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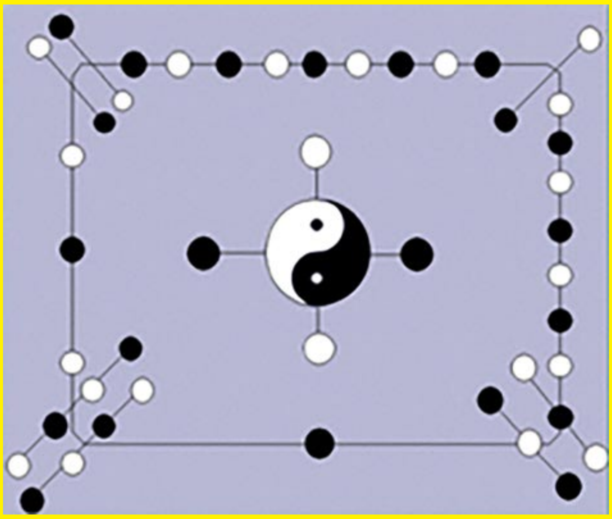

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September, 2021

Aims and Scope: The *mathematical combinatorics* is a subject that applying combinatorial notion to all mathematics and all sciences for understanding the reality of things in the universe, motivated by *CC Conjecture* of Dr.Linfan MAO on mathematical sciences. The **International J.Mathematical Combinatorics** (*ISSN 1937-1055*) is a fully refereed international journal, sponsored by the *MADIS of Chinese Academy of Sciences* and published in USA quarterly, which publishes original research papers and survey articles in all aspects of mathematical combinatorics, Smarandache multi-spaces, Smarandache geometries, non-Euclidean geometry, topology and their applications to other sciences. Topics in detail to be covered are:

Mathematical combinatorics;

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Topological graphs; Algebraic graphs; Random graphs; Combinatorial maps; Graph and map enumeration; Combinatorial designs; Combinatorial enumeration;

Differential Geometry; Geometry on manifolds; Low Dimensional Topology; Differential Topology; Topology of Manifolds;

Geometrical aspects of Mathematical Physics and Relations with Manifold Topology;

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Famous Words:

We must accept finite disappointment, but we never lose infinite hope.

By Mattin Luther King, a leader of the American civil rights

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Reality with Smarandachely Denied Axiom

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Abstract: Usually, one applies mathematics to hold on the reality of matters in the universe, i.e., mathematical reality and a few peoples firmly believe that classical mathematics could handle this things with no needs on its extension. However, it is not the case because contradictions exist everywhere but classical mathematics must be logically consistent without contradiction in the eyes of human beings. The fatal flaw in this view is it's priori assumption that the universe is uniform, and then can be characterized by homogeneous characters with classical mathematics such as differential equations. However, the universe including its matters are not uniform, even being messy. This fact implies that one should extend classical mathematics to an enveloping one for understanding matters in the universe and bearing maybe with contradictions, i.e., such systems in mathematics including with Smarandachely denied axioms. Certainly, an axiom is said Smarandachely denied if the axiom behaves differently, i.e., validated and invalided, or only invalidated but in at least two distinct ways in a system S. Such a system S is said Smarandace system. The main purpose of this paper is to introduce the Smarandachely denied axiom, show its contribution to reality and explain the role of its equivalent form, the Smarandache multispace for extending classical mathematics, i.e., mathematical combinatorics for understanding matters because each matter always inherits a topological structure by its nature in the universe.

Key Words: Reality, mathematical reality, CC conjecture, mathematical universe hypothesis, Smarandachely denied axiom, Smarandache system, Smarandache multispace, mathematical combinatorics.

AMS(2010): 03A05,03A10,05C22,51-02,70-02.

§1. Introduction

Usually, all matter in the universe are colorful, maybe with mystery and complex mechanism to the human eyes no matter it is living or not. We understand matters by the reality for promoting the survival and development of humans ourselves in harmony with nature. However, we are embarrassed hardly know their true face unless their surface characters before humans and even so, it could be also a false vision or hallucination, just feelings of humans. Then, *what is the reality of a matter*? The word *reality* of a matter $\mathcal{T}$ is its state as it actually exist, including everything that is and has been, no matter it is observable or comprehensible by humans. For

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humans ourselves, a natural question is *could we really hold on the reality of matters in the universe?* It should be noted that the answer is different in the scientific and the religious. For examples, *all matters are illusion of humans* claimed by Sakyamuni in his famous *Diamond Sutra* and *the universal truth can be restated but the restated truth is not the universal one* asserted by Laozi in his *Tao Te Ching*. However, nearly all scientists firmly believe that we can open the cover enveloped on matters, find their true face and then, hold on the natural laws of universe. Here, we do not discuss who's right or who's wrong. Even if we can really open the cover on matters in the universe, there is also a question on reality, i.e., *how to characterize the reality of matters?* The answer is nothing else but the science or particularly, the mathematical sciences.

We usually understand a matter by its characters or the system S of characters by knowing the state $x_v(t)$ on time t for elements $v \in S$. However, we can only observe the state, estimate and calculate the change rate of its elements on time t. Thus, we can not obtain directly the state x_v for $v \in S$ but a system of differential equations. We have to solve the differential equation system for hold on the states $x_v(t), v \in S$.

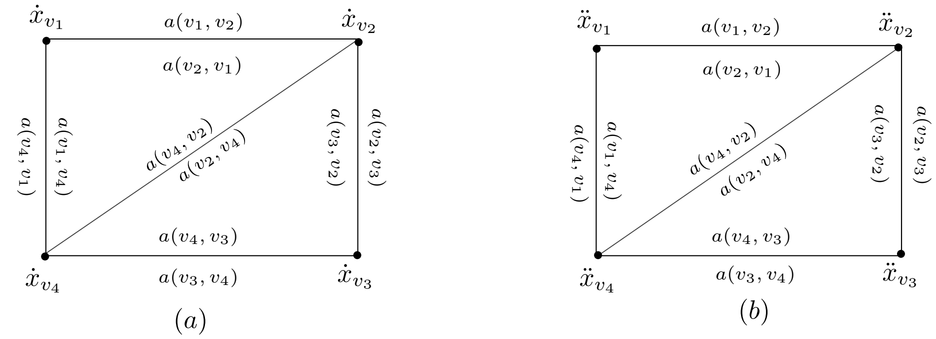

Figure 1. System States

For example, the differential equations of a 4-element system $S' = \{v_1, v_2, v_3, v_4\}$ shown in Figure 1(a) and (b) are respectively

$$\left\{\begin{array}{l} \dot{x}_{v_1} = F_1\left(a\left(v_2, v_1\right), a\left(v_4, v_1\right)\right) \\ \dot{x}_{v_2} = F_2\left(a\left(v_1, v_2\right), a\left(v_3, v_2\right), a\left(v_4, v_2\right)\right) \\ \dot{x}_{v_3} = F_3\left(a\left(v_2, v_3\right), a\left(v_4, v_3\right)\right) \\ \dot{x}_{v_4} = F_4\left(a\left(v_1, v_4\right), a\left(v_3, v_4\right), a\left(v_2, v_4\right)\right) \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{l} \ddot{x}_{v_1} = H_1\left(a\left(v_2, v_1\right), a\left(v_4, v_1\right)\right) \\ \ddot{x}_{v_2} = H_2\left(a\left(v_1, v_2\right), a\left(v_3, v_2\right), a\left(v_4, v_2\right)\right) \\ \ddot{x}_{v_3} = H_3\left(a\left(v_2, v_3\right), a\left(v_4, v_3\right)\right) \\ \ddot{x}_{v_4} = H_4\left(a\left(v_1, v_4\right), a\left(v_3, v_4\right), a\left(v_2, v_4\right)\right) \end{array}\right. \quad (1)$$

and then, we subjectively equate the state of $S'(t)$ with the solutions $x_{v_1}(t), x_{v_2}(t), x_{v_3}(t), x_{v_4}(t)$ of differential equations (1) without even knowing it maybe not true, where, x_{v_i} is the character of element v_i of the matter S, $a\left(v_i, v_j\right)$ is the action of element v_i on v_j, F_i, H_i are action functions for integers $1 \leq i, j \leq 4$, and $\dot{x}, \ddot{x}$ denote the first or second differentials of x on time t. It seems that a solution of differential equations (1) is a state of system S' because they are in causality. However, *is the converse is true also, i.e., any state of $S'(t)$ can be characterized by $x_{v_i}(t), 1 \leq i \leq 4$?* The answer is inconclusive because the solution $x_{v_i}(t), 1 \leq i \leq 4$ is the state of $S'(t)$ if and only if equations (1) are solvable and there is a bijection $\phi : S'(t) \leftrightarrow \{x_{v_1}(t), x_{v_2}(t), x_{v_3}(t), x_{v_4}(t)\}$, but we can not conclude so unless subjectively in mind on classical mathematics. Certainly, a mathematical system should be logically consistent without contradiction but lots of humans misunderstand this criterion, excluded contradictory systems in mathematics, which results in the limitation of mathematics on reality of matters. It should be noted that the most important thing is not excluded contradictions but how to let them coexist peacefully in mathematics for extending the limitation of classical mathematics and establish an envelope mathematics, in which classical mathematics only be its parts for understanding matters in the universe. For this objective, the Smarandachely denied axiom is a such one presented by F.Smarandache on geometry in 1969 following ([37],[40-41]).

