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The Variation of Electric Field With Respect to Darboux Triad in Euclidean 3-Space*

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Abstract: In this paper three electric fields are described via Darboux triad components in Euclidean 3-space. Later variations of three cases of electric field with respect to Darboux triad are studied. Finally Lorentz force equations are presented via electromagnetic magnetic curves with respect to Darboux triad in Euclidean 3-space.

Key Words: Geometric phase, Darboux frame, electric field.

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§1. Introduction

The geometric phase is described as the angle of rotation a light wave travelling in optic. The phenomenon of a geometric phase have many applications in condensed-matter physics, optics, particle physics, gravity, cosmology, chemical physics and mathematics [1-6]. The geometric phase is connected with parallel transport of the polarization along curved light [7-9].

Berry studied adiabatic phase and Pancharatnam's phase for polarized light [10]. Recently numerous authors presented the the electric field variation of along an optical fiber [11-14].

Balakrishnan *et al.* presented anholonomy density via Frenet triad in Euclidean 3-space $\mathcal{E}^3$ [15]. Three geometric phases and parallel transports for numerous frames have been investigated by Gürbüz in [16-20]. Balakrishnan introduced geometric phase for first class associated with some solitons for Darboux triad in $\mathcal{E}^3$ [21]. New classes associated with the nonlinear Schrödinger NLS equation for Darboux triad in $\mathcal{E}^3$ have been given in [22].

The electric polarization theory contains the geometric phase phenomenon [23]. Mukunda and Simon showed that the unit electric vector field $\mathbf{E}$ is written via the principal normal vector field $\mathbf{N}$ and the binormal vector field $\mathbf{B}$ of the Frenet triad $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ in Euclidean 3-space [24]. In this paper we express three electric fields via Darboux triad apparatus. Later evolutions of three electric fields are studied via Darboux triad in $\mathcal{E}^3$. Eventually Lorentz force equations are obtained via electromagnetic curves with respect to Darboux triad in $\mathcal{E}^3$.

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§2. Preliminaries

Let Γ_1 be a curve on a connected surface S with the arc length σ in $\mathcal{E}^3$. Apart from Frenet triad, at every point of curve, there is a Darboux triad $\{\mathbf{t}, \mathbf{g}, \mathbf{n}\}$. $\mathbf{t}$ is the tangent vector, $\mathbf{n}$ is the normal of surface and $\mathbf{g} = \mathbf{t} \times \mathbf{n}$. The spatial evolution of the Darboux triad $\{\mathbf{t}, \mathbf{g}, \mathbf{n}\}$ is given by [25]

$$\begin{bmatrix} \mathbf{t}_\sigma \\ \mathbf{g}_\sigma \\ \mathbf{n}_\sigma \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g^{(\varsigma)} & \kappa_n^{(\varsigma)} \\ -\kappa_g^{(\varsigma)} & 0 & \tau_g^{(\varsigma)} \\ -\kappa_n^{(\varsigma)} & -\tau_g^{(\varsigma)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{g} \\ \mathbf{n} \end{bmatrix} \tag{1}$$

$\kappa_g^{(\varsigma)}$ is the geodesic curvature, the normal curvature is $\kappa_n^{(\varsigma)}$ and $\tau_g^{(\varsigma)}$ is the geodesic torsion of the curve Γ_1. The time evolution of the Darboux triad $\{\mathbf{t}, \mathbf{g}, \mathbf{n}\}$ is given by

$$\begin{bmatrix} \mathbf{t}_u \\ \mathbf{g}_u \\ \mathbf{n}_u \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g^{(o)} & \kappa_n^{(o)} \\ -\kappa_g^{(o)} & 0 & \tau_g^{(o)} \\ -\kappa_n^{(o)} & -\tau_g^{(o)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{g} \\ \mathbf{n} \end{bmatrix} \tag{2}$$

where u denotes time and $\mathbf{t}_u = \frac{\partial \mathbf{t}}{\partial u}$.

A magnetic field is a closed 2-form $\mathcal{F}$ in $\mathcal{E}^3$. The Lorentz force Φ of a magnetic background $(\mathcal{E}^3, \langle , \rangle)$ is a (1,1) type skew-symmetric tensor and it is described as

$$\mathcal{F}(x, y) = \langle \Phi x, y \rangle$$

$x, y \in \chi(\mathcal{E}^3)$. A smooth curve Γ in $(\mathcal{E}^3, \langle , \rangle)$ is described as a magnetic curve of the dynamical system connected with the magnetic field $\mathcal{F}$ if its velocity vector field satisfies the following differential equation $\Gamma_{\sigma\sigma} = \Phi(\Gamma_\sigma)$. Divergence free vector fields and magnetic fields are one to one correspondence, the Lorentz force Φ concerned with the magnetic field $\mathbf{M}$ [26], [27]

$$\Phi(x) = \mathbf{M} \wedge x.$$

§3. Geometric Phase for First Case of Electric Field with Darboux Triad in $\mathcal{E}^3$

Balakrishan introduced first frame $\{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_2^*\}$ and first transformation ξ of curve evolution concerned with the NLS equation with respect to Darboux triad in $\mathcal{E}^3$ as following [21] :

$$\mathbf{P}_1 = \mathbf{t},\ \mathbf{P}_2 = \frac{\mathbf{g} + i\mathbf{n}}{\sqrt{2}} e^{i \int^\sigma \tau_g^{(\varsigma)} d\sigma'},\quad \mathbf{P}_2^* = \frac{\mathbf{g} - i\mathbf{n}}{\sqrt{2}} e^{-i \int^\sigma \tau_g^{(\varsigma)} d\sigma'} \tag{3}$$

$$\xi = \frac{\kappa_g + i\kappa_n}{\sqrt{2}} e^{i \int^\sigma \tau_g^{(\varsigma)} d\sigma'}. \tag{4}$$

The spatial evolution of the first frame $\{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_2^*\}$ is given by

$$\mathbf{P}_{1\sigma} = \xi^* \mathbf{P}_2 + \xi \mathbf{P}_2^*,\quad \mathbf{P}_{2\sigma} = -\xi \mathbf{P}_1,\quad \mathbf{P}_{2\sigma}^* = -\xi^* \mathbf{P}_1 \tag{5}$$

where ξ^* is the conjugate of ξ. Also temporal evolution of $\{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_2^*\}$ is

$$\mathbf{P}_{1u} = \mathbf{t}_u = -\lambda^*\mathbf{P}_2 - \lambda\mathbf{P}_2^* \tag{6}$$

$$\mathbf{P}_{2u} = \lambda\mathbf{P}_1 + i\mathcal{I}\mathbf{P}_2 \tag{7}$$

where $\mathcal{I}(\sigma, u)$ is a real function. From $\mathbf{P}_{2u\sigma} = \mathbf{P}_{2\sigma u}$, it can be obtained:

$$\mathcal{I}_\sigma = i\lambda\xi^* - i\lambda^*\xi. \tag{8}$$

where

$$\mathcal{AD}_1 d\sigma du = (\tau_{gu}^{(o)} - \tau_{gs}^{(\varsigma)})d\sigma du$$

is first anholonomy density measure for polarization plane of linearized light wave travelling along optic fiber in $\mathcal{E}^3$ [21].

