International J.Math. Combin. Vol.1(2021), 1-10

PBIB-Designs and Association Schemes from Minimum Neighborhood Sets of Certain Jump Sizes of Circulant Graphs

B. Chaluvaraju, Vidya T. and Shaikh Ameer Basha

(Department of Mathematics, Bangalore University, Jnana Bharathi Campus, Bengaluru -560 056, India)

E-mail: bchaluvaraju@gmail.com, vidyaraghut@gmail.com, shaikhameerbasha@gmail.com

Abstract: A set $S \subseteq V$ of a graph $G = (V, E)$ is a Neighborhood set of G if $G = \bigcup_{v \in S} \langle N(v) \rangle$, where $\langle N(v) \rangle$ is the subgraph induced by v and all vertices adjacent to v. The neighborhood number, $\eta(G)$ is the minimum cardinality of a neighborhood set of G. The minimum neighborhood set S with $|S| = \eta(G)$ is called η-set. Generally, the partially balanced incomplete block(PBIB)-Designs are obtained from the family of strongly regular graphs. Surprisingly, in this paper we obtain the PBIB-Designs and m-association schemes for $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$ arising from η -sets of certain jump sizes of circulant graphs.

Key Words: Association schemes, PBIB-designs, neighborhood sets, circulant graph.

AMS(2010): 05B05, 05C51, 05C69, 05E30, 51E05.

§1. Introduction

Let $G = (V, E)$ be a finite and undirected graph with no loops and multiple edges of vertex set V and edge set E. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph G, respectively. For graph-theoretical terminologies which are not defined here, we follow [15].

For a given positive integer p, let s_1, s_2, $\cdots$, s_t be a sequence of integers with $0 < s_1 < s_2 < \cdots < s_t < \frac{p+1}{2}$. The Circulant graph $C_p(s_1, s_2, \cdots, s_t)$ is the graph on p vertices labeled as v_1, v_2, $\cdots$, v_p with vertex v_i adjacent to each vertex $v_{(i \pm s_j)(mod\ p)}$ and the values s_j; $1 \leq j \leq t$ are called jump sizes.

The applications are mainly in pure mathematics and technology which mysteriously reflects the abstract concrete dichotomy of the theory of Circulant. Also, which are important in digital encoding; this is a wondrous technology it enables devices ranging from computers to music players to recover from errors in transmission and storage of data and restore the original data. For more details, we refer to [17].

Bose and Nair [3] introduced a class of binary, equi-replicate and proper designs, which are called partially balanced incomplete block (PBIB)-Designs. In these designs, all the elementary contrasts are not estimated with the same variance. The variances depend on the type of association between the treatments. There are many applications of PBIB-Designs in cluster

[1]Received January 13, 2021, Accepted March 2, 2021.

sampling, digital fingerprint codes and architecture of web solution. For more details on PBIB-Designs one can refer to [1], [6], [11], [12] and [13].

Given ν elements (objects or vertices), a relation satisfying the following conditions is said to be an association scheme with m classes:

(i) Any two elements are either first associates, or second associates, $\cdots$, or m^{th} associates, the relation of association being symmetric.

(ii) Each object x has n_k k^{th} associates, the number n_k being independent of x.

(iii) If two objects x and y are k^{th} associates, then the number of objects which are i^{th} associates of x and j^{th} associates of y is p^k_{ij} and is independent of the k^{th} associates x and y. Also $p^k_{ij} = p^k_{ji}$.

With the association scheme on ν objects, a PBIB-Design is an arrangement of ν objects into b sets (blocks) of size g where $g < \nu$ such that

(i) Every element is contained in exactly r blocks.

(ii) Each block contains g distinct elements.

(iii) Any two elements which are m^{th} associates occur together in exactly λ_m blocks.

The numbers ν, b, g, r, λ_1, λ_2, $\cdots$, λ_m are called the parameters of the first kind, whereas the numbers n_1, n_2, $\cdots$, n_m, p^k_{ij} $(i,\, j,\, k = 1,\, 2,\, \cdots,\, m)$ are called the parameters of the second kind.