Axiom 1.1 *An axiom is said Smarandachely denied if in the same space the axiom behaves differently, i.e., validated and invalided, or only invalidated but in at least two distinct ways.*

By Axiom 1.1, there are Smarandachely conceptions on geometry following.

Definition 1.2([16],[37]) *A Smarandache geometry is such a geometry that has at least one Smarandachely denied axiom.*

A conception closely related to Smarandache geometry is the Smarandache multispace defined in the following, which seems to be a generalization of Smarandache geometry but equivalent to Smarandache geometry by a geometrical view.

Definition 1.3([17],[37]) *Let* $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ *be m mathematical spaces, different two by two, i.e., for any two spaces* $(\Sigma_i;\mathcal{R}_i)$ *and* $(\Sigma_j;\mathcal{R}_j)$, $\Sigma_i \neq \Sigma_j$ *or* $\Sigma_i = \Sigma_j$ *but* $\mathcal{R}_i \neq \mathcal{R}_j$. *A Smarandache multispace* $\widetilde{\Sigma}$ *is a union* $\bigcup\limits_{i=1}^{m} \Sigma_i$ *with rules* $\widetilde{\mathcal{R}} = \bigcup\limits_{i=1}^{m} \mathcal{R}_i$ *on* $\widetilde{\Sigma}$, *i.e., the union of rules* $\mathcal{R}_i$ *on* Σ_i *for integers* $1 \leq i \leq m$, *denoted by* $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$.

The Smarandache multispace inherits a topological structure G^L with a generalization, i.e., continuity flow consisting of the element in mathematical combinatorics.

Definition 1.4([11-12]) *For an integer* $m \geq 1$, *let* $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ *be a Smarandache multispace consisting of m mathematical systems* $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$. *An inherited topological structure* $G^L\left[\widetilde{\Sigma};\widetilde{\mathcal{R}}\right]$ *of* $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ *is a labeled topological graph defined following:*

$$V\left(G^L\left[\widetilde{\Sigma};\widetilde{\mathcal{R}}\right]\right) = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_m\},$$

$$E\left(G^L\left[\widetilde{\Sigma};\widetilde{\mathcal{R}}\right]\right) = \{(\Sigma_i, \Sigma_j)\,|\Sigma_i \bigcap \Sigma_j \neq \emptyset,\ 1 \leq i \neq j \leq m\} \text{ with labeling}$$

$$L:\ \Sigma_i \to L(\Sigma_i) = \Sigma_i \quad \text{and} \quad L:\ (\Sigma_i, \Sigma_j) \to L(\Sigma_i, \Sigma_j) = \Sigma_i \bigcap \Sigma_j$$

for integers $1 \leq i \neq j \leq m$, *such as those shown in Figure* 2 *for the case of* $m = 4$ *and* $G \simeq K_4$.

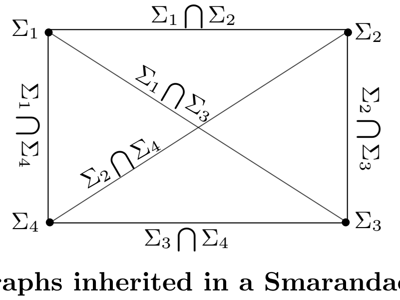

Figure 2. Graphs inherited in a Smarandache multispace

Now, *what is the contribution of Axiom* 1.1 *in extending of classical mathematics and what is its role with the reality?* The main purpose of this paper is to introduce the Smarandachely denied axiom, survey its contribution to geometry, and then from Smarandache multispace to mathematical combinatorics for extending classical mathematics to mathematical combinatorics for understanding the reality of matters in the universe because each matter always inherits a topological structure by its nature.

For terminologies and notations not mentioned here, we follow reference [4] for algebra, [5] for topological graphs, [16-18] and [37-38] for Smarandache geometry, multispaces and Smarandache systems.

§2. Smarandachely Denied Axiom to Geometry

Notice that the Smarandachely denied axiom is originally presented on geometry, which enables one to generalize geometry to Smarandache geometry concluding classical geometry as its parts. In a Smarandache geometry, the points, lines, planes, spaces, triangles, $\cdots$ are respectively called s-points, s-lines, s-planes, s-spaces, s-triangles, $\cdots$ in order to distinguish them from those in classical geometry. Although it is defined by Definition 1.2, an elementary but natural question is shown in the following.

Question 2.1 *Are there non-trivial Smarandache geometry constraint in logic?*

The answer is certainly *Yes*! For example, the axiom system of Euclidean geometry consists of 5 axioms: ① There is a straight line between any two points; ② A finite straight line can produce a infinite straight line continuously; ③ Any point and a distance can describe a circle; ④ All right angles are equal to one another; ⑤ If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles. Then, we are easily construct a Smarandache geometry following.

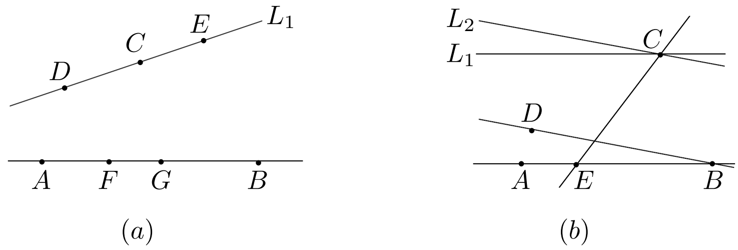

Figure 3. A non-trivial Smarandache geometry

Let $\mathbb{R}^2$ be a Euclidean plane and let A, B, C be three non-collinear points in $\mathbb{R}^2$. We define s-points $\mathcal{P}_s$ to be all usual Euclidean points in $\mathbb{R}^2$ and s-lines $\mathcal{L}_s$ to be any Euclidean line passing through one and only one of points A, B, C. We show the pair $\{P_s; L_s\}$ consists of a Smarandache geometry: 1) The axiom ① in Euclidean geometry is now replaced by *one s-line* or *no s-line* because through any two distinct s-points D, E collinear with one of A, B, C there is one s-line passing through them but through any two distinct s-points F, G lying on AB or non-collinear with one of A, B, C, there are no s-lines passing through them such as those shown in Figure 3(a); 2) The axiom ⑤ is now replaced by *one parallel* or *no parallel* because if we let L_1, L_2 be two s-lines passing through C with L_1 parallel but L_2 not parallel to AB in the Euclidean sense, then through any s-point D not lying on AB there are no s-lines parallel to L_1 but there is one s-lines parallel to L_2 if one of DB,DA or DC happens parallel to L_2. Otherwise, there are no s-lines passing through D parallel to L_2, see Figure 3(b) for details.

More examples of Smarandache geometry can be found in [3], [6-7], [10] and [40-41]. Certainly, the quantitative characterization on the real face of matters lead to the neutrosophic logic by Smarandachely denied axiom, which contributes the introduction of the degree of negation or partial negation of an axiom and, more general, of a scientific or humanistic proposition (theorem, lemma, etc.) in any field. It works somehow like the negation in fuzzy logic with a degree of truth, a degree of falsehood and a degree of truth, i.e. neither truth nor falsehood but unknown, ambiguous, indeterminate (see [40-41] for details).

As a particular case, the Euclidean, Lobachevsky-Bolyai-Gauss and Riemannian geometries may be united altogether by the Smarandache geometry in the same space because it can be partially Euclidean and partially non-Euclidean, and it seems connecting with the *relativity theory* and *parallel universes* because it includes the Riemannian geometry in a subspace but more generalized. H.Iseri [6-7] constructed the Smarandache 2-manifolds by using equilateral triangular disks on Euclidean plane $\mathbb{R}^2$, which can be come true by paper models in $\mathbb{R}^3$ for elliptic, Euclidean and hyperbolic cases and in paper [10], L.Mao advanced a new method for constructed Smarandache 2-manifold by combinatorial maps. Generally, it should be noted that Smarandache n-manifold for $n \geq 2$, i.e. combinatorial n-manifold and a differential theory on such manifolds were constructed by L.Mao in papers [11]. For $n = 1$, i.e., a curve in differential geometry is called *Smarandache curve* if it holds with Smarandachely denied axiom, which has been extensively researched and many researching, such as those of papers [2],[9],[36],[42]-[47], [50] were published in the *International Journal of Mathematical Combinatorics* after it suggested by L.Mao for the authors of [44]. In fact, nearly all geometries in classical mathematics such as those of Riemann geometry, Finsler geometry, Weyl geometry and Kahler geometry are particular cases of Smarandache geometry.

§3. Smarandachely Denied Axiom to Mathematical Systems

Although the Smarandachely denied axiom is originally to geometry for generalizing the 5th axiom of Euclidean geometry. Its notion can be generalized further to a generalized form on all mathematical systems by replacing the word *space* with *mathematical system* following.