$$\lambda = -\frac{(r+iw)}{\sqrt{2}}e^{i\int \tau_g^{(\varsigma)} d\sigma'} \tag{9}$$

satisfies Eqs.(6), (7) and (8). The time evolution of the Darboux triad is given by

$$\mathbf{t}_u = \varsigma_1^{(o)} \times \mathbf{t} = r\mathbf{g} + w\mathbf{n} \tag{10}$$

$$\mathbf{g}_u = \varsigma_1^{(o)} \times \mathbf{g} = -r\mathbf{t} + \tau_g^{(o)}\mathbf{n} \tag{11}$$

$$\mathbf{n}_u = \varsigma_1^{(o)} \times \mathbf{n} = -w\mathbf{t} - \tau_g^{(o)}\mathbf{g} \tag{12}$$

where $\varsigma_1^{(o)} = (\tau_g^{(o)}\mathbf{t} + B_1\mathbf{g} + C_1\mathbf{n})$, $r = C_1$, $w = -B_1$. Using Eq.(4) and Eq.(9), $\mathcal{I}_\sigma = \kappa_n^{(\varsigma)} r - \kappa_g^{(\varsigma)} w$. The time evolution of Darboux triad for first class can be written by

$$\mathbf{t}_u = r\mathbf{g} + w\mathbf{n} \tag{13}$$

$$\mathbf{g}_u = -r\mathbf{t} + (\int^{\sigma_1} \tau_{gu}^{(\varsigma)} d\sigma' - \mathcal{I})\mathbf{n} \tag{14}$$

$$\mathbf{n}_u = -w\mathbf{t} - (\int^{\sigma_1} \tau_{gu}^{(\varsigma)} d\sigma' - \mathcal{I})\mathbf{g} \tag{15}$$

and anholonomy density

$$\mathcal{AD}_1(\sigma, u) = -\mathcal{I}_\sigma = -r\kappa_n^{(\varsigma)} + w\kappa_g^{(\varsigma)}$$

for first class. Total phase $\mathcal{P}$ for first class with respect to Darboux triad in Euclidean 3-space is given by

$$\begin{aligned}\mathcal{P} &= -\int_{u_1}^{u_2}\int_{\sigma_0}^{\sigma_1} \mathcal{I}_\sigma d\sigma du = \int_{u_1}^{u_2}\int_{\sigma_0}^{\sigma_1} \langle \mathbf{t}, \mathbf{t}_\sigma \times \mathbf{t}_u \rangle \, d\sigma du \\ &= \int_{u_1}^{u_2}\int_{\sigma_0}^{\sigma_1} (-r\kappa_n^{(\varsigma)} + w\kappa_g^{(\varsigma)}) d\sigma du\end{aligned}$$

Also [22]

$$\begin{aligned} \mathbf{P}_{1u} &= -i\xi_\sigma^*\mathbf{P}_2 + i\xi_\sigma\mathbf{P}_2^* \\ \mathbf{P}_{2u} &= -i\xi_\sigma\mathbf{P}_1 + \mathcal{I}\mathbf{P}_2, \\ \mathbf{P}_{2u}^* &= i\xi_\sigma^* - \mathcal{I}\mathbf{P}_2^*,\ \mathcal{I} = i\xi\xi^*. \end{aligned}$$

From $\mathbf{P}_{1u\sigma} = \mathbf{P}_{1\sigma u}$ and $\mathbf{P}_{2u\sigma} = \mathbf{P}_{2\sigma u}$, the NLS equation system

$$\begin{aligned} \xi_u &= i\xi_{\sigma\sigma} + i\left|\xi\right|^2\xi \\ \xi_u^* &= -i\xi_{\sigma\sigma} - i\left|\xi\right|^2\xi. \end{aligned}$$

is obtained.

A optical fiber can be described by the curve $\Gamma_1(\sigma)$ on any surface with respect to Darboux triad in $\mathcal{E}^3$. The change of the electric field $\mathbf{E}_1$ can be written by

$$\mathbf{E}_{1\sigma} = \varphi_1\mathbf{t} + \varphi_2\mathbf{g} + \varphi_3\mathbf{n}. \tag{16}$$

Case 1. Assume that

$$\langle \mathbf{E}_1, \mathbf{t}\rangle = 0. \tag{17}$$

Using Eq.(16) and Eq.(17), it can be obtained

$$\varphi_1 = -\kappa_g\langle \mathbf{E}_1, \mathbf{g}\rangle - \kappa_n\langle \mathbf{E}_1, \mathbf{n}\rangle\ . \tag{18}$$

When no various loss mechanism along the optic fiber,

$$\langle \mathbf{E}_1, \mathbf{E}_1\rangle = const. \tag{19}$$

Using Eq.(16) and taking derivative with respect to σ of Eq.(19), it can be derived

$$\varphi_2\langle \mathbf{E}_1, \mathbf{g}\rangle = -\varphi_3\langle \mathbf{E}_1, \mathbf{n}\rangle. \tag{20}$$

Via Eq.(20), it can be obtained

$$\varphi_2 = \varpi\langle \mathbf{E}_1, \mathbf{n}\rangle,\quad \varphi_3 = -\langle \mathbf{E}_1, \mathbf{g}\rangle \tag{21}$$

The evolution for the polarization of light wave travelling from the point $\Gamma_1(\sigma_0)$ to the point $\Gamma_1(\sigma_1)$ along the $\Gamma_1 = \Gamma_1(\sigma)$ curve with respect to Darboux triad is given by the evolution of the electric field $\mathbf{E}_1$.

Consider $\langle \mathbf{E}_1, \mathbf{g}\rangle \neq 0$, $\langle \mathbf{E}_1, \mathbf{n}\rangle \neq 0$. Substituting Eqs.(18) and (21) in Eq.(16), the change of the electric field $\mathbf{E}_1$ is written by

$$\mathbf{E}_{1\sigma} = (-\kappa_g\langle \mathbf{E}_1, \mathbf{g}\rangle - \kappa_n\langle \mathbf{E}_1, \mathbf{n}\rangle)\mathbf{t} + \varpi\langle \mathbf{E}_1, \mathbf{n}\rangle\mathbf{g} - \varpi\langle \mathbf{E}_1, \mathbf{g}\rangle\mathbf{n} \tag{22}$$

where ϖ is a parameter. Using Eq.(20) for $\varpi = 0$, Eq.(22) is rewritten by

$$\mathbf{E}_{1\sigma} = (-\kappa_g \langle \mathbf{E}_1, \mathbf{g} \rangle - \kappa_n \langle \mathbf{E}_1, \mathbf{n} \rangle)\mathbf{t} \tag{23}$$