Bose [2] has initiated the study of strongly regular graph with parameters (p, l, σ, μ) of a finite simple graph on p vertices, regular of degree l (with $0 < l < p - 1$, so that there are both edges and nonedges), such that any two distinct vertices have σ common neighbors when they are adjacent, and μ common neighbors when they are nonadjacent. For more details on strongly regular graph and its related concepts, we refer to [4] and [14].

A set $S \subseteq V$ of a graph is a neighborhood set of G if $G = \bigcup_{v\in S}\langle N(v)\rangle$, where $\langle N(v)\rangle$ is the subgraph induced by v and all vertices adjacent to v. The neighborhood number $\eta(G)$ is the minimum cardinality of a neighborhood set of G. The minimum neighborhood set of S with $|S| = \eta(G)$ is called η-set. This concept was first introduced by Sampathkumar and Neeralagi [16]. For more details on neighborhood number and its related parameters, we refer to [7].

Slater [18] has introduced the concept of the number of dominating sets of G, which he denoted by $HED(G)$ in honor of Steve Hedetniemi. In this article, we will use $\tau_\eta(G)$ to denote the total number of η-set of a graph G. PBIB-Design associated with graph theoretic parameters are studied by [8], [9], [10] and [19].

§2. $\eta(G)$ and $\tau_\eta(G)$ for Different Jump Sizes of Circulant Graphs

2.1 Circulant Graph $C_p(1)$

The circulant graph with jump size one is also known as cycle C_p for $p \geq 3$, that is, $C_p(1) \cong C_p$ with $p \geq 3$.

Remark 2.1 The circulant graphs $C_4(1)$ and $C_5(1)$ are the strongly regular graphs.

Theorem 2.1 *For any circulant graph $G_1 = C_p(1)$ with $p \geq 4$ vertices*

(*i*) $\eta(G_1) = \left\lceil \dfrac{p}{2} \right\rceil$;

(*ii*) $\tau_\eta(G_1) = \begin{cases} 2 & \text{if } p \text{ is even}, \\ p & \text{if } p \text{ is odd.} \end{cases}$

Proof (*i*) The proof is due to [16].

(*ii*) Let $G_1 = C_p(1)$ be a circulant graph with $p \geq 4$ vertices labeled as $v_1, v_2, \cdots, v_p$. We have two cases for discussion:

Case 1. If p is even then by (i), we have $\eta(G_1) = \dfrac{p}{2}$ with two disjoint η-sets of G_1 as $\{v_1, v_3, v_5, \cdots, v_{(p-1)}\}$ and $\{v_2, v_4, v_6, \cdots, v_p\}$. Hence $\tau_\eta(G_1) = 2$.

Case 2. If p is odd then by (i), we have $\eta(G_1) = \left\lceil \dfrac{p}{2} \right\rceil$ with p number of η-sets of G_1 as $\{v_i, v_{(i+1)(mod\ p)}, v_{(i+3)(mod\ p)}, \cdots, v_{(i+\frac{p-1}{2})(mod\ p)}\}$; $1 \leq i \leq p$. Hence $\tau_\eta(G_1) = p$. □

2.2 Circulant Graph $C_p(\lfloor \frac{p}{2} \rfloor)$

The circulant graph $C_p(\lfloor \frac{p}{2} \rfloor)$, $p \geq 3$ is a circulant graph with jump size $\lfloor \frac{p}{2} \rfloor$.

Remark 2.2 The following results hold by definition of circulant graph:

(*i*) $C_p(\lfloor \frac{p}{2} \rfloor) \cong C_p(1); p = 2n-1, n \geq 2$;

(*ii*) The circulant graph $C_p(\lfloor \frac{p}{2} \rfloor)$ with $p = 2n$, $n \geq 1$ vertices contain n times of K_2's and they are disconnected, which are not strongly regular.

