice that $S(L_n)$ has $5n-2$ vertices and $6n-4$ edges. Let $v_1, v_2, \cdots, v_{2n}$ be the vertices of L_n and $u_1, u_2, \cdots, u_{3n-2}$ be the vertices newly added in each edges of L_n, respectively . We name the vertices and edges as shown in the Figure 8.

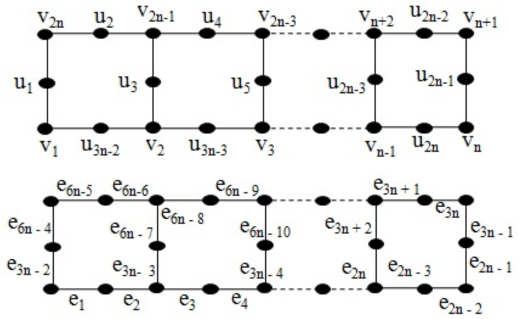

Figure 8 Vertex and Edge naming of the exactly 1-subdivision of ladder $S(L_n)$

Then, an M modulo N graceful labeling of vertices in the exactly 1-subdivision of ladder $S(L_n)$ is given by

$f(u_i) = [i-1]N$, $i = 1$ to $3n-2$;
$f(v_i) = (3n-4+i)N + \mathrm{M}$, $i = 1$ to n;
$f(v_{n+i}) = (5n-5+i)N + \mathrm{M}$, $i = 1$ to n

and an M modulo N graceful labeling of edges in the exactly 1-subdivision of ladder $S(L_n)$ is deduced by

$f^*(e_{2i-1}) = (2i-2)N + \mathrm{M}$, $i = 1$ to $n-1$;
$f^*(e_{2i}) = (2i-1)N + \mathrm{M}$, $i = 1$ to $n-1$;
$f^*(e_{3n-1-i}) = (3n-2-i)N + \mathrm{M}$, $i = 1$ to n;
$f^*(e_{3n-4+3i}) = (3n-5+3i)N + \mathrm{M}$, $i = 1$ to n;
$f^*(e_{3n-3+3i}) = (3n-4+3i)N + \mathrm{M}$, $i = 1$ to $n-1$;
$f^*(e_{3n-2+3i}) = (3n-3+3i)N + \mathrm{M}$, $i = 1$ to $n-1$.

This completes the proof. □

Example 2.8 A 2 modulo 5 graceful labeling on the exactly 1-subdivision of ladder $S(L_9)$ is shown in Figure 9.

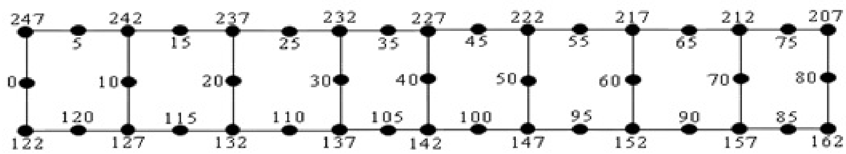

Figure 9

Example 2.9 A 4 modulo 9 graceful labeling on the exactly 1-subdivision of ladder $S(L_4)$ is shown in Figure 10.