The Fermi-Walker derivative of the electric field $\mathbf{E}_1$ with respect to Darboux triad in $\mathcal{E}^3$ is given by

$$^{DFW}\mathbf{E}_{1\sigma} = \mathbf{E}_{1\sigma} - \langle \mathbf{t}, \mathbf{E}_1 \rangle \mathbf{t}_\sigma + \langle \mathbf{t}_\sigma, \mathbf{E}_1 \rangle \mathbf{t}. \tag{24}$$

The electric field $\mathbf{E}_1$ is the Fermi-Walker parallel transport if and only if

$$^{DFW}\mathbf{E}_{1\sigma} = 0. \tag{25}$$

Using Eqs.(17), (24) and (25) it can be obtained

$$\mathbf{E}_{1\sigma} = \langle \mathbf{t}_\sigma, \mathbf{E}_1 \rangle \mathbf{n}. \tag{26}$$

The electric field vector $\mathbf{E}_1$ with aid of the Darboux triad apparatus $\mathbf{g}$ and $\mathbf{n}$ is expressed by

$$\mathbf{E}_{1\sigma}(\sigma) = \Omega(\sigma)\frac{(\mathbf{g} + i\mathbf{n})}{\sqrt{2}} + \Omega^*(\sigma)\frac{\mathbf{g} - i\mathbf{n}}{\sqrt{2}}. \tag{27}$$

where $\mathbf{E}_1\mathbf{E}_1^* = 1$ and $|\Omega(\sigma)|^2 + |\Omega^*(\sigma)|^2 = 1, \mathbf{E}_1^*$ is complex conjugate of $\mathbf{E}_1$.

$$\mathcal{P} = \int^{\sigma_1} \tau_g^{(\varsigma)} d\sigma'$$

is the change phase of the polarization light injected into this fiber with respect to Darboux triad in $\mathcal{E}^3$.

$$\Omega(\sigma) = e^{i\int^{\sigma_1} \tau_g^{(\varsigma)} d\sigma'} \Omega(\sigma_0)$$

$$\Omega^*(\sigma) = e^{-i\int^{\sigma_1} \tau_g^{(\varsigma)} d\sigma'} \Omega^*(\sigma_0)$$

with the polarization coefficients are

$$\Omega(\sigma_0) = \left(\frac{\mathbf{g} + i\mathbf{n}}{\sqrt{2}}\right)^* \mathbf{E}_1(\sigma_0)$$

$$\Omega^*(\sigma_0) = \left(\frac{\mathbf{g} - i\mathbf{n}}{\sqrt{2}}\right)^* \mathbf{E}_1(\sigma_0).$$

Also via $\mathbf{P}_2$, $\mathbf{P}_2^*$, $\Omega(\sigma_0)$ and $\Omega^*(\sigma_0)$, the electric field $\mathbf{E}_1(\sigma)$ is expressed as

$$\mathbf{E}_1(\sigma) = \mathbf{P}_2\Omega(\sigma_0) + \mathbf{P}_2^*\Omega^*(\sigma_0) \tag{28}$$

Respectively, taking derivative with respect to σ and the time u of Eq.(28), the spatial and temporal evolutions of the electric field $\mathbf{E}_1$ for Darboux triad are derived as following:

$$\begin{aligned}\mathbf{E}_{1\sigma} &= \mathbf{P}_{2\sigma}\Omega(\sigma_0)+\mathbf{P}^*_{2\sigma}\Omega^*(\sigma_0)\\ \mathbf{E}_{1u} &= \mathbf{P}_{2u}\Omega(\sigma_0)+\mathbf{P}^*_{2u}\Omega^*(\sigma_0).\end{aligned}$$

From compatibility condition $\mathbf{E}_{1\sigma u} = \mathbf{E}_{1u\sigma}$, the nonlinear Schrödinger NLS equation system.

The Lorentz force equation $\Phi^{(t)}$ of the electric field vector $\mathbf{E}_1$ is given by

$$\Phi^{(t)}\mathbf{E}_1 = \mathbf{E}_{1\sigma} = \mathbf{M}^{(t)} \times \mathbf{E}_1 \tag{29}$$

and

$$\begin{aligned}\left\langle \Phi^{(t)}\mathbf{E}_1, \mathbf{t}\right\rangle &= -\left\langle \mathbf{E}_1, \Phi^{(t)}\mathbf{t}\right\rangle,\ \left\langle \Phi^{(t)}\mathbf{E}_1, \mathbf{g}\right\rangle = -\left\langle \mathbf{E}_1, \Phi^{(t)}\mathbf{g}\right\rangle && (30)\\ \left\langle \Phi^{(t)}\mathbf{E}_1, \mathbf{n}\right\rangle &= -\left\langle \mathbf{E}_1, \Phi^{(t)}\mathbf{n}\right\rangle. && (31)\end{aligned}$$

The trajectory of travelling particle along the magnetic field $\mathbf{M}^{(t)}$ with respect to Darboux triad in $\mathcal{E}^3$ is described as electromagnetic trajectory. If $\mathbf{DEM}^{(t)}$ curve follows the magnetic trajectory, it is described as the Darboux electromagnetic curve in $\mathcal{E}^3$. With the help of Eqs. (30), (31) the Lorentz force Φ^t equations in the Darboux force equations of the $\mathbf{DEM}^{(t)}$ curve of the Γ_1 are given by

$$\begin{bmatrix}\Phi^{(t)}(\mathbf{t})\\ \Phi^{(t)}(\mathbf{g})\\ \Phi^{(t)}(\mathbf{n})\end{bmatrix} = \begin{bmatrix}0 & -\kappa_g^{(\varsigma)} & -\kappa_n^{(\varsigma)}\\ \kappa_g^{(\varsigma)} & 0 & -\varpi\\ \kappa_n^{(\varsigma)} & \varpi & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{g}\\ \mathbf{n}\end{bmatrix} \tag{32}$$

$\mathbf{DEM}^{(t)}$ curve of the Γ_1 is a magnetic trajectory of the magnetic field $\mathbf{M}^{(t)}$ divergence free field iff $\mathbf{M}^{(t)}$ is given by in the following

$$\mathbf{M}^{(t)} = -\varpi\mathbf{t} + \kappa_n\mathbf{g} - \kappa_g\mathbf{n}$$

§4. Geometric Phase for Second Case of Electric Field with Darboux Triad in $\mathcal{E}^3$

Respectively, the second frame $\{\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_2^*\}$ and second transformation ϕ associated with the NLS equation via Darboux triad is given by [22]

$$\begin{aligned}\mathbf{Q}_1 &= \mathbf{g}, && (33)\\ \mathbf{Q}_2 &= \frac{\mathbf{t}+i\mathbf{n}}{\sqrt{2}}e^{i\int^{\sigma_1}\kappa_n^{(\varsigma)}d\sigma'},\ \mathbf{Q}_2^* = \frac{\mathbf{t}-i\mathbf{n}}{\sqrt{2}}e^{-i\int^{\sigma_1}\kappa_n^{(\varsigma)}d\sigma'} && (34)\end{aligned}$$

$$\phi = \frac{(-\kappa_g^{(\varsigma)} + i\tau_g^{(\varsigma)})}{\sqrt{2}}e^{i\int^{\sigma}\kappa_n^{(\varsigma)}d\sigma'} \tag{35}$$

Using Eqs.(33) and (34) the spatial evolution of the frame $\{\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_2^*\}$ is given by

$$\begin{aligned}\mathbf{Q}_{1\sigma} &= \phi^*\mathbf{Q}_2 + \phi\mathbf{Q}_2^* \\ \mathbf{Q}_{2\sigma} &= -\phi\mathbf{Q}_1 \\ \mathbf{Q}_{2\sigma}^* &= -\phi^*\mathbf{Q}_1\end{aligned}$$

where $\phi^* = \frac{(-\kappa_g^{(\varsigma)} - i\tau_g^{(\varsigma)})}{\sqrt{2}} e^{-i\int^{\sigma} \kappa_n^{(\varsigma)} d\sigma}$.