Axiom 3.1(Generalized Smarandachely denied axiom) *An axiom is said generalized Smarandachely denied if in the same mathematical system the axiom behaves differently, i.e., validated and invalided, or only invalidated but in at least two distinct ways and then, a Smarandache system is such a mathematical system that has at least one Smarandachely denied axiom.*

For example, a Lie group G in classical mathematics is a Smarandachely denied system because an element $a \in G$ is both a point on manifold G, also an element in group G. Thus, if we let Axiom I be an axiom that points have no neighborhood $\mathcal{C}$ on G, Axiom II be that

a^{-1} is not exist for $\forall a \in G$ and Axiom III to be that the group operations $(a,b) \to a \cdot b$ or $a \to a^{-1}$ are not C^∞-mapping. Then, Axioms I, II and III are invalidated in a Lie group, which implies G is a Smarandache system and in general, the result following can be verified.

Proposition 3.2 *Let* $(\mathcal{S}_1, \mathcal{R}_1), (\mathcal{S}_2, \mathcal{R}_2), \cdots, (\mathcal{S}_n, \mathcal{R}_n)$ *be* n *systems in classical mathematics with* $\mathcal{S}_i \neq \mathcal{S}_j$ *or* $\mathcal{S}_i = \mathcal{S}_j$ *but* $\mathcal{R}_i \neq \mathcal{R}_j$ *for an integers* $n \geq 2$*, where* $\mathcal{S}_i$ *is a set,* $\mathcal{R}_i \subset \mathcal{S}_i \times \mathcal{S}_i$ *for integers* $1 \leq i \leq n$*. Then, the union*

$$(\Sigma; \Pi) = (\mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots \cup \mathcal{S}_n; \mathcal{R}_1 \cup \mathcal{R}_2 \cup \cdots \cup \mathcal{R}_n)$$

is a Smarandache system.

Proof Similarly to the case of Lie group, define Axiom i to be $(a,b) \notin \mathcal{R}_i$ for $\forall a, b \in \mathcal{S}_i$ in $(\Sigma; \Pi)$ if $(a,b) \in \mathcal{R}_i$ in $(\mathcal{S}_i, \mathcal{R}_i)$ for integers $1 \leq i \leq n$. Then, each Axiom i is invalidated in $(\Sigma; \Pi)$, i.e., invalidated in at least two distinct ways. □

Notice that the Smarandache system $\left(\widetilde{\mathcal{S}}; \widetilde{\mathcal{R}}\right)$ in Proposition 3.2 is in fact a Smarandache multispace by Definition 1.3, appearing not only in mathematics but also in physics, for instance the unmatter composed of particles and anti-particles [39], and generally, the generalized Smarandachely denied axiom is equivalent to Smarandache multispace.

Proposition 3.3 *A mathematical system* (Σ, Π) *is generalized Smarandachely denied if and only if it is a Smarandache multispace.*

Proof The proof on necessity of the result is similar to the decomposition shown in [22], divided into two cases following.

Case 1. There is an axiom $\mathscr{A}$ in (Σ, Π) that behaves both validated and invalided. Define

$$\Sigma_1 = \{x \in \Sigma \text{ hold with Axiom } \mathscr{A}\}, \quad \Sigma_2 = \{y \in \Sigma \text{ hold not with Axiom } \mathscr{A}\}.$$

Then, $\Sigma = \Sigma_1 \bigcup \Sigma_2$, i.e., (Σ, Π) is a Smarandache multispace.

Case 2. There is an axiom in $\mathscr{A}$ in (Σ, Π) that behaves invalidated but in distinct ways $W_1, W_2, \cdots, W_s, s \geq 2$. Define $\Sigma_i = \{x \in \Sigma \text{ hold not with Axiom } \mathscr{A} \text{ in way } W_i\}$ for integers $1 \leq i \leq s$ and $\Sigma_0 = \Sigma \setminus \bigcup\limits_{i=1}^{s} \Sigma_i$, where Σ_0 maybe an empty set. Then

$$\Sigma = \left(\bigcup_{i=1}^{n} \Sigma_i\right) \bigcup \Sigma_0 \tag{3}$$

is a Smarandache multispace.

The proof on sufficiency of the result is similar to the proof of Proposition 3.2. Let (Σ, Π) be a Smarandache multispace with $\Sigma = \Sigma_1 \bigcup \Sigma_2 \bigcup \cdots \bigcup \Sigma_n$, $\Pi = \mathcal{R}_1 \bigcup \mathcal{R}_2 \bigcup \cdots \bigcup \mathcal{R}_n$ and $(\Sigma_i; \mathcal{R}_i)$ being a mathematical space. Define Axiom $\mathcal{A}_i = \{x \notin \Sigma_i \text{ if } x \in \Sigma_i\}$ for integers $1 \leq i \leq n$. Then, each axiom $\mathcal{A}_i$ in Σ behaves both validated, invalided, and also invalidated in n distinct ways. □

Notice that there are many achievements of Smarandache multispaces in extending classical mathematics with applying to physics. For example, multigroups, multirings, multifields, multialgebra and multistructure,$\cdots$, etc. discussed in [1], [14]-[15], [17] and [48]-[49].

§4. Smarandachely Denied Axiom to Reality

4.1. Mathematical Reality. Proposition 3.3 enables one to discuss the reality of matters in the universe by generalized Smarandachely denied axiom. Particularly, the reality discussed by differential equations in physics, i.e., the mathematical reality is the reality on a matter or not. The conclusion following is a little surprising for the usual view on classical mathematics.

Proposition 4.1 *If $\mathscr{R}_M$, $\mathscr{R}$ are respectively the mathematical reality or the reality of a matter in the universe. Then $\mathscr{R}_M \subseteq \mathscr{R}$ and furthermore, there are many examples hold with $\mathscr{R}_M \neq \mathscr{R}$.*

Proof The relation $\mathscr{R}_M \subseteq \mathscr{R}$ is obvious. For the inequality $\mathscr{R}_M \neq \mathscr{R}$, many cases show the mathematical reality is not the reality, even the ridiculous on reality of a matter in sometimes, for instance the equations (1) are non-solvable, i.e., we can not obtain the state of system $S'(t)$. Even the equations (1) are solvable, we can not conclude their solutions describing the state of system S'. For example, let H_1, H_2, H_3, H_4 and H_1', H_2', H_3', H_4' be two groups of running horses constraint with running on respectively 4 straight lines

$$
① \begin{cases} x+y &= 2 \\ x+y &= -2 \\ x-y &= -2 \\ x-y &= 2 \end{cases} \qquad \text{or} \qquad ② \begin{cases} x=y \\ x+y=2 \\ x=1 \\ y=1 \end{cases}
$$

on the Euclidean plane $\mathbb{R}^2$ such as those shown in Figure $4(a), (b)$ and (c).

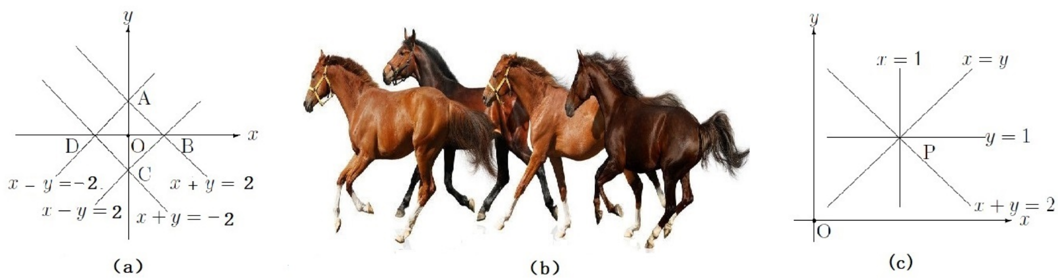

Figure 4. Four horses on the plane

Clearly, the first system is non-solvable because $x + y = -2$ is contradictious to $x + y = 2$, and so that for the equation $x - y = -2$ to $x - y = 2$ but the second system is solvable with $(x, y) = (1, 1)$. *Could we conclude that the running states of horses H_1', H_2', H_3', H_4' are a point $(1, 1)$, i.e., staying on the point $(1, 1)$ and H_1, H_2, H_3, H_4 are nothing*? The answer is certainly not because all of the horses are running on the Euclidean plane $\mathbb{R}^2$. However, we know nothing on the state by the solution of the two equation systems because the solvability of systems ① or ② only implies the orbits intersection, not the running state of the group of horses. □

Then, *what is wrong in the example of the two groups of horses?* The wrong appears with the assumption that the solution of ① or ② characterizes the running state of horses

H'_1, H'_2, H'_3, H'_4 or H_1, H_2, H_3, H_4. Certainly, each equation in systems ① or ② really characterizes the running state of one horse but we can not equate the running states of the four horses with the solution of ① and ② because they are different in objective, and maybe contradictory. Generally, there is contradiction maybe if we characterize a group of matters within a same space. Indeed, we can eliminate the contradiction by characterizing them with different variables of spaces, for instance in parallel spaces one by one. However, they are really a system with relations on its elements, we should know their global state of the system, not isolated one on its elements in the Euclidean plane $\mathbb{R}^2$. This case implies also that the non-solvable systems of equations characterize matters also if each of them was established on the characters of matters in the universe but the solution is not the state of the matter when it consists of characters more than 2, which implies the classical mathematics is incomplete for understanding the reality of matters, particularly, the biological systems in the universe.