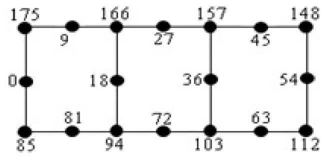

Figure 10

Corollary 2.10 *An algorithm on* M *modulo* N *graceful labeling of the exactly* 1*-subdivision of a ladder graph is presented in the following.*

```
♯include<iostream.h>
♯include<conio.h>
void main()
{
clrscr();
int i,j,n,M,N,Y;
cout≪"M modulo N graceful labeling of the exactly 1-Subdivision of Ladder graph";
cout≪endl≪"Enter the N value: N = ";
cin≫N;
cout≪"Enter the n value: n = ";
cin≫n;
cout≪"Want to find particular M if yes enter 1 :";
cin≫Y;
if(Y==1)
{
cout≪"Enter the M value: M = ";
cin≫M;
goto X;
}
for(M=1;M<=N;M++)
{
X:
cout≪M≪" modulo "≪N≪" graceful labeling of vertices on the exactly 1-Subdivision of Ladder graph:";
cout≪endl;
for(i=1;i<=((3*n)-2);i++)
{
cout≪" f(u"≪i≪")="≪(i-1)*N;
}
for(i=1;i<=n;i++)
{
cout≪" f(v"≪i≪")="≪(3*n-4+i)*N+M;
```

```
}
for(i=1;i<=n;i++)
{
cout≪" f(v"≪n+i≪")="≪(5*n-5+i)*N+M;
}
cout≪endl≪M≪" modulo "≪N≪" graceful labeling of edges on the exactly 1-Subdivision
of Ladder graph:";
cout≪endl;
for(i=1;i<=n-1;i++)
{
cout≪" f*(e"≪(2*i)-1≪")="≪((2*i)-2)*N+M;
cout≪" f*(e"≪(2*i)≪")="≪((2*i)-1)*N+M;
}
for(i=1;i<=n;i++)
{
cout≪" f*(e"≪(3*n)-i-1≪")="≪((3*n)-i-2)*N+M;
}
for(i=1;i<=n;i++)
{
cout≪" f*(e"≪(3*n)-4+3*i≪")="≪((3*n)+(3*i)-5)*N+M;
if(i<=(n-1))
{
cout≪" f*(e"≪(3*n)-3+3*i≪")="≪((3*n)+3*i-4)*N+M;
cout≪" f*(e"≪(3*n)-2+3*i≪")="≪((3*n)+3*i-3)*N+M;
} }
if(Y==1)
{
goto R;
} }
R:
getch();
}
```

§3. Conclusion

We conclude that ladder graph and the exactly 1-subdivision of ladder graph are M modulo N graceful labeling. The given algorithms designed for automatically assign M modulo N graceful labeling on ladder graph and the exactly 1-subdivision of ladder graph.

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On the p-Groups of the Algebraic Structure of $\mathbb{D}_{2^n} \times \mathbb{C}_8$

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Abstract: In this paper, the explicit formulae is given for the number of distinct fuzzy subgroups of the cartesian product of the dihedral group of order 2^n with a cyclic group of order eight, where $n > 3$.

Key Words: Finite p-groups, nilpotent group, fuzzy subgroups, dihedral Group, inclusion-exclusion principle, maximal subgroups.

AMS(2010): 05C78, 05C85.

§1. Introduction

This paper is a follow up from[1]. In this work the distinct number of fuzzy subgroups for the Nilpotent p-Group of $\mathbb{D}_{2^n} \times \mathbb{Z}_8$ is found.

§2. Methodology

The method that will be used in counting the chains of fuzzy subgroups of an arbitrary finite p-group G is described. Suppose that $M_1, M_2, \cdots, M_t$ are the maximal subgroups of G, and denote by $h(G)$ the number of chains of subgroups of G which ends in G. By simply applying the technique of computing $h(G)$, using the application of the Inclusion-Exclusion Principle, we have that:

$$h(G) = 2\left(\sum_{r=1}^{t} h(M_r) - \sum_{1\leq r_1<r_2\leq t} h(M_{r_1}\cap M_{r_2}) + \cdots + (-1)^{t-1} h\left(\bigcap_{r=1}^{t} M_r\right)\right) \tag{1.1}$$

In [2], (1.1) was used to obtain the explicit formulas for some positive integers n.

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Theorem A(Marius) *The number of distinct fuzzy subgroups of a finite p-group of order* p^n *which have a cyclic maximal subgroup is*

(*i*) $h(\mathbb{Z}_{p^n})=2^n$;

(*ii*) $h(\mathbb{Z}_p\times\mathbb{Z}_{p^{n-1}})=h(M_{p^n})=2^{n-1}[2+(n-1)p]$.

§3. The Number of Fuzzy Subgroups for $\mathbb{Z}_8\times\mathbb{Z}_8$

Lemma 3.1 *Let G be Abelian such that* $G=\mathbb{Z}_4\times\mathbb{Z}_4$. *Then,* $h(G)=2h(\mathbb{Z}_2\times\mathbb{Z}_{2^2})=48$.