Theorem 2.2 *For any circulant graph $G_2 = C_p(\lfloor \frac{p}{2} \rfloor)$ with $p \geq 4$ vertices,*

(*i*) $\eta(G_2) = \left\lceil \dfrac{p}{2} \right\rceil$;

(*ii*) $\tau_\eta(G_2) = \begin{cases} 2^p & \text{if } n \text{ is even} \\ p & \text{if } p \text{ is odd.} \end{cases}$

Proof (*i*) Let $G_2 = C_p(\lfloor \frac{p}{2} \rfloor)$ be any circulant graph with $p \geq 4$ vertices labeled as $v_1, v_2, \cdots, v_p$. We have the following two cases:

Case 1. If $p = 2n; n \geq 2$, then $C_p(\lfloor \frac{p}{2} \rfloor)$ is bipartite graph with n number of $K_2's$. Hence

$$\eta(C_p(\lfloor \frac{p}{2} \rfloor)) = \frac{p}{2}.$$

Case 2. If $p = 2n-1; n \geq 2$, then by Theorem 2.1(i), neighborhood number of $C_p(1)$ is $\lceil \frac{p}{2} \rceil$ and by Remark 2.2(*i*), $C_p(\lfloor \frac{p}{2} \rfloor) \cong C_p(1)$;
$p = 2n-1, n \geq 2$. Therefore, we have

$$\eta(C_p(\lfloor \frac{p}{2} \rfloor)) = \lceil \frac{p}{2} \rceil, \ p \geq 4.$$

(*ii*) For the values of $\tau_\eta(G_2)$, we have

Case 1. If $p = 2n; n \geq 2$ then by (i), we have $\eta(C_p(\lfloor \frac{p}{2} \rfloor)) = \dfrac{p}{2}$ with 2^p disjoint η-sets. Hence

$$\tau_\eta(G_1) = 2^p.$$

Case 2. If $p = 2n + 1; n \geq 2$, then the proof follows from Theorem 2.1(ii). □

2.3 Circulant Graph with Odd Jump Sizes

The circulant graph $C_p(1, 3, \cdots, \lfloor \frac{p}{2} \rfloor)$, $p \geq 6$ is a circulant graph with odd jump sizes.

Remark 2.3 If the sequence is of an odd jump size from 1 to $\lfloor \frac{p}{2} \rfloor$, then $C_p(1, 3, \cdots, \lfloor \frac{p}{2} \rfloor)$ is strongly regular graph.

Theorem 2.3 *For any circulant graph* $G_3 = C_p(1, 3, \cdots, \lfloor \frac{p}{2} \rfloor)$ *with* $p = 4n-2$ *or* $4n-1$, $n \geq 2$,

(*i*) $\eta(G_3) = \lfloor \frac{p}{2} \rfloor$;

(*ii*) $\tau_\eta(G_3) = \begin{cases} 2 & \text{if } p = 4n-2;\ n \geq 2 \\ 2p & \text{if } p = 4n-1;\ n \geq 2. \end{cases}$

Proof (i) Let $G_3 = C_p(1, 3, \cdots, \lfloor \frac{p}{2} \rfloor)$ be a circulant graph with vertices labeled as $v_1, v_2, \cdots, v_p$, where $p = 4n - 2$ or $4n - 1$, $n \geq 2$. We have

Case 1. If $p = 4n - 2; n \geq 2$, then $C_p(1, 3, 5, \cdots, \lfloor \frac{p}{2} \rfloor) \cong K_{\frac{p}{2}, \frac{p}{2}}$. Hence

$$\eta(C_p(1, 3, 5, \cdots, \left\lfloor \frac{p}{2} \right\rfloor)) = \frac{p}{2}.$$

Case 2. If $p = 4n - 1, n \geq 2$, then there are

$$S = \{v_i, v_{(i+1)(mod\ p)}, v_{(i+2)(mod\ p)} \cdots, v_{(i+\lfloor \frac{p}{2} \rfloor - 1)(mod\ p)}\}$$

and

$$S = \{v_i, v_{(i+2)(mod\ p)}, v_{(i+4)(mod\ p)} \cdots, v_{(i+2\lfloor \frac{p}{2} \rfloor - 2)(mod\ p)}\},$$

where $1 \leq i \leq p$ are the minimum neighborhood sets, containing $\lfloor \frac{p}{2} \rfloor$ elements. Therefore,

$$\eta(C_p(1, 3, 5, \cdots, \left\lfloor \frac{p}{2} \right\rfloor)) = \left\lfloor \frac{p}{2} \right\rfloor.$$