Notice that a cosmologist, Max Tegmark proposed a hypothesis on the universe once, called the *mathematical universe hypothesis* following, spread widely in the scientific community.

Hypothesis 4.2(Max Tegmark,[47]) *The physical universe is not merely described by mathematics but a mathematical structure, i.e.,* $\mathscr{R}_M = \mathscr{R}$.

Certainly, the mathematical universe hypothesis is essentially a duplication of the *Theory of Everything*. However, Proposition 4.1 concludes classical mathematics is incomplete for understanding matters in the unverse, which implies that the incorrectness of the mathematical universe hypothesis, or in other words, the first step to making this hypothesis work should be the extending of classical mathematics, i.e., including contradictions, let them peaceful coexistence in mathematics, and then it maybe set up on the universe.

4.2. Non-Solvable Equation System. Certainly, each linear equation $ax + by = c$ with $ab \neq 0$ is in fact a point set $L_{ax+by=c} = \{(x, y)|ax + by = c\}$ in $\mathbb{R}^2$, such as those shown in Figure 4(a) and (c) for the linear systems ① and ② with

$$L_{x+y=2} \bigcap L_{x+y=-2} \bigcap L_{x-y=2} \bigcap L_{x-y=-2} = \emptyset, \qquad L_{x=y} \bigcap L_{x+y=2} \bigcap L_{x=1} \bigcap L_{y=1} = \{(1, 1)\}$$

in the Euclidean plane $\mathbb{R}^2$. Generally, the solution manifold of an equation

$$\mathscr{F}(x_1, x_2, \cdots, x_n, y) = 0, \ n \geq 1 \tag{4}$$

is defined to be an n-manifold $S_{\mathscr{F}} = (x_1, x_2, \cdots, x_n, y(x_1, x_2, \cdots, x_n)) \subset \mathbb{R}^{n+1}$ if it is solvable. Otherwise, $\emptyset$ in geometry and a system

$$(ES_m) \quad \begin{cases} \mathscr{F}_1(x_1, x_2, \cdots, x_n, y) = 0 \\ \mathscr{F}_2(x_1, x_2, \cdots, x_n, y) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mathscr{F}_m(x_1, x_2, \cdots, x_n, y) = 0 \end{cases} \tag{5}$$

of equations with initial values $\mathscr{F}_i(0), \ 1 \leq i \leq m$ in Euclidean space $\mathbb{R}^{n+1}$ is solvable or not dependent on $\bigcap_{i=1}^{m} S_{\mathscr{F}_i} \neq \emptyset$ or $= \emptyset$ in it geometrical meaning, where $S_{\mathscr{F}_i} \neq \emptyset$ for integers

$1 \leq i \leq m$.

Then, *what is the reality of a matter $\mathcal{T}$?* Generally, let $\mu_1, \mu_2, \cdots, \mu_n$ be known and $\nu_i, i \geq 1$ unknown characters at time t for a matter $\mathcal{T}$. Then, $\mathcal{T}$ should be understood by

$$\mathcal{T} = \left(\bigcup_{i=1}^{n} \{\mu_i\}\right) \bigcup \left(\bigcup_{k \geq 1} \{\nu_k\}\right) \qquad (6)$$

in logic but with an approximation $\mathcal{T}^\circ = \bigcup_{i=1}^{n} \{\mu_i\}$ for $\mathcal{T}$ by humans at time t, which is nothing else but the Smarandache system or multispace by Proposition 3.3. The example of 4 horses run in the plane shows that applying the solution of equations (5), i.e., $\bigcap_{i=1}^{m} S_{\mathscr{F}_i}$ to the state of a system S maybe cause a ridiculous conclusion, particularly in the case of the non-solvable. In fact, the state of a matter $\mathcal{T}$ described by the system equation (5) is not $\bigcap_{i=1}^{m} S_{\mathscr{F}_i}$ but the equality (6), i.e., Smarandache multispace. We should extend the conception of solution of equations (5).

Definition 4.3([20]) *The $\vee$-solvable, $\wedge$-solvable and non-solvable spaces of equations* (5) *are defined respectively by*

$$\bigcup_{i=1}^{m} S_{\mathscr{F}_i}, \quad \bigcap_{i=1}^{m} S_{\mathscr{F}_i} \quad \textit{and} \quad \bigcup_{i=1}^{m} S_{\mathscr{F}_i} - \bigcap_{i=1}^{m} S_{\mathscr{F}_i}.$$

What is the importance of the $\vee$-solvable and $\wedge$-solvable space? Clearly, the $\vee$-solvable space of (5) shown the state of the system characterized by (5). For example, the state of the four horses should be the $\vee$-solvable space $L_{x+y=2} \bigcup L_{x+y=-2} \bigcup L_{x-y=2} \bigcup L_{x-y=-2}$, not the $\wedge$-solvable space $L_{x+y=2} \bigcap L_{x+y=-2} \bigcap L_{x-y=2} \bigcap L_{x-y=-2}$, the usual solution for system ①, and should be the $\vee$-solvable space $L_{x=y} \bigcup L_{x+y=2} \bigcup L_{x=1} \bigcup L_{y=1}$, not the $\wedge$-solvable space $L_{x=y} \bigcap L_{x+y=2} \bigcap L_{x=1} \bigcap L_{y=1}$, the usual solution for system ②. *And under what conditions does the $\wedge$-solvable space is the same of the $\vee$-solvable space?* If we understand matters in the universe by systems S, the answer of this question dependents on the unit elements in S, i.e., the $\wedge$-solvable space is the same of the $\vee$-solvable space only if the system of equations is characterizing an unit element of S or view the behavior of S as a particle because the solution of equations is pointing to the state of the element or the particle. Otherwise, the solution of equations (5) would be pointing to unit elements more than 2, maybe contradictory without solutions in the usual meaning.

By Definition 1.4, the $\vee$-solvable space of (5) inherits a topological structure $G^L\left[\widetilde{S}_{\mathscr{F}}\right]$ with vertex set $V\left(G^L\left[\widetilde{S}_{\mathscr{F}}\right]\right)$ and edge set $V\left(G^L\left[\widetilde{S}_{\mathscr{F}}\right]\right)$ respectively defined by

$$V\left(G^L\left[\widetilde{S}_{\mathscr{F}}\right]\right) = \{S_{\mathscr{F}_i}; 1 \leq i \leq m\};$$

$$E\left(G^L\left[\widetilde{S}_{\mathscr{F}}\right]\right) = \left\{\left(S_{\mathscr{F}_i}, S_{\mathscr{F}_j}\right) \text{ if } S_{\mathscr{F}_i} \bigcap S_{\mathscr{F}_j} \neq \emptyset, 1 \leq i \neq j \leq m\right\} \text{ and labelling}$$

$$L: S_{\mathscr{F}_i} \to S_{\mathscr{F}_i} \text{ and } L: \left(S_{\mathscr{F}_i}, S_{\mathscr{F}_j}\right) \to S_{\mathscr{F}_i} \bigcap S_{\mathscr{F}_j},$$

where $1 \leq i \neq j \leq m$. For example, the topological structures of the equation systems ① and ② are respectively shown in Figure 5(a) and (b) with points A, B, C, D and P shown in Figure 4(a) and (c).

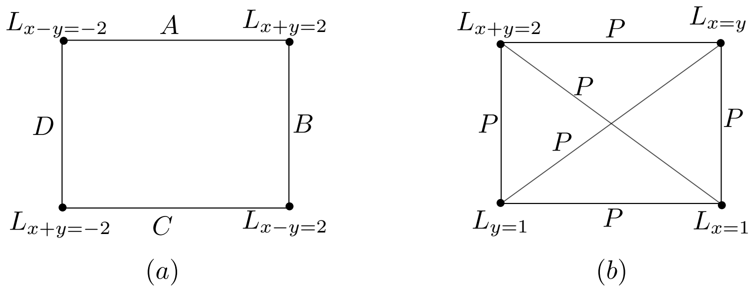

Figure 5. The inherited graphs of the running horses

Thus, the state of a system S characterized by equations (5) can be characterized by its inherits topological structure $G^L\left[\widetilde{S}_{\mathscr{F}}\right]$ on time t, i.e., *G-solution of equations* of (5) and obtain a general conclusion that *all equations characterizing the state of a system S must have a G-solution whether they are solvable or not*, which enables one to get the state $G^L\left[\widetilde{S}_{\mathscr{F}}\right](t)$ of the system S on times, no longer dependent on the solvability of the state equations of system S. For details on G-solutions of non-solvable systems of linear equations, ordinary differential equations and partial differential equations, the reader is referred to the references [19-21] and [24].

4.3. Stability. One advantage of the G-solution on equations of a system S is that we can generally define the stability of S by $G^L\left[\widetilde{S}_{\mathscr{F}}\right](t)$, not dependent on its usual solvability. In classical mathematics, a system of equations is called stable or asymptotically stable if for all solutions $Y(t)$ of the equations with $|Y(t_0) - X(t_0)| < \delta(\varepsilon)$ exists for all $t \geq t_0$, then $|Y(t) - X(t)| < \varepsilon$ for $\forall \varepsilon > 0$ or furthermore, $\lim\limits_{t\to 0} |Y(t) - X(t)| = 0$. However, if the equations are non-solvable, the classical theory of stability is failed to apply. Then *how can we hold on the stability of system characterized by equations* (5)*, maybe non-solvable?* For a system S characterized by (5), we can generalize the stability of the usual to the ω-stable by G-solution of equations (5), not dependent on its usual solvability.