Consider

$$\begin{aligned}\mathbf{Q}_{1u} &= \mathbf{g}_u = a_2\mathbf{Q}_2 + b_2\mathbf{Q}_2^* + c_2\mathbf{Q}_1 && (36)\\ \mathbf{Q}_{2u} &= h_2\mathbf{Q}_2 + f_2\mathbf{Q}_2^* + \vartheta\mathbf{Q}_1. && (37)\end{aligned}$$

From $\langle \mathbf{Q}_{1u}, \mathbf{Q}_1\rangle = 0 \Rightarrow c_2 = 0$, $\langle \mathbf{Q}_{1u}, \mathbf{Q}_2\rangle = b_2$, $\langle \mathbf{Q}_{2u}, \mathbf{Q}_1\rangle = \vartheta \Rightarrow b_2 = -\vartheta$, $\langle \mathbf{Q}_{2u}, \mathbf{Q}_2\rangle = f_2 = 0$, $\langle \mathbf{Q}_{2u}^*, \mathbf{Q}_2\rangle = -h_2 \Rightarrow h_2 = -f_2^*$ and $a_2 = -\vartheta^*$.
Eqs.(36) and (37) are rewritten by

$$\begin{aligned}\mathbf{Q}_{1u} &= \mathbf{g}_u = -\vartheta^*\mathbf{Q}_2 - \vartheta\mathbf{Q}_2^* && (38)\\ \mathbf{Q}_{2u} &= \vartheta\mathbf{Q}_1 + i\mathcal{J}\mathbf{Q}_2 && (39)\end{aligned}$$

with $\mathcal{J}(\sigma, u)$ a real function. From $\mathbf{Q}_{2u\sigma} = \mathbf{Q}_{2\sigma u}$ the followings are obtained

$$\begin{aligned}\phi_u &= -\vartheta_\sigma + i\mathcal{J}\phi \\ \mathcal{J}_\sigma &= i\vartheta\phi^* - i\vartheta^*\phi. && (40)\end{aligned}$$

When $\mathbf{t}$ and $\mathbf{n}$ rotates around $\mathbf{g}$ with $\kappa_n^{(\varsigma)}(\sigma)$, a geometric phase $\mathcal{P} = \int\limits_{\sigma_0}^{\sigma_1} \kappa_n^{(\varsigma)}(\sigma)d\sigma'$ arises between $\mathbf{t}$, $\mathbf{n}$ and corresponding nonrotating Darboux triad in $\mathcal{E}^3$.
When the linearized light wave travelling moves from u_1 to u_2 along the curve in optic fiber, a geometric phase $\mathcal{P} = \int\limits_{u_1}^{u_2} \kappa_n^{(o)}(u)du$ arises between natural Darboux triad and nonrotating Darboux triad in Euclidean 3-space. The rotation angles of polarization plane can be given by

$$\begin{aligned}\mathcal{P}_1 &= \kappa_n^{(\varsigma)}(\sigma, u)\Delta\sigma + \kappa_n^{(o)}(\sigma + \Delta\sigma, u)\Delta u \\ \mathcal{P}_2 &= \kappa_n^{(o)}(\sigma, u)\Delta u + \kappa_n^{(\varsigma)}(\sigma, u + \Delta u)\Delta\sigma\end{aligned}$$

The phase difference are given by $\delta\mathcal{P} = \mathcal{P}_1 - \mathcal{P}_2 = \mathcal{AD}_2(\sigma, u)\Delta\sigma\Delta u$.
$\mathcal{AD}_2 = (\kappa_{n\sigma}^{(\varsigma)} - \kappa_{nu}^{(o)})$ is second anholonomy density measure for polarization plane of linearized light wave travelling along optic fiber for second case in $\mathcal{E}^3$. Also

$$\vartheta = -\frac{(l + iw)}{\sqrt{2}} e^{i\int^{\sigma_1} \kappa_n^{(\varsigma)} d\sigma'} \tag{41}$$

satisfies Eqs.(39) and (40). The time evolution of Darboux triad for second class is given by

[22]

$$\begin{aligned} \mathbf{t}_u &= \varsigma_2^{(o)} \times \mathbf{t} = -\, l\mathbf{g} + \kappa_n^{(o)} \mathbf{n} && (42) \\ \mathbf{g}_u &= \varsigma_2^{(o)} \times \mathbf{g} = l\mathbf{t} + w\mathbf{n} && (43) \\ \mathbf{n}_u &= \varsigma_2^{(o)} \times \mathbf{n} = -\kappa_n^{(o)} \mathbf{t} - w\mathbf{g} && (44) \end{aligned}$$

where $\varsigma_2^{(o)} = A_2\mathbf{t} - \kappa_n^{(\varsigma)}\mathbf{g} + C_2\mathbf{n}$, $l = -C_2$, $w = -A_2$.

Using Eqs.(35), (40) and (41) it can be obtained

$$\mathcal{J}_\sigma = -(\tau_g^{(\varsigma)} l + \kappa_g^{(\varsigma)} w). \tag{45}$$

From Eqs.(34), (39), (42), (43) and (44), the time evolution of Darboux triad for second class with Eq.(43) is given by

$$\mathbf{t}_u = l\mathbf{g} + (\int^{\sigma_1} \kappa_{nu}^{(\varsigma)} d\sigma^{'} - \mathcal{J})\mathbf{n}$$

$$\mathbf{n}_u = -(\int^{\sigma_1} \kappa_{nu}^{(\varsigma)} d\sigma^{'} - \mathcal{J})\mathbf{t} - w\mathbf{g}$$

and the anholonomy density $\mathcal{AD}_2(\sigma, u) = -\mathcal{J}_\sigma = (\tau_g^{(\varsigma)} l + \kappa_g^{(\varsigma)} w)$ for second class. Total phase $\mathcal{P}$ for second class with respect to Darboux triad in $\mathcal{E}^3$ is given by

$$\begin{aligned} \mathcal{P} &= -\int_{u_1}^{u_2} \int_{\sigma_0}^{\sigma_1} \mathcal{J}_\sigma d\sigma du = \int_{u_1}^{u_2} \int_{\sigma_0}^{\sigma_1} (\tau_g^{(\varsigma)} l + \kappa_g^{(\varsigma)} w) d\sigma du \\ &= \int_{u_1}^{u_2} \int_{\sigma_0}^{\sigma_1} \langle \mathbf{g}, \mathbf{g}_\sigma \times \mathbf{g}_u \rangle \, d\sigma du. \end{aligned}$$