(ii) For the values of $\tau_\eta(G_3)$, we have

Case 1. If $p = 4n - 2; n \geq 2$, then by (i), we have $\eta(G_3) = \dfrac{p}{2}$ with two disjoint η-sets of G_3 as $\{v_1, v_3, v_5, \cdots, v_{(p-1)}\}$ and $\{v_2, v_4, v_6, \cdots, v_p\}$. Hence,

$$\tau_\eta(G_3) = 2.$$

Case 2. If $p = 4n - 1, n \geq 2$ then by (i), we have $\eta(G_3) = \left\lceil \dfrac{p}{2} \right\rceil$ with $2p$ number of η-sets. Hence

$$\tau_\eta(G_3) = 2p.$$ □

2.4 Circulant Graph with Even Jump Sizes

The circulant graph $C_p(2,4,\cdots,\lfloor\frac{p}{2}\rfloor)$; $p\geq 6$ is a circulant graph with even jump sizes.

Remark 2.4 The circulant graphs $C_5(2)$, $C_6(2)$, $C_8(2,4)$, $C_{10}(2,4)$, $C_{12}(2,4,6)$ are few examples of strongly regular graphs.

Theorem 2.4 *For any circulant graph $G_4=C_p(2,4,\cdots,\lfloor\frac{p}{2}\rfloor)$ with $p=4n$ or $4n+1$, $n\geq 2$,*

$$(i)\quad \eta(G_4)=\begin{cases}2 & \text{if } p=4n;\ n\geq 2\\ 4 & \text{if } p=4n+1;\ n\geq 2.\end{cases}$$

$$(ii)\quad \tau_\eta(G_4)=\begin{cases}(\frac{p}{2})^2 & \text{if } p=4n;\ n\geq 2\\ 4p & \text{if } p=4n+1;\ n\geq 2.\end{cases}.$$

Proof (i) Let $G_4=C_p(2,4,\cdots,\lfloor\frac{p}{2}\rfloor)$ be a circulant graph with $p=4n$ or $4n+1$, $n\geq 2$. We have the following two cases for discussion:

Case 1. If $p=4n$; $n\geq 2$, the greatest common divisor of $2,4,6\cdots,\lfloor\frac{p}{2}\rfloor=2\neq 1$. Hence G_4 has two disconnected blocks $B_1=\{v_1,v_3,\cdots,v_{p-1}\}$ and $B_2=\{v_2,v_4,\cdots,v_p\}$ for $1\leq i\leq p$ and each block is complete graph $K_{p/2}$. This implies, $\eta(G_4)=2$.

Case 2. If $p=4n+1$, $n\geq 2$, then the result follows from Theorem 2.3(i).

(ii) For the values of $\tau_\eta(G_4)$, we have the following two cases:

Case 1. If $p=4n$, $n\geq 2$, then by (i), there exists η-sets $\{v_i,v_j\}$ of G_4, such that $v_i\in B_1$ and $v_j\in B_2$. Thus, it follows that

$$\tau_\eta(G_4)=(\frac{p}{2})^2.$$

Case 2. If $p=4n+1$, $n\geq 2$, then the result is similar to Theorem 2.1(ii). □

2.5 Circulant Graph $C_p(1,2,\cdots,\lfloor\frac{p}{2}\rfloor)$

The circulant graph with $p\geq 3$ vertices and having jump size $1,2,\cdots,\lfloor\frac{p}{2}\rfloor$ is known as the complete graph K_p, that is,

$$C_p\left(1,2,\cdots,\left\lfloor\frac{p}{2}\right\rfloor\right)\cong K_p.$$

Remark 2.5 The complete graph K_p is strongly regular for all $p\geq 3$. The status of the trivial singleton graph K_1 is unclear. Opinions differ on if K_2 is a strongly regular graph, since it has no well-defined μ parameter, it is preferable to consider as not to be a strongly regular.

Theorem 2.5 *For any circulant graph $G_5=C_p(1,2,\cdots,\lfloor\frac{p}{2}\rfloor)$ with $p\geq 3$ vertices,*

(i) $\eta(G_5)=1$;
(ii) $\tau_\eta(G_5)=p$.