Proof By the use of GAP (Group Algorithms and Programming), G has three maximal subgroups in which each of them is isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_{2^2}$. Hence, we have that

$$\begin{aligned}\frac{1}{2}h(G) &= 3h(\mathbb{Z}_2\times\mathbb{Z}_{2^2})-3h(\mathbb{Z}_2\times\mathbb{Z}_{2^2})+h(\mathbb{Z}_2\times\mathbb{Z}_{2^2})\\ &= h(\mathbb{Z}_2\times\mathbb{Z}_4).\end{aligned}$$

And by Theorem A, $h(\mathbb{Z}_2\times\mathbb{Z}_{2^2})=24$, which implies that $h(\mathbb{Z}_4\times\mathbb{Z}_4)=48$. □

Corrolary 3.2 *Following Lemma* 3.1, $h(\mathbb{Z}_4\times\mathbb{Z}_{2^5}), h(\mathbb{Z}_4\times\mathbb{Z}_{2^6}), h(\mathbb{Z}_4\times\mathbb{Z}_{2^7})$ *and* $h(\mathbb{Z}_4\times\mathbb{Z}_{2^8})=$ 1536*,* 4096*,* 10496 *and* 26112*, respectively.*

Theorem 3.3 *Let* $G=\mathbb{Z}_{2^n}\times\mathbb{Z}_8$*, then* $h(G)=\frac{1}{3}(2^{n+1})(n^3+12n^2+17n-24)$.

Proof The three maximal subgroups of G have the following properties :

One is isomorphic to $\mathbb{Z}_8\times\mathbb{Z}_{2^{n-1}}$, while two are isomorphic to $\mathbb{Z}_4\times\mathbb{Z}_{2^n}$. We have

$$\begin{aligned}\frac{1}{2}h(G) &= 2h(\mathbb{Z}_4\times\mathbb{Z}_{2^n})+h(\mathbb{Z}_8\times\mathbb{Z}_{2^{n-1}})-3h(\mathbb{Z}_4\times\mathbb{Z}_{2^{n-1}})+h(\mathbb{Z}_4\times\mathbb{Z}_{2^{n-1}})\\ &= 2h(\mathbb{Z}_4\times\mathbb{Z}_{2^n})+h(\mathbb{Z}_8\times\mathbb{Z}_{2^{n-1}})-2h(\mathbb{Z}_4\times\mathbb{Z}_{2^{n-1}})\\ &= h(\mathbb{Z}_8\times\mathbb{Z}_{2^{n-1}})+2h(\mathbb{Z}_4\times\mathbb{Z}_{2^n})-h(\mathbb{Z}_4\times\mathbb{Z}_{2^{n-1}}).\end{aligned}$$

Hence,

$$\begin{aligned}h(G) &= 4h(\mathbb{Z}_4\times\mathbb{Z}_{2^n})-4h(\mathbb{Z}_4\times\mathbb{Z}_{2^{n-1}})+2h(\mathbb{Z}_8\times\mathbb{Z}_{2^{n-1}})\\ &= 4h(\mathbb{Z}_4\times\mathbb{Z}_{2^n})+4h(\mathbb{Z}_4\times\mathbb{Z}_{2^{n-1}})+8h(\mathbb{Z}_4\times\mathbb{Z}_{2^{n-2}})-16h(\mathbb{Z}_4\times\mathbb{Z}_{2^{n-3}})\\ &\quad +32h(\mathbb{Z}_4\times\mathbb{Z}_{2^{n-3}})-32h(\mathbb{Z}_4\times\mathbb{Z}_{2^{n-4}})+16h(\mathbb{Z}_8\times\mathbb{Z}_{2^{n-4}})\\ &= 4h(\mathbb{Z}_4\times\mathbb{Z}_{2^n})+4h(\mathbb{Z}_4\times\mathbb{Z}_{2^{n-1}})+8h(\mathbb{Z}_4\times\mathbb{Z}_{2^{n-2}})+16h(\mathbb{Z}_4\times\mathbb{Z}_{2^{n-3}})\\ &\quad +32h(\mathbb{Z}_4\times\mathbb{Z}_{2^{n-4}})-64h(\mathbb{Z}_4\times\mathbb{Z}_{2^{n-5}})+32h(\mathbb{Z}_8\times\mathbb{Z}_{2^{n-5}})\\ &\quad +\cdots-2^{j+1}h(\mathbb{Z}_4\times\mathbb{Z}_{2^{n-j}})+2^jh(\mathbb{Z}_8\times\mathbb{Z}_{2^{n-j}})\ (\text{for } n-j=3)\end{aligned}$$