The quantum geometric phase for second class of curve evolution with respect to Darboux triad in $\mathcal{E}^3$ is obtained

$$\mathcal{P} = i \int_{\sigma_0}^{\sigma_1} d\sigma \frac{\partial}{\partial \sigma} \int_{u_1}^{u_2} \langle \mathbf{Q}_{2u}, \mathbf{Q}_2^* \rangle \; du.$$

Also [22]

$$\begin{aligned} \mathbf{Q}_{1u} &= -i\phi_\sigma^* \mathbf{Q}_2 + i\phi_\sigma \mathbf{Q}_2^*, \quad \mathbf{Q}_{2u} = -i\phi_\sigma \mathbf{Q}_1 + \mathcal{J}\mathbf{Q}_2, \\ \mathbf{Q}_{2u}^* &= i\phi_\sigma^* \mathbf{Q}_1 - \mathcal{J}\mathbf{Q}_2^*, \; \mathcal{J} = i\phi\phi^* \end{aligned}$$

From $\phi_{1u\sigma} = \phi_{1\sigma u}$ and $\phi_{2u\sigma} = \phi_{2\sigma u}$, the NLS equation

$$\phi_u = i\phi_{\sigma\sigma} + i \mid \phi \mid^2 \phi$$

is obtained.

A optical fiber can be described by a curve $\Gamma_2(\sigma)$ with respect to Darboux triad in $\mathcal{E}^3$. The direction of electric field $\mathbf{E}_2$ is given by the direction of the state of the linearly polarized light wave injected to the fiber with respect to Darboux triad in $\mathcal{E}^3$. The change of the electric field

$\mathbf{E}_2$ with respect to Darboux frame in $\mathcal{E}^3$ can be given by

$$\mathbf{E}_{2\sigma} = \zeta_1 \mathbf{t} + \zeta_2 \mathbf{g} + \zeta_3 \mathbf{n}. \tag{46}$$

Case 2. Assume that

$$\langle \mathbf{E}_2, \mathbf{g} \rangle = 0. \tag{47}$$

Using Eqs. (46) and (47), it can be written by

$$\zeta_2 = -\kappa_g \langle \mathbf{E}_2, \mathbf{t} \rangle - \tau_g \langle \mathbf{E}_2, \mathbf{n} \rangle \tag{48}$$

Consider

$$\langle \mathbf{E}_2, \mathbf{E}_2 \rangle = const. \tag{49}$$

Taking derivative with respect to σ of Eq.(49), the followings are obtained

$$\zeta_1 \langle \mathbf{E}_2, \mathbf{t} \rangle = -\zeta_3 \langle \mathbf{E}_2, \mathbf{n} \rangle \tag{50}$$

$$\zeta_1 = \chi \langle \mathbf{E}_2, \mathbf{n} \rangle, \quad \zeta_3 = -\chi \langle \mathbf{E}_2, \mathbf{t} \rangle \tag{51}$$

where χ is a parameter.

Using Eq.(20) and $\langle \mathbf{E}_2, \mathbf{t} \rangle \neq 0$, $\langle \mathbf{E}_2, \mathbf{n} \rangle \neq 0$. Substituting Eqs. (48) and (51) in (46), the evolution of the electric field vector $\mathbf{E}_2$ with respect to Darboux triad is given by

$$\mathbf{E}_{2\sigma} = \chi \langle \mathbf{E}_2, \mathbf{n} \rangle \mathbf{t} + (-\kappa_g \langle \mathbf{E}_2, \mathbf{t} \rangle - \tau_g \langle \mathbf{E}_2, \mathbf{n} \rangle)\mathbf{g} - \chi \langle \mathbf{E}_2, \mathbf{t} \rangle \mathbf{n} \tag{52}$$

Via Eq.(52) for $\chi = 0$,

$$\mathbf{E}_{2\sigma} = (-\kappa_g \langle \mathbf{E}_2, \mathbf{t} \rangle - \tau_g \langle \mathbf{E}_2, \mathbf{n} \rangle)\mathbf{g} \tag{53}$$

The modified Fermi-Walker derivative for the electric field vector $\mathbf{E}_2$ with respect to Darboux triad for second class is described by

$$^{DmFW}\mathbf{E}_{2\sigma} = \mathbf{E}_{2\sigma} - \langle \mathbf{g}, \mathbf{E}_2 \rangle \mathbf{g}_\sigma + \langle \mathbf{g}_\sigma, \mathbf{E}_2 \rangle \mathbf{g} \tag{54}$$

The electric field $\mathbf{E}_2$ is the modified Fermi-Walker parallel if and only if

$$^{DmFW}\mathbf{E}_{2\sigma} = 0. \tag{55}$$

Via Eqs.(47), (54) and (55), one obtains $\mathbf{E}_{2\sigma} = \langle \mathbf{g}_\sigma, \mathbf{E}_2 \rangle \mathbf{n}$.

The electric field vector $\mathbf{E}_2$ with aid of the Darboux triad apparatus $\mathbf{t}$ and $\mathbf{n}$ can be expressed by

$$\mathbf{E}_2(\sigma) = \Upsilon(\sigma)\frac{(\mathbf{t} + i\mathbf{n})}{\sqrt{2}} + \Upsilon^*(\sigma)\frac{\mathbf{t} - i\mathbf{n}}{\sqrt{2}}. \tag{56}$$

where $\mathbf{E}_2\mathbf{E}_2^* = 1$ and $|\Upsilon(\sigma)|^2 + |\Upsilon^*(\sigma)|^2 = 1$.