Proof Let $G_5=C_p(1,2,\cdots,\lfloor\frac{p}{2}\rfloor)$ be a circulant graph with $p\geq 3$ vertices. Then,

(i) The proof is due to [16].

(ii) By Theorem 2.1(i), we have $\eta(G_5) = 1$ and η-sets of G_5 are $\{v_i\}; 1 \leq i \leq p$. Thus

$$\tau_\eta(G_5) = p. \qquad \square$$

§3. Matrix Representation of Circulant Graphs via Association Schemes

The matrix representations of certain classes of circulant graphs are shown as in the following table:

<table>
<tr><th colspan="2">Circulant Graph ($n \geq 2$)</th><th>Relations for Association Scheme</th><th>Matrix Respresentation.</th></tr>
<tr><td rowspan="2">G_1</td><td>$p = 2n$</td><td rowspan="2">Two distinct vertices are said to be first associates, if their jump size is one as well as adjacent and k^{th} associates where ($2 \leq k \leq \lfloor \frac{p}{2} \rfloor$), if their jump size is k as well as non adjacent.</td><td>Type 1</td></tr>
<tr><td>$p = 2n + 1$</td><td>Type 2</td></tr>
<tr><td rowspan="2">G_2</td><td>$p = 2n$</td><td rowspan="2">Two distinct vertices are said to be k^{th} associates, where ($1 \leq k \leq \lfloor \frac{p-2}{2} \rfloor$), if their jump size is k as well as non adjacent and are $\lfloor \frac{p}{2} \rfloor^{th}$ associates if their jump size $\lfloor \frac{p}{2} \rfloor$ as well as adjacent.</td><td>Type 1</td></tr>
<tr><td>$p = 2n + 1$</td><td>Type 2</td></tr>
<tr><td rowspan="2">G_3</td><td>$p = 4n - 2$</td><td rowspan="2">Two distinct vertices are said to be odd associates, if their jump size are odd as well as adjacent and even associates if their jump size are even as well as non adjacent</td><td>Type 1</td></tr>
<tr><td>$p = 4n - 1$</td><td>Type 2</td></tr>
<tr><td rowspan="2">G_4</td><td>$p = 4n$</td><td rowspan="2">Two distinct vertices are said to be even associa-tes, if their jump size is even as well as adjacent and are odd associates, if their jump size are odd as well as non-adjacent</td><td>Type 1</td></tr>
<tr><td>$p = 4n + 1$</td><td>Type 2</td></tr>
<tr><td colspan="2">G_5</td><td>Two distinct vertices v_i and v_j of V are k^{th} associates, $1 \leq k \leq \lfloor \frac{p}{2} \rfloor$, if $|x - y| = k$ and are adjacent.</td><td>Type 1
Type 2</td></tr>
</table>

Table 1. Relation defining association schemes with matrix representation.

In continuation of the relation from the above table, the following types of tables can be constructed for the association schemes and they are:

Type 1. The matrix representation of circulant graph for $p(\geq 2)$ even, with an association scheme is as follows:

Elements	**Association scheme**					
	First	Second	$\cdots$	k	$\cdots$	$\frac{p}{2}$
v_1	$v_p,\ v_2$	$v_{p-1},\ v_3$	$\cdots$	$v_{(p-(k-1))(mod\ p)}$, $v_{(1+k)(mod\ p)}$	$\cdots$	$v_{1+\frac{p}{2}}$
v_2	$v_1,\ v_3$	$v_p,\ v_4$	$\cdots$	$v_{(p-(k-2))(mod\ p)}$, $v_{(2+k)(mod\ p)}$	$\cdots$	$v_{2+\frac{p}{2}}$
v_3	$v_2,\ v_4$	$v_1,\ v_5$	$\cdots$	$v_{(p-(k-3))(mod\ p)}$, $v_{(3+k)(mod\ p)}$	$\cdots$	$v_{3+\frac{p}{2}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_i	$v_{(i-1)(mod\ p)}$, $v_{(i+1)(mod\, p)}$	$v_{(i-2)(mod\ p)}$, $v_{(i+2)(mod\, p)}$	$\cdots$	$v_{(p-(k-i))(mod\, p)}$, $v_{(i+k)(mod\, p)}$	$\cdots$	$v_{(i+\frac{p}{2})(mod\, p)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_p	$v_{p-1},\ v_1$	$v_{p-2},\ v_2$	$\cdots$	$v_{p-k},\ v_k$	$\cdots$	$v_{\frac{p}{2}}$

Table 2. Association schemes of circulant graphs for $p\ (\geq 2)$ is even.