Definition 4.4([20-21]) *Let $G_1^L\left[\widetilde{S}_{\mathscr{F}}\right](t)$ and $G_2^L\left[\widetilde{S}_{\mathscr{F}}\right](t)$ be two G-solutions of equations* (5) *and let $\omega : G^L\left[\widetilde{S}_{\mathscr{F}}\right](t) \to \mathbb{R}$ be an index function. Then, the G-solution of equations* (5) *is said to be ω-stable if there exists a number $\delta(\varepsilon)$ for any number $\varepsilon > 0$ such that*

$$\left|\omega\left(G_1^L\left[\widetilde{S}_{\mathscr{F}}\right](t)\right) - \omega\left(G_2^L\left[\widetilde{S}_{\mathscr{F}}\right](t)\right)\right| < \varepsilon$$

for $t \geq t_0$ or furthermore, asymptotically ω-stable if

$$\lim_{t\to\infty}\left|\omega\left(G_1^L\left[\widetilde{S}_{\mathscr{F}}\right](t)\right) - \omega\left(G_2^L\left[\widetilde{S}_{\mathscr{F}}\right](t)\right)\right| = 0$$

if the initial values hold with

$$\left|\omega\left(G_1^L\left[\widetilde{S}_{\mathscr{F}}\right](t_0)\right) - \omega\left(G_2^L\left[\widetilde{S}_{\mathscr{F}}\right](t_0)\right)\right| < \delta(\varepsilon).$$

If the index function ω is linear, we can further introduce the sum-stability of systems characterized by equations (5).

Definition 4.5([20-22], [24],[26]) *A G-solution is said to be sum-stable or asymptotically sum-stable if all solutions $\mathbf{x}_i(t), 1 \leq i \leq m$ of equations of* (5) *exists for $t \geq t_0$ and*

$$\left\| \sum_{i=1}^{m} \mathbf{x}_i(t) - \sum_{i=1^m} \mathbf{y}_i(t) \right\| < \varepsilon,$$

or furthermore,

$$\lim_{t \to t_0} \left\| \sum_{i=1}^{m} \mathbf{x}_i(t) - \sum_{i=1^m} \mathbf{y}_i(t) \right\| = 0$$

if $\|\mathbf{x}_i(t_0) - \mathbf{y}_i(t_0)\| < \varepsilon$, where, $\varepsilon > 0$ is a real number and $\|\mathbf{X}\|$ denotes the norm of vector $\mathbf{X}$.

The sum-stability of ordinary differential equations and partial differential equations are researched in references [17] and [20]. For example, the following result on the sum-stability of linear ordinary differential equations was obtained.

Theorem 4.5([20-21]) *A G-solution of system* (5) *of linear homogenous differential equations is asymptotically sum-stable if and only if* $\mathrm{Re}\alpha_i < 0$ *for $\overline{\beta}_i(t)e^{\alpha_i t} \in S_{\mathscr{F}_i}$, $1 \leq i \leq m$ of linear basis, where $\overline{\beta}_i(t)$ is a polynomial of degree less than $k-1$ on t if α_i is a k-fold root of characteristic equation of the $\mathscr{F}_i(x_1, x_2, \cdots, x_n, y) = 0$.*

§5. Mathematical Combinatorics

The mathematical combinatorics is such a subject that applying the combinatorial notion, i.e. CC Conjecture in [13] to all other mathematics and all other sciences for understanding the reality of things in the universe, happens to share the same view of Smarandache's notion, particularly, the Smarandache multispace by a combinatorial view.

5.1. CC Conjecture. Notice that the Smarandache multispace can be viewed as a combinatorial theory because it discuss the combination of spaces not only one as in classical mathematics. Essentially, the CC conjecture is also a notion for extending classical mathematics presented by imitating the 2-cell partition on surface initially following.

Conjecture 5.1([13],16) *Any mathematical science can be reconstructed from or made by combinationization.*

It should be noted that this conjecture claims that we can select finite combinatorial rulers and axioms to reconstruct or make generalization for mathematical sciences likewise Euclidean geometry, abstract groups, vector space or rings such that the mathematics as a combinatorial generalization of the classical one, and we can make the combinationization of different branches in mathematics and extend them over topological structures because the classical mathematics can be viewed as a mathematics over a topological point in space.

Why is the CC conjecture important for extending classical mathematics? Because of the limitation of humans ourselves in understanding manner, i.e., only partially or locally understanding likewise the philosophical implications in the fable of the blind men and an elephant, we have to combine all the partially or locally understanding on matters in the universe, and because the universe is a combinatorial one in human eyes, we should have such a mathematics going with the understanding manner, i.e., mathematical combinatorics. Then, *do we really need a proof on the CC conjecture?* No, not need! Because we have known all matters in the universe are combined of elementary particles, and all livings are combined of cells and genes. If we approve the scientific understanding on matters in the universe is by mathematics, then we should extend classical mathematics to a combinatorial one catering to the understanding manner, i.e., mathematical combinatorics because we can then understand matters form the partial or local to the global by mathematics. Certainly, there are many achievements for extending classical mathematics by CC conjecture after it was presented in 2006. For example, combinatorial Euclidean space, combinatorial Riemannian submanifolds, combinatorial principal fiber bundles and combinatorial gravitational fields, $\cdots$, etc. are researched. For the contribution of CC conjecture to classical mathematics, the reader is refereed to [18], which was motivated by the combinatorial notion, particularly, the CC conjecture.

What is the relation of the CC conjecture with Smarandachely denied axiom? By Proposition 3.3, the generalized Smarandachely denied axiom is equivalent to the Smarandache systems on mathematical science and the CC conjecture is a notion on mathematical systems over topological structures, they are essentially equal but the direction of CC conjecture is a little clearer on extending classical mathematics because we already have the theory on topological graphs which is essentially the combinationization of 2-dimensional manifolds ([5], [16], [18]) and we can apply all achievements in the classical to mathematical combinatorics on elements, i.e., continuity flows, i.e., extending elements in classical mathematics over topological structures.

5.2. Continuity Flow. For understanding matters in the universe, classical mathematics provides a quantitative analysis on their appearance in front of humans with various hypothesis on interaction of units. However, if one observer could shrinks his body smaller as it needs, he will enters the interior space of the observed matter and observes the matter in a microcosmic level. Then, *what will he see?* He will find the observed matter is nothing else but a piece of net in the space such as the neural net of brain partially shown in Figure 6. This case implies also the necessity of mathematical combinatorics, i.e., catering for science in a microcosmic level.

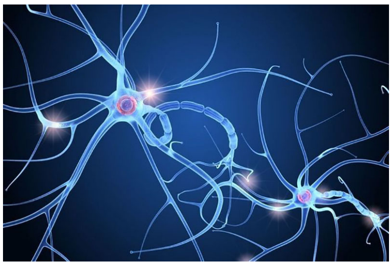

Figure 6. Neural net

Notice that such a topological structure G^L is inherited in a matter or its characterizing by Definition 1.4 on Smarandache multispace, also alluded by the traditional Chinese medicine which applies 12 meridians to heal a patient, characterizes the state of a human, i.e., an inherited topological structure in a living body of human. However, the topological structure G^L defined in Definition 1.4 has no direction on its edge, i.e., the actions of Σ_i on Σ_j and Σ_i on Σ_i both are equal but the flows in the 12 meridians of a human body all have directions and generally, all actions should have directions in the nature. Whence, we should generalized the topological structure in Definition 1.4 from no to with directions on edges, constraint with the conservation laws on all its vertices as flows in nature, which leads to the continuity flows.

Definition 5.2([28]) *A continuity flow* $\left(\overrightarrow{G};L,A\right)$ *is an oriented embedded graph* $\overrightarrow{G}$ *in a topological space* $\mathscr{S}$ *associated with a mapping* $L: v \to L(v),\ (v,u) \to L(v,u)$, 2 *end-operators* $A_{vu}^{+}: L(v,u) \to L^{A_{vu}^{+}}(v,u)$ *and* $A_{uv}^{+}: L(u,v) \to L^{A_{uv}^{+}}(u,v)$ *on a Banach space* $\mathscr{B}$ *over field* $\mathscr{F}$

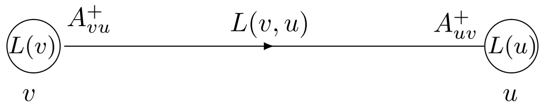

Figure 7

with $L(v,u) = -L(u,v)$ *and* $A_{vu}^{+}(-L(v,u)) = -L^{A_{vu}^{+}}(v,u)$ *for* $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ *holding with continuity equation*

$$\sum_{u \in N_G(v)} L^{A_{vu}^{+}}(v,u) = L(v) \quad for \quad \forall v \in V\left(\overrightarrow{G}\right),$$

where $L(v)$ *is the surplus flow on vertex* v.