Here $\Upsilon(\sigma)$ and $\Upsilon^*(\sigma)$ are

$$\Upsilon(\sigma) = e^{i\int^{\sigma_1}\kappa_n^{(\varsigma)}d\sigma'}\Upsilon(\sigma_0),\ \Upsilon^*(\sigma) = e^{-i\int^{\sigma_1}\kappa_n^{(\varsigma)}d\sigma'}\Upsilon^*(\sigma_0) \tag{57}$$

and the polarization coefficients are

$$\Upsilon(\sigma_0) = \left(\frac{\mathbf{t}+i\mathbf{n}}{\sqrt{2}}\right)^*\mathbf{E}_2(\sigma_0),\ \Upsilon^*(\sigma_0) = \left(\frac{\mathbf{t}-i\mathbf{n}}{\sqrt{2}}\right)^*\mathbf{E}_2(\sigma_0) \tag{58}$$

Via Eqs.(34) and (57), Eq.(56) is re-expressed by

$$\mathbf{E}_2(\sigma) = \mathbf{Q}_2\Upsilon(\sigma_0) + \mathbf{Q}_2^*\Upsilon^*(\sigma_0). \tag{59}$$

Respectively, the spatial and temporal evolutions of the electric field $\mathbf{E}_2$ for Darboux triad are derived as following:

$$\begin{aligned}\mathbf{E}_{2\sigma} &= \mathbf{Q}_{2\sigma}\Upsilon(\sigma_0) + \mathbf{Q}_{2\sigma}^*\Upsilon^*(\sigma_0)\\ \mathbf{E}_{2u} &= \mathbf{Q}_{2u}\Upsilon(\sigma_0) + \mathbf{Q}_{2u}^*\Upsilon^*(\sigma_0).\end{aligned}$$

From compatibility condition $\mathbf{E}_{2\sigma u} = \mathbf{E}_{2u\sigma}$, the NLS equation system connected with the electric field $\mathbf{E}_2$ is derived.

Geometric phase for polarized light injected into a fiber with respect to Darboux triad for second case in $\mathcal{E}^3$ is given by

$$\mathcal{P} = \int^{\sigma_1}\kappa_n^{(\varsigma)}d\sigma'.$$

Consider the Lorentz force equation $\Phi^{(g)}$ for second case of the electric field vector

$$\Phi^{(g)}\mathbf{E}_2 = \mathbf{E}_{2\sigma} = \mathbf{M}^{(g)}\times\mathbf{E}_2 \tag{60}$$

and

$$\begin{aligned}\left\langle\Phi^{(g)}\mathbf{E}_2,\mathbf{t}\right\rangle &= -\left\langle\mathbf{E}_2,\Phi^{(g)}\mathbf{t}\right\rangle, \left\langle\Phi^{(g)}\mathbf{E}_2,\mathbf{g}\right\rangle = -\left\langle\mathbf{E}_2,\Phi^{(g)}\mathbf{g}\right\rangle, \quad (61)\\ \left\langle\Phi^{(g)}\mathbf{E}_2,\mathbf{n}\right\rangle &= -\left\langle\mathbf{E}_2,\Phi^{(g)}\mathbf{n}\right\rangle. \quad (62)\end{aligned}$$

The trajectory of travelling particle along the magnetic field $\mathbf{M}^{(g)}$ with respect to Darboux triad is described as the electromagnetic trajectory. If $\mathbf{DEM}^{(g)}$ curve follows the magnetic trajectory, it is described as the Darboux electromagnetic curve. With the help of Eqs. (61) and(62), the Lorentz force Φ^g in the Darboux triad of the $\mathbf{DEM}^{(g)}$ curve of Γ_2 are given by

$$\begin{bmatrix}\Phi^{(g)}(\mathbf{t})\\ \Phi^{(g)}(\mathbf{g})\\ \Phi^{(g)}(\mathbf{n})\end{bmatrix} = \begin{bmatrix}0 & -\kappa_g^{(\varsigma)} & -\chi\\ \kappa_g^{(\varsigma)} & 0 & \tau_g\\ \chi & -\tau_g^{(\varsigma)} & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{g}\\ \mathbf{n}\end{bmatrix} \tag{63}$$

Via Eq. (63), the vector field divergence free $\mathbf{M}^{(g)}$ is given by

$$\mathbf{M}^{(g)} = \chi \mathbf{g} + \tau_g t - \kappa_g \mathbf{n}.$$

§5. Geometric Phase for Third Case of Electric Field with Darboux Triad

The third frame $\{\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_2^*\}$ and the third transformation ψ for third class of curve evolution concerned with the NLS equation with respect to Darboux triad in $\mathcal{E}^3$ are given by [22]

$$\mathbf{R}_1 = \mathbf{n}, \tag{64}$$

$$\mathbf{R}_2 = \frac{\mathbf{t} + i\mathbf{g}}{\sqrt{2}} e^{i \int^{\sigma_1} \kappa_g^{(\varsigma)} d\sigma'}, \ \mathbf{R}_2^* = \frac{\mathbf{t} - i\mathbf{g}}{\sqrt{2}} e^{-i \int^{\sigma_1} \kappa_g^{(\varsigma)} d\sigma'} \tag{65}$$

$$\psi = \frac{(\kappa_n^{(\varsigma)} + i\tau_g^{(\varsigma)})}{\sqrt{2}} e^{i \int^{\sigma} \kappa_g^{(\varsigma)} d\sigma'} \tag{66}$$

Using Eqs. (65) and (66), the spatial evolution of $\{\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_2^*\}$ is given by [22]

$$\mathbf{R}_{1\sigma} = -\psi^* \mathbf{R}_2 - \psi \mathbf{R}_2^*, \ \mathbf{R}_{2\sigma} = \psi \mathbf{R}_1, \ \mathbf{R}_{2\sigma}^* = \psi^* \mathbf{R}_1 \tag{67}$$

where $\psi^* = \frac{(\kappa_n^{(\varsigma)} - i\tau_g^{(\varsigma)})}{\sqrt{2}} e^{-i \int^{\sigma} \kappa_g^{(\varsigma)} d\sigma'}$.

Consider

$$\mathbf{R}_{1u} = \mathbf{n}_u = a_3 \mathbf{R}_2 + b_3 \mathbf{R}_2^* + c_3 \mathbf{R}_1, \tag{68}$$

$$\mathbf{R}_{2u} = h_3 \mathbf{R}_2 + f_3 \mathbf{R}_2^* + \eta \mathbf{R}_1. \tag{69}$$

From $\langle \mathbf{R}_{1u}, \mathbf{R}_1 \rangle = 0 \Rightarrow c_3 = 0$, $\langle \mathbf{R}_{1u}, \mathbf{R}_2 \rangle = b_3$, $\langle \mathbf{R}_{2u}, \mathbf{R}_1 \rangle = \eta \Rightarrow b_3 = -\eta$, $\langle \mathbf{R}_{2u}, \mathbf{R}_2 \rangle = f_3 = 0, \langle \mathbf{R}_{1u}, \mathbf{R}_2^* \rangle = a_3 \Rightarrow \eta^* = a_3$, $\langle \mathbf{R}_{2u}^*, \mathbf{R}_2 \rangle = -h_3 \Rightarrow h_3 = -f_3^*$.
Eqs. (68) and (69) can be rewritten by

$$\mathbf{R}_{1u} = \mathbf{n}_u = -\eta^* \mathbf{R}_2 - \eta \mathbf{V}_2^*, \tag{70}$$

$$\mathbf{R}_{2u} = \eta \mathbf{R}_1 + i\mathcal{L}\mathbf{R}_2 \tag{71}$$

with $\mathcal{L}(\sigma, u)$ is a real function. From $\mathbf{R}_{2u\sigma} = \mathbf{R}_{2\sigma u}$ the followings can be derived by