By Table 2, the parameters of second kind are given by $n_i = 2$ for $1 \leq i \leq \frac{p}{2} - 1$ and $n_{\frac{p}{2}} = 1$.

With the association scheme for the Table 2, we have the matrix representation of the circulant graph $C_p(s_1, s_2, \cdots, s_t)$; $p\,(\geq 2)$ vertices is

$$P^k = \begin{pmatrix} p_{11}^k & p_{12}^k & \cdots & p_{1\,\frac{p}{2}}^k \\ p_{21}^k & p_{22}^k & \cdots & p_{2\,\frac{p}{2}}^k \\ \vdots & \vdots & \vdots & \vdots \\ p_{\frac{p}{2}1}^k & p_{\frac{p}{2}2}^k & \cdots & p_{\frac{p}{2}\,\frac{p}{2}}^k \end{pmatrix}.$$

Therefore, the possible values of k in the matrix P^k are given below:

If $k = 1$, then

(i) $p_{ij}^1 = 1$ for $1 \leq i \leq \frac{p}{2} - 1$, $j = i + 1$;

(ii) $p_{ij}^1 = 1$ for $1 \leq j \leq \frac{p}{2} - 1$, $i = 1 + j$.

If $2 \leq k \leq \frac{p}{2} - 1$, then

(i) $p_{ij}^k = 1$ for $1 \leq j \leq \frac{p}{2} - 1$ as well as $i + j = k$, $j = k + i$ and $i + j = p - k$;

(ii) $p_{ij}^k = 1$ for $1 \leq j \leq \frac{p}{2} - 1$, $i = k + j$ and $i + j = p - k$.

If $k = \frac{p}{2}$, then $p_{ij}^k = 2$ for $1 \leq i \leq \frac{p}{2} - 1$ and $j = k - i$ with the remaining entries zero.

Type 2. The matrix representation of circulant graph for $p(\geq 2)$ odd, with an association scheme is as follows:

Elements	**Association scheme**					
	First	Second	$\cdots$	k	$\cdots$	$\frac{p-1}{2}$
v_1	$v_p,\ v_2$	$v_{p-1},\ v_3$	$\cdots$	$v_{(p-(k-1))(mod\ p)}$, $v_{(1+k)(mod\ p)}$	$\cdots$	$v_{1+\frac{p-1}{2}},\ v_{1+\frac{p-1}{2}+1}$
v_2	$v_1,\ v_3$	$v_p,\ v_4$	$\cdots$	$v_{(p-(k-2))(mod\ p)}$, $v_{(2+k)(mod\ p)}$	$\cdots$	$v_{2+\frac{p-1}{2}},\ v_{2+\frac{p-1}{2}+1}$
v_3	$v_2,\ v_4$	$v_1,\ v_5$	$\cdots$	$v_{(p-(k-3))(mod\ p)}$, $v_{(3+k)(mod\ p)}$	$\cdots$	$v_{3+\frac{p-1}{2}},\ v_{3+\frac{p-1}{2}+1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_i	$v_{(i-1)(mod\ p)}$, $v_{(i+1)(mod\ p)}$	$v_{(i-2)(mod\ p)}$, $v_{(i+2)(mod\ p)}$	$\cdots$	$v_{(p-(k-i))(mod\ p)}$, $v_{(i+k)(mod\ p)}$	$\cdots$	$v_{(i+\frac{p-1}{2})(mod\ p)}$, $v_{(i+\frac{p-1}{2}+1)(mod\ p)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_p	$v_{p-1},\ v_1$	$v_{p-2},\ v_2$	$\cdots$	$v_{p-k},\ v_k$	$\cdots$	$v_{\frac{p-1}{2}},\ v_{\frac{p-1}{2}+1}$

Table 3. Association schemes of circulant graph with $p(\geq 3)$ is odd.