Particularly, if $L(v) = \dot{x}_v$, *constants* $\mathbf{v}_v, v \in V\left(\overrightarrow{G}\right)$, *the continuity flow* $\left(\overrightarrow{G};L,A\right)$ *is respectively said to be a complex flow, an action flow, and* $\overrightarrow{G}$*-flow if* $A = \mathbf{1}_{\mathscr{V}}$, *where,* $\dot{x}_v = dx_v/dt$, x_v *is a variable on vertex* v *and* $\mathbf{v}$ *is a vector in* $\mathscr{V}$ *for* $\forall v \in E\left(\overrightarrow{G}\right)$.

For a given graph family $\mathscr{G} = \{\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots, \overrightarrow{G}_m\}$, a Banach space $\mathscr{B}$ and a field $\mathscr{F}$, we denote by $\mathscr{G}_{\mathscr{B}}$ all continuity flows generated by $\overrightarrow{G} \in \mathscr{G}$ with Banach space $\mathscr{B}$, field $\mathscr{F}$ and abbreviate a continuity flow $\left(\overrightarrow{G};L,A\right)$ to $\overrightarrow{G}^L$ in the context. Notice that a continuity flow replaced the Banach space $\mathscr{B}$ and field $\mathscr{F}$ by number field $\mathbb{R}$ with end-operators $1_{\mathbb{R}}$ on the ends of edges in $\overrightarrow{G}$, then a continuity flow $\left(\overrightarrow{G};L,A\right)$ is nothing else but the usual network N discussed in graph theory, and the complex flow is the complex network discussed on complex system in this case. Until today, the $\overrightarrow{G}$-flows and action flows are extensively studied in [22], [26-28] and [30], and for characterizing the livings, the harmonic flow is introduced [24] by replacing vectors $\mathbf{v} \in \mathscr{B}$ on edges in $\overrightarrow{G}$ by complex vectors $\mathbf{v} - i\mathbf{v}$, which is an abstraction and also, a generalization of the 12 meridians in traditional Chinese medicine.

There is a natural question on the continuity flow, i.e., *why does it labels vertices and edges by vectors, not as the usual numbers in continuity flow?* Because the state of units of matters is diversity and multiple directions evolving in the universe, we can characterize its evolution by vectors in multi-dimensions, and *why let vertices constraint with conservation laws?* Because a matter is only a kind form of energy which holds with the conservation law in nature. Whence, the continuity flow $\left(\overrightarrow{G};L,A\right)$ is essentially a generalization of Smarandache multi-space in the microcosmic level combined with the energy flow's character.

5.3. Mathematical Element. Certainly, a continuity flow $\left(\overrightarrow{G};L,A\right)$ is a digraph embedded in a topological space $\mathcal{S}$ by the view of combinatorics, i.e., a structure in space such as those shown in Figure 6. We can research it on vertex, edges or a cluster of vertices, hold on its locally characters as the usual in mathematics. However, *is it really a mathematical element itself* *as the usual number, vector, matrix, $\cdots$ with operations such as the addition, multiplication, differential or integral?* The answer is affirmative, i.e., it can be really viewed as a mathematical element.

Clearly, the continuity flow $\left(\overrightarrow{G};L,A\right)$ is a vector if there is only one vertex in $\overrightarrow{G}$, which consists of the elements in linear algebra. *Could we view a continuity flow as a vector and then establish mathematics on continuity flows?* The answer is yes with operations addition $+$ and multiplication $\cdot$ defined following.

$$\overrightarrow{G}^{L}+\overrightarrow{G'}^{L'} = \left(\overrightarrow{G}\setminus\overrightarrow{G'}\right)^{L}\bigcup\left(\overrightarrow{G}\bigcap\overrightarrow{G'}\right)^{L+L'}\bigcup\left(\overrightarrow{G'}\setminus\overrightarrow{G}\right)^{L'}, \quad (7)$$

$$\overrightarrow{G}^{L}\cdot\overrightarrow{G'}^{L'} = \left(\overrightarrow{G}\setminus\overrightarrow{G'}\right)^{L}\bigcup\left(\overrightarrow{G}\bigcap\overrightarrow{G'}\right)^{L\cdot L'}\bigcup\left(\overrightarrow{G'}\setminus\overrightarrow{G}\right)^{L'}, \quad (8)$$

$$\lambda\cdot\overrightarrow{G}^{L} = \overrightarrow{G}^{\lambda\cdot L}, \quad (9)$$

where $\lambda\in\mathscr{F}$ and $L:(v,u)\to L(v,u)\in\mathscr{B}, L':(v,u)\to L'(v,u)\in\mathscr{B}$ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$ or $E\left(\overrightarrow{G'}\right)$ such that

$$L+L':(v,u)\to\left(L(v,u)+L'(v,u)\right),$$
$$L\ \cdot\ L':(v,u)\to\left(L(v,u)\cdot L'(v,u)\right),$$
$$\lambda\cdot L(v,u)=\lambda\cdot L(v,u)$$

with substituting end-operator $A:(v,u)\to A^{+}_{vu}(v,u)+(A')^{+}_{vu}(v,u)$ or $A:(v,u)\to A^{+}_{vu}(v,u)\cdot(A')^{+}_{vu}(v,u)$ for $(v,u)\in E\left(\overrightarrow{G}\bigcap\overrightarrow{G'}\right)$ in $\overrightarrow{G}^{L}+\overrightarrow{G'}^{L'}$ or $\overrightarrow{G}^{L}\cdot\overrightarrow{G'}^{L'}$. Then, we can define the usual element in mathematics. For example, the *sum* and *product*

$$a_1\overrightarrow{G}_1^{L_1}+a_2\overrightarrow{G}_2^{L_2}+\cdots+a_n\overrightarrow{G}_n^{L_n} = \left(\bigcup_{i=1}^{n}G_i\right)^{a_1L_1+a_2L_2+\cdots+a_nL_n},$$

$$\left(a_1\overrightarrow{G}_1^{L_1}\right)\cdot\left(a_2\overrightarrow{G}_2^{L_2}\right)\cdots\left(a_n\overrightarrow{G}_n^{L_n}\right) = \left(\bigcup_{i=1}^{n}G_i\right)^{a_1L_1\cdot a_2L_2\cdots a_nL_n}$$

and the *polynomial*

$$a_0+a_1\overrightarrow{G}^{L}+a_2\overrightarrow{G}^{L^2}+\cdots+a_n\overrightarrow{G}^{L^n}=\overrightarrow{G}^{a_0+a_1L+a_2L^2+\cdots+a_nL^n}$$

with units $\mathbf{O}$ and $\mathbf{I}$ in $(\mathscr{G}_{\mathscr{B}};+)$, respectively and $(\mathscr{G}_{\mathscr{B}};\cdot)$ such that

$$\mathbf{O}+\overrightarrow{G}^{L}=\overrightarrow{G}^{L}+\mathbf{O}=\overrightarrow{G}^{L},\ \ \mathbf{I}\cdot\overrightarrow{G}^{L}=\overrightarrow{G}^{L}\cdot\mathbf{I}=\overrightarrow{G}^{L}$$

and inverse flows $-\overrightarrow{G}^{L}$, $\overrightarrow{G}^{L^{-1}}$. We then get

Theorem 5.3([33-34]) *Let $\mathscr{G}$ be a graph family with Banach space $\mathscr{B}$ and field $\mathscr{F}$. Then, $(\mathscr{G}_{\mathscr{B};+,\cdot})$ is a linear space with operations* $(7)-(9)$.

Furthermore, we introduce metric on $\mathscr{G}_{\mathscr{B}}$ if $\mathscr{B}$ is a *normed space* following.

Definition 5.4([26-27]) *Let $(\mathscr{B};+,\cdot)$ be a normed space over field $\mathscr{F}$ with norm $\|\mathbf{v}\|$, $\mathbf{v}\in\mathscr{B}$ and $\overrightarrow{G}^L\in\mathscr{G}_{\mathscr{B}}$. The norm of $\overrightarrow{G}^L$ is defined by*

$$\left\|\overrightarrow{G}^L\right\| = \sum_{(v,u)\in E\left(\overrightarrow{G}\right)} \|L(v,u)\|,$$

i.e., the norm $\|\ \|$ is a mapping with $\|\ \|:\mathscr{G}^t_{\mathscr{B}}\to\mathbb{R}^+$.

Then, we get the conclusion following.