$$\psi_u = \eta_\sigma + i\mathcal{L}_\sigma \psi, \tag{72}$$

$$\mathcal{L}_\sigma = i\eta^* \psi - i\eta \psi^*. \tag{73}$$

When $\mathbf{t}$ and $\mathbf{g}$ rotates around $\mathbf{n}$ with $\kappa_g^{(\varsigma)}(\sigma)$, a geometric phase $\mathcal{P} = \int\limits_{\sigma_0}^{\sigma_1} \kappa_g^{(\varsigma)} d\sigma$ arises between $\mathbf{t}$, $\mathbf{g}$ and corresponding nonrotating Darboux triad in $\mathcal{E}^3$. When the linearized light wave travelling moves from u_1 to u_2 along the curve in optic fiber, a geometric phase $\mathcal{P} = \int\limits_{u_1}^{u_2} \kappa_g^{(o)} du$ develops between natural Darboux triad and nonrotating Darboux triad in $\mathcal{E}^3$. The rotation angles of polarization plane can be given by

$$\mathcal{P}_1 = \kappa_g^{(\varsigma)}(\sigma, u)\Delta\sigma + \kappa_g^{(o)}(\sigma + \Delta\sigma, u)\Delta u$$

$$\mathcal{P}_2 = \kappa_g^{(o)}(\sigma, u)\Delta u + \kappa_g^{(\varsigma)}(\sigma, u + \Delta u)\Delta\sigma.$$

Phase difference are given as $\delta\mathcal{P} = \mathcal{P}_1 - \mathcal{P}_2 = \mathcal{AD}_3(\sigma, u)\Delta\sigma\Delta u$, where $\mathcal{AD}_3 = (\kappa_{g\sigma}^{(\varsigma)} - \kappa_{gu}^{(o)})$ is third anholonomy density measure for polarization plane of linearized light wave travelling along optic fiber for third class in $\mathcal{E}^3$. Also

$$\eta = -\frac{(j + iz)}{\sqrt{2}} e^{i\int^{\sigma_1} \kappa_g^{(\varsigma)} d\sigma'} \tag{74}$$

satisfies Eqs.(70), (71) and (73). The time evolution of Darboux triad is given by [22]

$$\mathbf{t}_u = \varsigma_3^{(o)} \times \mathbf{t} = \kappa_g^{(o)}\mathbf{g} - jn, \tag{75}$$

$$\mathbf{g}_u = \varsigma_3^{(o)} \times \mathbf{g} = -\kappa_g^{(o)}\mathbf{t} - z\mathbf{n}, \tag{76}$$

$$\mathbf{n}_u = \varsigma_3^{(o)} \times \mathbf{n} = j\mathbf{t} + z\mathbf{g} \tag{77}$$

where $\varsigma_3^{(o)} = A_3\mathbf{t} + B_3\mathbf{g} + \kappa_g^{(o)}\mathbf{n}$, $z = -A_3$, $j = B_3$. Using Eqs. (66), (74) it can be obtained

$$\mathcal{L}_\sigma = j\tau_g^{(\varsigma)} - \kappa_n^{(\varsigma)} z. \tag{78}$$

The time evolution of Darboux triad for third class connected with the NLS equation is given by

$$\mathbf{n}_u = j\mathbf{t} + z\mathbf{g} \tag{79}$$

$$\mathbf{t}_u = -j\mathbf{n} - \left(\int^{\sigma_1} \kappa_{gu}^{(\varsigma)} d\sigma' - \mathcal{L}\right)\mathbf{g} \tag{80}$$

$$\mathbf{g}_u = \left(\mathcal{L} - \int^{\sigma_1} \kappa_{gu}^{(\varsigma)} d\sigma'\right)\mathbf{t} - z\mathbf{n}. \tag{81}$$

The anholonomy density $\mathcal{AD}_3$ for third class with respect to Darboux frame in Euclidean 3-space:

$$\mathcal{AD}_3(\sigma, u) = -\mathcal{L}_\sigma = \kappa_n^{(\varsigma)} z - j\tau_g^{(\varsigma)}. \tag{82}$$

and the total phase $\mathcal{P}$ for third class with respect to Darboux triad in $\mathcal{E}^3$ is given by

$$\begin{aligned}\mathcal{P} &= -\int_{u_1}^{u_2}\int_{\sigma_0}^{\sigma_1} \mathcal{L}_\sigma d\sigma du \\ &= \int_{u_1}^{u_2}\int_{\sigma_0}^{\sigma_1} (\kappa_n^{(\varsigma)} z - j\tau_g^{(\varsigma)}) d\sigma du = \int_{u_1}^{u_2}\int_{\sigma_0}^{\sigma_1} \langle \mathbf{n}, \mathbf{n}_\sigma \times \mathbf{n}_u \rangle \, d\sigma du.\end{aligned}$$

The quantum geometric phase is given

$$\mathcal{P} = i\int_{\sigma_0}^{\sigma_1} d\sigma \frac{\partial}{\partial\sigma} \int_{u_1}^{u_2} \langle \mathbf{R}_{2u}, \mathbf{R}_2^* \rangle \, du.$$

A optical fiber can be described by a curve $\Gamma_3(\sigma)$ with respect to Darboux frame in $\mathcal{E}^3$. The direction of electric field $\mathbf{E}_3$ denotes the direction of the state of the linearly polarized light wave injected to fiber with respect to Darboux frame in $\mathcal{E}^3$. The change of the electric field $\mathbf{E}_3$ with respect to Darboux triad can be written by

$$\mathbf{E}_{3\sigma} = \pi_1 \mathbf{t} + \pi_2 \mathbf{g} + \pi_3 \mathbf{n}. \tag{83}$$

Case 3. Assume that

$$\langle \mathbf{E}_3, \mathbf{n} \rangle = 0. \tag{84}$$

From Eq.(84)

$$\langle \mathbf{E}_{3\sigma}, \mathbf{n} \rangle = \langle \mathbf{E}_3, \kappa_n \mathbf{t} + \tau_g \mathbf{g} \rangle$$

$$\pi_3 = \kappa_n \langle \mathbf{E}_3, \mathbf{t} \rangle + \tau_g \langle \mathbf{E}_3, \mathbf{g} \rangle \tag{85}$$

Also

$$\langle \mathbf{E}_3, \mathbf{E}_3 \rangle = const. \tag{86}$$

Using Eq.(83) and taking derivative with respect to σ of Eq.(86), it can be obtained

$$\pi_1 \langle \mathbf{E}_3, \mathbf{t} \rangle = -\pi_2 \langle \mathbf{E}_3, \mathbf{g} \rangle \tag{87}$$

$$\pi_1 = \epsilon \langle \mathbf{E}_3, \mathbf{g} \rangle, \quad \pi_2 = -\epsilon \langle \mathbf{E}_3, \mathbf{t} \rangle \tag{88}$$

where ϵ is a parameter. The evolution in the polarization of light wave travelling from the point $\Gamma_3(\sigma_0)$ to $\Gamma_3(\sigma_1)$ along curve with respect to Darboux triad is given by the evolution of the electric field $\mathbf{E}_3$. $\langle \mathbf{E}_3, \mathbf{t} \rangle \neq 0$, $\langle \mathbf{E}_3, \mathbf{n} \rangle \neq 0$. Substituting Eqs. (85) and (88) in (83), the Eq.(83) is rewritten by

$$\mathbf{E}_{3\sigma} = \epsilon \langle \mathbf{E}_3, \mathbf{g} \rangle \mathbf{t} - \epsilon \langle \mathbf{E_2}, \mathbf{t} \rangle \mathbf{g} + (\kappa_n \langle \mathbf{E}_3, \mathbf{t} \rangle + \tau_g \langle \mathbf{E}_3, \mathbf{g} \rangle) n \tag{89}$$