By Table 3 the parameters of second kind are given by $n_i = 2$ for $1 \leq i \leq \frac{p-1}{2}$ and $n_{\frac{p}{2}} = 1$.

With the association scheme for the Table 3, we have the matrix representation of the Circulant graph $C_p(s_1, s_2, \cdots, s_t)$; $p\ (\geq 3)$ vertices is

$$P^k = \begin{pmatrix} p^k_{11} & p^k_{12} & \cdots & p^k_{1\ \frac{p-1}{2}} \\ p^k_{21} & p^k_{22} & \cdots & p^k_{2\ \frac{p-1}{2}} \\ \vdots & \vdots & \vdots & \vdots \\ p^k_{(\frac{p-1}{2})1} & p^k_{(\frac{p-1}{2})2} & \cdots & p^k_{(\frac{p-1}{2})\ (\frac{p-1}{2})} \end{pmatrix}.$$

Therefore, the possible values of k in the matrix P^k are given below:

If $k = 1$, then

(i) $p^1_{ij} = 1$ for $1 \leq i \leq \frac{p-1}{2} - 1$ $j = i + 1$;
(ii) $p^1_{ij} = 1$ for $1 \leq j \leq \frac{p-1}{2} - 1$, $i = 1 + j$;
(iii) $p^1_{ij} = 1$ for $i = \frac{p-1}{2}$, $j = \frac{p-1}{2}$.

If $2 \leq k \leq \frac{p-3}{2}$, then

(i) $p^k_{ij} = 1$ for $1 \leq i \leq \frac{p-3}{2}$ as well as $i + j = k$, $j = k + i$ and $i + j = p - k$;
(ii) $p^k_{ij} = 1$ for $1 \leq j \leq \frac{p-3}{2}$ as well as $i = k + j$ and $i + j = p - k$.

If $k = \frac{p-1}{2}$, then

(i) $p^k_{ij} = 1$, for $1 \leq i \leq \frac{p-3}{2}$, $j = \frac{p-1}{2} - i$;
(ii) $p^k_{ij} = 1$, for $1 \leq i \leq \frac{p-1}{2}$, $j = \frac{p+1}{2} - i$ with remaining entries are all zero.

§4. The Parameters of PBIB-Designs

By considering Theorems 2.1 to 2.5, Tables 1 and 2 and the possible values of k in the matrix P^k using two different types, we have the parameters of PBIB-Design as follows:

Circuant Graph		Parameters of PBIB-Designs					
		p	b	g	r	λ_m	
G_1	$p = 2n,\ n \geq 2$	p	2	$\frac{p}{2}$	1	1 if m is even	0 if m is odd
	$p = 2n+1,\ n \geq 2$	p	p	$\lceil \frac{p}{2} \rceil$	$\lceil \frac{p}{2} \rceil$	$\lceil \frac{p}{2} \rceil - \lceil \frac{m-2}{2} \rceil$ if m is even	$\lceil \frac{m}{2} \rceil$ if m is odd
G_2	$p = 2n,\ n \geq 2$	p	2^p	$\frac{p}{2}$	2^{p-1}	2^{p-2} if $1 \leq m < \frac{p}{2}$	0 if m $= \frac{p}{2}$
	$p = 2n+1,\ n \geq 2$	p	p	$\lceil \frac{p}{2} \rceil$	$\lceil \frac{p}{2} \rceil$	$\lceil \frac{p}{2} \rceil - \lceil \frac{m-2}{2} \rceil$ if m is even	$\lceil \frac{m}{2} \rceil$ if m is odd
G_3	$p = 4n-2,\ n \geq 2$	p	2	$\frac{p}{2}$	1	1 if m is even	0 if m is odd
	$p = 4n-1,\ n \geq 2$	p	$2p$	$\lfloor \frac{p}{2} \rfloor$	$p-1$	$p + \frac{m}{2} - 5$ if m is even	$\lfloor \frac{p}{2} \rfloor - \lceil \frac{m}{2} \rceil$ if m is odd
G_4	$p = 4n,\ n \geq 2$	p	$\frac{p^2}{4}$	2	$\frac{p}{2}$	0 if m is even	0 if m is odd
	$p = 4n+1,\ n \geq 2$	p	$4p$	4	16	Problem 5.1	
G_5		p	p	1	1	0	

Table 4. Parameters of PBIB-designs.