Theorem 5.5([28]) *Let $\mathscr{G}$ be a graph family with Banach space $\mathscr{B}$ and field $\mathscr{F}$. Then, $(\mathscr{G}_{\mathscr{B};+,\cdot})$ is a Banach space with operations $(7)-(9)$.*

An operator $f:\overrightarrow{G}_1^{L_1}\to\overrightarrow{G}_2^{L_2}$ on $\mathscr{G}_{\mathscr{B}}$ is *G-isomorphic* if it holds with conditions: ① there is an isomorphism $\varphi:\overrightarrow{G}_1\to\overrightarrow{G}_2$ of graph and ② $L_2=f\circ\varphi\circ L_1$ for $\forall(v,u)\in E\left(\overrightarrow{G}_1\right)$. Particularly, let $\varphi=\mathrm{id}_{\overrightarrow{G}}$, such an operator is determined by equation $L_2=f\circ L_1$, which enables one to define the function on continuity flows by $f\left(\overrightarrow{G}^{L[t]}\right)=\overrightarrow{G}^{f(L[t])}$, get $\lim\limits_{t\to t_0} f\left(\overrightarrow{G}^{L[t]}\right)=f\left(\overrightarrow{G}_0^{L[t_0]}\right)$ if f respect to L and L respect to t both are continuous and

$$e^{\overrightarrow{G}^{L[t]}}=\mathbf{I}+\frac{\overrightarrow{G}^{L[t]}}{1!}+\frac{\overrightarrow{G}^{2L[t]}}{2!}+\cdots+\frac{\overrightarrow{G}^{nL[t]}}{n!}+\cdots.$$

Furthermore, we generalize the differential and integral in calculus to $\mathscr{G}_{\mathscr{B}}$ by

$$\frac{df}{dt}=\lim_{\Delta t\to 0}\frac{f\left(\overrightarrow{G'}^{L'}[t+\Delta t]\right)-f\left(\overrightarrow{G}^L[t]\right)}{\overrightarrow{G'}^{L'}[t+\Delta t]-\overrightarrow{G}^L[t]}$$

if f is a G-isomorphic operator on $\mathscr{G}_{\mathscr{B}}$ with $f\left(\overrightarrow{G'}^{L'}[t+\Delta t]\right)\to f\left(\overrightarrow{G}^L[t]\right)$ if $\Delta t\to 0$ and

$$\int F\left(\overrightarrow{G}^L[t]\right)dt \;=\; f\left(\overrightarrow{G}^L[t]\right)+C \;\text{ if }\; \frac{df}{dt}\left(\overrightarrow{G}^L[t]\right)=F\left(\overrightarrow{G}^L[t]\right),$$

and then, we know formulae on differential and integral operators, i.e.,

$$\int\left(\frac{df}{dt}\left(\overrightarrow{G}^L[t]\right)\right)dt=f\left(\overrightarrow{G}^L[t]\right)+C,\quad \frac{df}{dt}\left(\int\left(f\left(\overrightarrow{G}^L[t]\right)\right)dt\right)=f\left(\overrightarrow{G}^L[t]\right)$$

and the solution

$$X[t]=e^{\int\overrightarrow{G}^{L_{c1}}dt}\cdot\left(\int\overrightarrow{G}^{L_{c0}}\cdot e^{-\int\overrightarrow{G}^{L_{c1}}dt}dt+C\right)$$

of ordinary differential equation

$$\frac{dX}{dt}=\overrightarrow{G}^{L_{c_1}}[t]\cdot X+\overrightarrow{G}^{L_{c_0}}[t]$$

in $\mathscr{G}_{\mathscr{B}}$ as the usual in calculus. Furthermore, we introduce linear functionals on $\mathscr{G}_{\mathscr{B}}$ and extend the fundamental results in functionals. For example, the Fréchet and Riesz representation theorem on linear continuous functionals following.

Theorem 5.6([22],[26],[30]) *Let* $\mathbf{T} : \overrightarrow{G}^{\mathscr{V}} \to \mathbb{C}$ *be a linear continuous functional, where* $\mathscr{V}$ *is a Hilbert space. Then there is a unique* $\overrightarrow{G}^{\widehat{L}} \in \overrightarrow{G}^{\mathscr{V}}$ *such that* $\mathbf{T}\left(\overrightarrow{G}^{L}\right) = \left\langle \overrightarrow{G}^{L}, \overrightarrow{G}^{\widehat{L}} \right\rangle$ *for* $\forall \overrightarrow{G}^{L} \in \overrightarrow{G}^{\mathscr{V}}$.

And then, *could we establish a dynamics on continuity flows?* This question was asked for establishing the graph dynamics in [13] and the answer is affirmative. For example, we know a result on dynamics of continuity flows following.

Theorem 5.7([30]) *If* $L(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))(v,u)$ *is a Lagrangian on edge* (v,u) *and* $\mathscr{L}\left[L(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))\right] : (v,u) \to \mathscr{L}\left[L(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))(v,u)\right]$ *is a differentiable functional on a continuity flow* $\overrightarrow{G}^{L}[t]$ *for* $(v,u) \in E\left(\overrightarrow{G}\right)$ *with* $[\mathscr{L}, A] = \mathbf{0}$ *for* $A \in \mathscr{A}$*, then*

$$\frac{\partial \overrightarrow{G}^{\mathscr{L}}}{\partial x_i} - \frac{d}{dt}\frac{\partial \overrightarrow{G}^{\mathscr{L}}}{\partial \dot{x}_i} = \mathbf{O}, \qquad 1 \le i \le n.$$

Particularly, if the Lagrangian $\mathscr{L}\left[\overrightarrow{G}^{L}[t]\right]$ of a continuity flow $\overrightarrow{G}^{L}[t]$ is independent on (v,u), we know a conclusion on the Euler-Lagrange equations of continuity flows following.

Corollary 5.8(Euler-Lagrange) *If the Lagrangian* $\mathscr{L}\left[\overrightarrow{G}^{L}[t]\right]$ *of a continuity flow* $\overrightarrow{G}^{L}[t]$ *is independent on* (v,u)*, i.e., all Lagrangians* $L(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))(v,u)$*,* $(v,u) \in E\left(\overrightarrow{G}\right)$ *are synchronized, then the dynamic behavior of* $\overrightarrow{G}^{L}[t]$ *can be characterized by* n *equations*

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0, \qquad 1 \le i \le n,$$

which are essentially equivalent to the Euler-Lagrange equations of bouquet $\overrightarrow{B}_1^{L} \in \overrightarrow{B}_{1\mathscr{B}}$*, i.e, dynamic equations on a particle* P.

All of the above-mentioned works shows that a continuity flow is really a mathematical element, likewise elements in classical mathematics, which can be applied to characterize and holds with the reality of matters in natural manner of philosophy. For example, [25], [33] on the structure of elementary particles. For more results on continuity flows with applications to reality of matters in the universe, the reader is refereed to references [22]-[35].

§6. Conclusion

Holding on the reality of matters is an eternal topic of humans, not only beautiful in its mathematical forms but the reality [35], which also provides us an endless resource of thought for scientific research and then, approximating the reality of matters in the universe. Although it is limited of a human life and this work will never go to the end for humans. Consequently, we need to extend our knowing constantly on matters in the universe because this process is step by step, an infinite process by the understanding paradigm (6) of humans. Then, *what we are surveyed in this paper?* We introduce Smarandachely denied axiom on space or systems for the understanding matters because they are not homogenous or equal in the eyes of humans, show its a generalized form equalizing to Smarandache multispace which is an appropriately understanding of matters by philosophy and then, explain how to extend classical mathematics to mathematical combinatorics by CC conjecture. All of the discussions firmly convince one that Smarandachely denied axiom is an important axiom or notion for extending today's science which will further pushes humans to know the truth of matters in the universe, i.e., the reality.

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A New Characterization of Ruled Surfaces According to q-Frame Vectors in Euclidean 3-Space

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Abstract: In this work, we study new families of ruled surfaces generated by q- frame vectors called quasi vectors in 3-dimensional Euclidean space. First, the characterizations of these ruled surfaces such as first and second fundamental forms, Gaussian and mean curvatures are given. After we work on the ruled surfaces generated by the general vector field and give the same caharacterizations for these surfaces whose director is general vector field, we investigate some geometric properties such as developability, minimality, striction curve, and distribution parameter. Lastly, we visualize the surfaces whose directors are tangent, q-normal, q-binormal and general vector field by taking two different curves.

Key Words: Gaussian curvature, mean curvature, quasi frame, ruled surface.

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§1. Introduction

In order to understand what's going on around us, we need to work on the surfaces. Therefore, it is important to have an idea about how to construct the surfaces. Considering the structural advantage of ruled surfaces and the ease of constructing their geometries, ruled surfaces are one of the most attractive surfaces to work on. The ruled surface is a special type of surface which is generated by the motion of a straight line (ruling) along a curve.

After these surfaces were found and investigated by Gaspard Mongea, Ravani and Ku studied ruled surface and examined some properties of them in 1991. Some of the studies have been done by Aydemir and Kasap in 2005, Sarioglugil and Tutar in 2007, Ali et. al. in 2013, Senturk and Yuce in 2015, Unluturk et. al., in 2016, Dede et al. in 2017, Kaymanli in 2020 and Gozutok et al., in 2020 in Euclidean space [1], [4], [5], [7], [9], [12]-[14], [17] while Turgut and Hacisalihoglu in 1998, Kimm and Yoon in 2004, Tosun and Gungor in 2005, Orbay and Aydemir in 2019 and in 2010, Kaymanli et. al., in 2020 in Minkowski space [2], [8], [10], [11], [15], [16].

In this work, we study new families of ruled surfaces generated by q- frame vectors called quasi vectors in 3-dimensional Euclidean space. First, the characterizations of these ruled surfaces such as first and second fundamental forms, Gaussian and mean curvatures are given.

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After we work on the ruled surfaces generated by the general vector field and give the same caharacterizations for these surfaces whose director is general vector field, we investigate some geometric properties such as developability, minimality, striction curve, and distribution parameter. Lastly, we visualize the surfaces whose directors are tangent, q-normal, q-binormal and general vector field by taking two different curves.