$$
\begin{aligned}
&= 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + 2^{n-3}h(\mathbb{Z}_8 \times \mathbb{Z}_{2^3}) - 2^{n-1}h(\mathbb{Z}_4 \times \mathbb{Z}_{2^3}) + \sum_{k=1}^{n-3}[2^{k+1}h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-k}}) \\
&= 2^{n+2}[n^2 + 5n + 3] + \sum_{k=1}^{n-3} h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-k}}) \\
&= 2^{n+2}((n^2 + 5n + 3) + \frac{1}{6}(n-3)(n^2 + 9n + 14)) \\
&= \frac{1}{3}(2^{n+1})(n^3 + 12n^2 + 17n - 24)
\end{aligned}
$$

if $n > 2$. This completes the proof. □

Theorem 3.4 *Suppose that $G = \mathbb{D}_{2^3} \times \mathbb{C}_8$. Then, $h(G) = 5376$.*

Proof A calculation shows that

$$
\begin{aligned}
\frac{1}{2}h(G) \quad = \quad & h(\mathbb{D}_{2^3} \times \mathbb{Z}_4) + 2h(\mathbb{Z}_{2^3} \times \mathbb{Z}_2 \times \mathbb{Z}_2) - 4h(\mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2) \\
& + h(\mathbb{Z}_8 \times \mathbb{Z}_4) - 6h(\mathbb{Z}_8 \times \mathbb{Z}_2) - 2h(\mathbb{Z}_4 \times \mathbb{Z}_4) + 8h(\mathbb{Z}_4 \times \mathbb{Z}_2) \\
& + h(\mathbb{Z}_{2^3}) = 2688,
\end{aligned}
$$

which implies that $h(G) = 2 \times 2688 = 5376$. This completes the proof. □

Theorem 3.5 *Let $G = \mathbb{D}_{2^5} \times \mathbb{Z}_8$. Then, $h(G) = 111136$.*

Proof A calculation shows that

$$
\begin{aligned}
\frac{1}{2}h(G) \quad = \quad & h(\mathbb{D}_{2^5} \times \mathbb{Z}_{2^2}) + 2h(\mathbb{D}_{2^4} \times \mathbb{Z}_{2^3}) - 4h(\mathbb{D}_{2^4} \times \mathbb{Z}_{2^2}) + h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^3}) \\
& - 2h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^2}) - 2h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^3}) + 8h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^2}) + h(\mathbb{Z}_{2^4}) \\
& - 4h(\mathbb{Z}_{2^3}) = 55568,
\end{aligned}
$$

which implies that $h(G) = 2 \times 55568 = 111136$. □

Theorem 3.6 *Suppose that $G = \mathbb{D}_{2^6} \times \mathbb{Z}_8$. Then, $h(G) = 492864$.*

Proof A calculation shows that

$$
\begin{aligned}
\frac{1}{2}h(G) \quad = \quad & h(\mathbb{D}_{2^6} \times \mathbb{Z}_4) + 2h(\mathbb{D}_{2^5} \times \mathbb{Z}_{2^3}) - 4h(\mathbb{D}_{2^5} \times \mathbb{Z}_4) + h(\mathbb{Z}_{2^5} \times \mathbb{Z}_{2^3}) \\
& - 2h(\mathbb{Z}_{2^5} \times \mathbb{Z}_{2^2}) - 2h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^3}) + 8h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^2}) + h(\mathbb{Z}_{2^5}) - 4h(\mathbb{Z}_{2^4}) = 246432,
\end{aligned}
$$

which implies that $h(G) = 2 \times 246432 = 492864$. □

Theorem 3.7 *Let $G = \mathbb{D}_{2^n} \times \mathbb{C}_2$, the nilpotent group formed by the cartesian product of the dihedral group of order 2^n and a cyclic group of order 2. Then, the number of distinct fuzzy subgroups of G is given by $h(G) = 2^{2n}(2n+1) - 2^{n+1}, n > 3$.*