§5. Conclusion and Open Problems

Generally, the PBIB-Designs are obtained from the families of strongly regular graphs. Interestingly, in this paper we determined the total number of η - sets, the PBIB-Designs and its association schemes arising from the η - sets of certain circulant graphs. Finally, we pose some open problems as follows:

Problem 5.1 *Generalize the* λ_m*, using PBIB-Designs associated with* $G_6 = C_p(2, 4, \cdots, \lfloor \frac{p}{2} \rfloor)$*;* $p = 4n + 1,\ n \geq 2$.

Problem 5.2 *Find all the strongly regular graphs for even and odd jump sizes of the circulant graphs.*

Acknowledgement

Thanks are due to Prof. N.D. Soner for his help and valuable suggestions in the preparation of this paper.

References

[1] Anu Sharma, Cini Varghese and Seema Jaggi, A web solution for partially balanced incomplete block experimental designs, *Computers and Electronics in Agriculture*, 99, 132–134, 2013.

[2] R. C. Bose, Strongly regular graphs, partial geometries and partially balanced designs, *Pacific J. Math.*, 13, 389–419, 1963.

[3] R. C. Bose and K. R. Nair, Partially balanced incomplete block design, *Sankhya*, 4, 337-372, 1939.

[4] A. E. Brouwer, A. V. Ivanov and M. H. Klin, Some new strongly regular graphs, *Combinatorica*, 9, 339–344, 1989.

[5] A. E. Brouwer and D. M. Mesner, The connectivity of strongly regular graphs, *Eur. J. Combin.*, 6, 215–216, 1985.

[6] A. E. Brouwer and H. A. Wilbrink, *Block Designs, Handbook of Incidence Geometry*, Elsevier, pp. 349-382, 1995.

[7] B. Chaluvaraju, Some parameters on neighborhood number of a graph, *Electronic Notes in Discrete Mathematics*, Elsevier B.V., 33, 139–146, 2009.

[8] B. Chaluvaraju, H. S. Boregowda and Shaikh Ameer Basha, PBIB-Designs associated with (α, β)-sets of circulant graphs, (communicated).

[9] B. Chaluvaraju, S. A. Diwakar and Shaikh Ameer Basha, PBIB-Designs associated with minimum dominating sets of circulant graphs, (communicated).

[10] B. Chaluvaraju and N. Manjunath, PBIB-Designs and association schemes arising from minimum bi-connected dominating sets of some special classes of graphs, *Afrika Matematika*, Springer, 29(1-2), 47-63, 2018.

[11] C. J. Colbourn, J. H. Dinitz, *Handbook of Combinatorial Designs*, CRC Press, 1996.

[12] M. N. Das and N. C. Giri, *Designs and Analysis of Experiments*, Wiley Eastern Limited, New Delhi, 1986.

[13] A. Dey, *Incomplete Block Designs.* World Scientific Publishing Co. Pvt. Ltd., Singapore, 2010.

[14] Dragan Marušič, Strongly regularity and circulant graphs, *Disc. Math.*, 78, 119–125, 1989.

[15] F. Harary, *Graph Theory*, Addison-Wesley, Reading Mass, 1969.

[16] E. Sampathkumar and Prabha S. Neeralagi, The neighbourhood number of a graph, *Indian Journal of Pure Applied Mathematics*, 16(2), 126–132, 1985.

[17] A. Sangeetha Devi, M. M. Shanmugapriya, Application of 2-dominator coloring in graphs using MATLAB, *J. Comp. and Math. Sci.*, 7 (4), 168–174, 2016.

[18] P. J. Slater, The Hedetniemi number of a graph, *Congr. Numer.*, 139, 65–75, 1999.

[19] H. B. Walikar, H. S. Ramane, B. D. Acharya, H. S. Shekarappa and S. Arumugam, Partially balanced incomplete block design arising from minimum dominating sets of paths and cycles, *AKCE J. Graphs Combin.*, 4(2), 223–232, 2007.