Via Eq.(89) for $\epsilon = 0$,

$$\mathbf{E}_{3\sigma} = (\kappa_n \langle \mathbf{E}_3, \mathbf{t} \rangle + \tau_g \langle \mathbf{E}_3, \mathbf{g} \rangle) n \tag{90}$$

The modified Fermi-Walker derivative for the electric field $\mathbf{E}_3$ with respect to Darboux triad for third class is described by

$${}^{DmFW}\mathbf{E}_{3\sigma} = \mathbf{E}_{3\sigma} - \langle \mathbf{n}, \mathbf{E}_2 \rangle \mathbf{n}_\sigma + \langle \mathbf{n}_\sigma, \mathbf{E}_2 \rangle \mathbf{n} \tag{91}$$

The electric field $\mathbf{E}_3$ is the Fermi-Walker parallel if and only if

$${}^{DmFW}\mathbf{E}_{3\sigma} = 0. \tag{92}$$

Via (84), (91), (92), one obtains $\mathbf{E}_{3\sigma} = \langle \mathbf{n}_\sigma, \mathbf{E}_2 \rangle \mathbf{n}$.

The electric field vector $\mathbf{E}_3$ with respect to the Darboux triad apparatus $\mathbf{t}$ and $\mathbf{g}$ can be written by

$$\mathbf{E}_3(\sigma) = \Sigma(\sigma) \frac{(\mathbf{t} + i\mathbf{g})}{\sqrt{2}} + \Sigma^*(\sigma) \frac{\mathbf{t} - i\mathbf{g}}{\sqrt{2}}. \tag{93}$$

where $\mathbf{E}_3\mathbf{E}_3^* = 1$ and $|\Sigma(\sigma)|^2 + |\Sigma^*(\sigma)|^2 = 1$. Here

$$\Sigma(\sigma) = e^{i\int^{\sigma_1} \kappa_g^{(\varsigma)} d\sigma'}\Sigma(\sigma_0),\ \Sigma^*(\sigma) = e^{-i\int^{\sigma_1} \kappa_g^{(\varsigma)} d\sigma'}\Sigma^*(\sigma_0). \tag{94}$$

The polarization coefficients are

$$\Sigma(\sigma_0) = \left(\frac{\mathbf{t} + i\mathbf{g}}{\sqrt{2}}\right)^* \mathbf{E}_3(\sigma_0)$$

$$\Sigma^*(\sigma_0) = \left(\frac{\mathbf{t} - i\mathbf{g}}{\sqrt{2}}\right)^* \mathbf{E}_3(\sigma_0).$$

Eq.(93) is re-expressed as the following

$$\mathbf{E}_3(\sigma) = \mathbf{R}_2\Sigma(\sigma_0) + \mathbf{R}_2^*\Sigma^*(\sigma_0) \tag{95}$$

When taking derivative with respect to σ and the time u of Eq. (95), the spatial and temporal evolutions of the electric field $\mathbf{E}_3$ for Darboux triad are derived as follows

$$\begin{aligned}
\mathbf{E}_{3\sigma} &= \mathbf{R}_{2\sigma}\Sigma(\sigma_0) + \mathbf{R}_{2\sigma}^*\Sigma^*(\sigma_0) \\
\mathbf{E}_{3u} &= \mathbf{R}_{2u}\Sigma(\sigma_0) + \mathbf{R}_{2u}^*\Sigma^*(\sigma_0)
\end{aligned}$$

From compatibility condition $\mathbf{E}_{3\sigma u} = \mathbf{E}_{3u\sigma}$, the nonlinear Schrödinger equation NLS system connected with the electric field $\mathbf{E}_3$ is obtained.

$$\mathcal{P} = \int^{\sigma_1} \kappa_g^{(\varsigma)} d\sigma'$$

is the change phase of the polarization light injected into a fiber for third case of the electric field with respect to Darboux frame in $\mathcal{E}^3$. Consider the Lorentz force equation $\Phi^{(n)}$ for third case of the electric field vector

$$\begin{aligned}
\Phi^{(n)}\mathbf{E}_3 &= \mathbf{E}_{3\sigma} = \mathbf{M}^{(n)} \times \mathbf{E}_3, \\
\left\langle \Phi^{(n)}\mathbf{E}_3, \mathbf{t} \right\rangle &= -\left\langle \mathbf{E}_3, \Phi^{(n)}\mathbf{t} \right\rangle, \left\langle \Phi^{(n)}\mathbf{E}_3, \mathbf{g} \right\rangle = -\left\langle \mathbf{E}_3, \Phi^{(n)}\mathbf{g} \right\rangle \\
\left\langle \Phi^{(n)}\mathbf{E}_3, \mathbf{n} \right\rangle &= -\left\langle \mathbf{E}_3, \Phi^{(n)}\mathbf{n} \right\rangle.
\end{aligned}$$

The trajectory of travelling particle along the magnetic field $\mathbf{M}^{(n)}$ with respect to Darboux frame is described as the electromagnetic trajectory. If the curve $\mathbf{DEM}^{(n)}$ follows the magnetic trajectory, it is described as the Darboux electromagnetic curve. With the help of Eq. (61) the Darboux Lorentz force equations along the optic fiber for third case the electric field are obtained

$$\begin{bmatrix} \Phi^{(n)}(\mathbf{t}) \\ \Phi^{(n)}(\mathbf{g}) \\ \Phi^{(n)}(\mathbf{n}) \end{bmatrix} = \begin{bmatrix} 0 & -\epsilon & \kappa_n^{(\varsigma)} \\ \epsilon & \tau_g^{(\varsigma)} & 0 \\ -\kappa_n^{(\varsigma)} & -\tau_g^{(\varsigma)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{g} \\ \mathbf{n} \end{bmatrix} \tag{96}$$

$\mathbf{DEM}^{(n)}$ curve of the Γ_3 is the magnetic trajectory of the magnetic field $\mathbf{M}^{(n)}$ iff the vector field divergence free $\mathbf{M}^{(n)}$ is given by

$$\mathbf{M}^{(n)} = -\kappa_n^{(\varsigma)}\mathbf{g} - \epsilon\mathbf{n} + \tau_g^{(\varsigma)}\mathbf{t}.$$

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