§4. The Number of Fuzzy Subgroups for $\mathbb{D}_{2^n} \times \mathbb{C}_8$

Theorem 4.1 *Suppose that $G = \mathbb{D}_{2^n} \times \mathbb{C}_8$. Then, the number of distinct fuzzy subgroups of G is given by*

$$\begin{aligned} 2^{2(n-1)}(6n+113) \quad &+ \quad 2^n \left[13 - 6n - 2n^2 + 3\sum_{j=1}^{n-3} 2^{(j-1j)}(2n+1-2j) \right] \\ &+ \quad \frac{1}{3} 2^{n+2} \Big[(n-1)^3 + (n-2)^3 + 24n^2 - 38n - 30 \\ &+ \quad \sum_{k=1}^{n-5} 2^k \left[(n-2-k)^3 + 12(n-2-k)^2 + 17(n-k) - 58 \right] \Big] \end{aligned}$$

Proof A calculation shows that

$$\begin{aligned} h(\mathbb{D}_{2^n} \times \mathbb{C}_8) \quad &= \quad 2h(\mathbb{Z}_{2^{n-1}}) + 2h(\mathbb{D}_{2^n} \times Z_4) + 2h(\mathbb{D}_{2^{n-1}} \times \mathbb{C}_8) \\ &\quad +4h(Z_{2^{n-2}} \times \mathbb{C}_8) + 2^4 h(Z_{2^{n-3}} \times \mathbb{C}_8) + 2^6 h(\mathbb{Z}_{2^{n-4}} \times \mathbb{C}_8) - 2^8 h(\mathbb{Z}_{2^{n-5}} \times \mathbb{Z}_{2^3}) \\ &\quad -4h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) + 2^{10} h(\mathbb{Z}_{2^{n-5}}) \times \mathbb{Z}_{2^2} - 2^9 h(\mathbb{Z}_{2^{n-5}}) - 2^9 h(\mathbb{D}_{2^{n-4}} \times \mathbb{C}_{2^2}) \\ &\quad +2^8 h(\mathbb{D}_{2^{n-4}} \times \mathbb{C}_{2^3}) \\ &= \quad 2^n + 2h(\mathbb{D}_{2^n} \times \mathbb{C}_4) + 2h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^3}) + 2^2 h(\mathbb{Z}_{2^{n-2}} \times \mathbb{Z}_{2^3}) \\ &\quad -2^{2(n-3)} h(\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^3}) + 2^{2(n-2)} h(\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} - 2^2 h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) \\ &\quad -2^{2n-5} h(\mathbb{Z}_{2^2}) - 2^{2n-5} h(\mathbb{D}_{2^3} \times \mathbb{Z}_{2^2}) + 2^{2(n-3)} h(\mathbb{D}_{2^3} \times \mathbb{Z}_{2^3}) \\ &\quad +3\sum_{i=1}^{n-5} 2^{2ij} h(\mathbb{Z}_{2^{n-2-i}} \times \mathbb{Z}_{2^3}) \end{aligned}$$

as required. □

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Famous Words

Generally, we understand a matter by the composite of elementary particles with a thinking patter following

$$\text{Matter} \stackrel{Decompose}{\longrightarrow} \text{Microcosmic Particles} \stackrel{Abstract}{\longrightarrow} \text{Complex Network}$$

where the complex network is an inherit structure of the matter on microcosmic particles and different subjects discuss microcosmic behaviors of particles. Until today, we lack of effective methods, even lack of such a mathematics on complex network or complex system which can not enables us hold on the whole matter T in theory unless all its microcosmic particle are in synchronization. In this case, we can hardly conclude that a scientific conclusion is true in the whole universe because it is understanding only by humans ourself, an intelligent creature happily born on the earth. (Extracted from the paper: Science's dilemma – a review on science with applications, *Progress in Physics*, Vol.15, 3(2019), 78-85)

By Dr.Linfan MAO, a Chinese mathematician, philosophical critic.

Author Information

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Abstract: Authors are requested to provide an abstract of not more than 250 words, latest Mathematics Subject Classification of the American Mathematical Society, Keywords and phrases. Statements of Lemmas, Propositions and Theorems should be set in italics and references should be arranged in alphabetical order by the surname of the first author in the following style:

Books

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Research papers

[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.

[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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