§2. **Preliminaries**

In this section, we give some background information about Frenet frame and how to construct q-frame. Let $\alpha(s)$ be a space curve with a non-vanishing second derivative. The Frenet frame is written as

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \ \mathbf{b} = \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|}, \ \mathbf{n} = \mathbf{b} \wedge \mathbf{t}.$$

The curvature κ and the torsion τ are given by

$$\kappa = \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3}, \tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \wedge \alpha''\|^2}.$$

The well-known Frenet formulas are given by

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{bmatrix} = v \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{1}$$

where $v = \|\alpha'(s)\|$.

Besides Frenet frame, we use another frame called q-frame consists of unit tangent vector $\mathbf{t}$, the q-normal $\mathbf{n}_q$ and the q-binormal vector $\mathbf{b}_q$ along a space curve $\alpha(t)$.

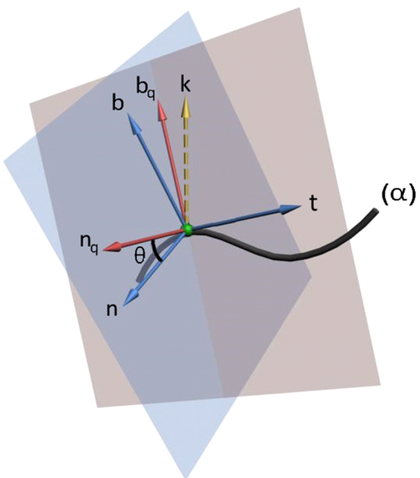

Figure 1 The q-frame and Frenet frame

The q-frame $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q, \mathbf{k}\}$ is defined by

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \mathbf{n}_q = \frac{\mathbf{t} \wedge \mathbf{k}}{\|\mathbf{t} \wedge \mathbf{k}\|}, \mathbf{b}_q = \mathbf{t} \wedge \mathbf{n}_q \tag{2}$$

shown in Figure 1, where $\mathbf{k}$ is the projection vector [3].

Without loss of generality, we chose the projection vector $\mathbf{k} = (0, 0, 1)$ in this study. However, the q-frame is singular in all cases where $\mathbf{t}$ and $\mathbf{k}$ are parallel. Thus, in those cases where $\mathbf{t}$ and $\mathbf{k}$ are parallel the projection vector $\mathbf{k}$ can be chosen as $\mathbf{k} = (0, 1, 0)$ or $\mathbf{k} = (1, 0, 0)$.

In order to define a relation between q-frame and Frenet frame, we pick Euclidean angle θ between the principal normal $\mathbf{n}$ and q-normal $\mathbf{n}_q$ vectors. Then the relation matrix may be expressed as

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{3}$$

or

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}. \tag{4}$$

Let $\alpha(s)$ be a curve that is parameterized by arc length s. Differentiating (3) with respect to s, then substituting (4) into the results gives the variation equations of the q-frame in the following form

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_q' \\ \mathbf{b}_q' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}, \tag{5}$$

where the q-curvatures are

$$\begin{aligned} k_1 &= \langle \mathbf{t}', \mathbf{n}_q \rangle \\ k_2 &= \langle \mathbf{t}', \mathbf{b}_q \rangle \\ k_3 &= \langle \mathbf{n}_q', \mathbf{b}_q \rangle . \end{aligned} \tag{6}$$

The parametric equation of ruled surface $\varphi(s, v)$ is given as

$$\varphi(s, v) = \alpha(s) + vX(s), \tag{7}$$

where $\alpha(s)$ is a curve and $X(s)$ is a generator vector. The distribution parameter of the ruled surface is identified by (see [6], [14])

$$P_X = \frac{\det(\alpha_s, X, X_s)}{\langle X_s, X_s \rangle}. \tag{8}$$

The striction point on the ruled surface is the foot of the common perpendicular line successive rulings on the main ruling. It is given as

$$\beta_X(s) = \alpha(s) - \frac{\langle \alpha_s, X_s \rangle}{\langle X_s, X_s \rangle} X(s) \tag{9}$$

Let M be a regular surface given with the parameterization $\varphi(s, v)$ in E^3. The tangent space of M at an arbitrary point is spanned by the vectors φ_s and φ_v. The coefficients of the first fundamental form of M are defined as

$$E = \langle \varphi_s, \varphi_s \rangle, F = \langle \varphi_s, \varphi_v \rangle, G = \langle \varphi_v, \varphi_v \rangle, \tag{10}$$

where $\langle , \rangle$ is the Euclidean inner product. Then the unit normal vector field of M is defined as

$$N = \frac{\varphi_s \wedge \varphi_v}{||\varphi_s \wedge \varphi_v||}. \tag{11}$$

The coefficients of the second fundamental form of M are defined as

$$e = \langle \varphi_{ss}, N \rangle, f = \langle \varphi_{sv}, N \rangle, g = \langle \varphi_{vv}, N \rangle. \tag{12}$$

The Gaussian curvature and the mean curvature of M are given by

$$K = \frac{eg - f^2}{EG - F^2} \tag{13}$$

and

$$H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)}, \tag{14}$$

respectively.

Theorem 2.1([12]) *The ruled surface is developable if and only if* $P_X = 0$.

Theorem 2.2 *The ruled surface is minimal if and only if* $H = 0$.

§3. Ruled Surfaces Generated by q-Frame Vectors

The ruled surfaces generated by q-frame vectors t, n_q, b_q are given as

$$\begin{aligned} \phi^t(s, v) &= \alpha(s) + vt(s), \\ \phi^{n_q}(s, u) &= \alpha(s) + un_q(s), \\ \phi^{b_q}(s, z) &= \alpha(s) + zb_q(s), \end{aligned}$$

respectively. The ruled surface generated by general vector filed X is written as

$$\phi^X(s, w) = \alpha(s) + wX(s) \tag{15}$$

where $X(s) = x_1(s)t + x_2(s)n_q + x_3(s)b_q$.

Theorem 3.1 *The distribution parameters of surfaces ϕ^t, ϕ^{n_q} and ϕ^{b_q} are*

$$P_t = 0, \quad P_{n_q} = \frac{k_3}{k_1^2 + k_3^2} \quad and \quad P_{b_q} = \frac{k_3}{k_2^2 + k_3^2},$$

respectively.

Theorem 3.2 *The striction curves on the ruled surfaces ϕ^t, ϕ^{n_q} and ϕ^{b_q} are given by*

$$\beta_t(s) = \alpha(s), \quad \beta_{n_q}(s) = \alpha(s) + \frac{k_1}{k_1^2 + k_3^2} n_q \quad and \quad \beta_{b_q}(s) = \alpha(s) + \frac{k_2}{k_2^2 + k_3^2} b_q,$$

respectively.

Theorem 3.3 *The distribution parameter and striction curve of ϕ^X are calculated as*

$$P_X = \frac{-x_3(k_1x_1 + x_2' - k_3x_3) + x_2(k_1x_1 + k_3x_2 + x_3')}{(x_1' - k_1x_2 - k_2x_3)^2 + (k_1x_1 + x_2' - k_3x_3)^2 + (k_2x_1 + k_3x_2 + x_3')^2},$$

$$\beta_X(s) = \alpha(s) - \frac{(x_1' - k_1x_2 - k_2x_3)(x_1t + x_2n_q + x_3b_q)}{(x_1' - k_1x_2 - k_2x_3)^2 + (k_1x_1 + x_2' - k_3x_3)^2 + (k_2x_1 + k_3x_2 + x_3')^2}$$

respectively.

Proof Taking derivative of $\alpha(s)$ and $X(s)$ with respect to s, we can easily find $\alpha'(s) = \mathbf{t}$ and

$$X'(s) = (x_1' - x_2k_1 - x_3k_2)\,\mathbf{t} + (x_1k_1 + x_2' - x_3k_3)\,\mathbf{n}_q + (x_1k_2 + x_2k_3 + x_3')\,\mathbf{b}_q,$$

respectively. With the help of the obtained equation and equations (8) and (9), an algebraic calculus gives us desired results. □

Theorem 3.4 *The Gaussian curvatures of surfaces ϕ^t, ϕ^{n_q} and ϕ^{b_q} are*

$$K_t = 0, \quad K_{n_q} = 0 \quad and \quad K_{b_q} = \frac{-k_3^2}{(k_3^2z^2 + (1 - k_2z)^2)^2},$$

respectively.

Theorem 3.5 *The mean curvatures of surfaces ϕ^t, ϕ^{n_q} and ϕ^{b_q} are*

$$H_t = \frac{-k_3}{2v\sqrt{k_1^2 + k_2^2}}, \quad H_{n_q} = \frac{k_2(1 - 2k_1u + (k_1^2 + k_3^2)u^2)}{2(k_3^2u^2 + (1 - k_1u)^2)^{3/2}}$$

and

$$H_{b_q} = \frac{-k_1}{2(k_3^2z^2 + (k_2z - 1)^2)},$$

respectively.

Theorem 3.6 *The Gaussian and mean curvatures of ϕ^X are*

$$\begin{aligned} H_X &= \frac{1}{2W\sqrt{(Bx_3-Cx_2)^2+(Cx_1-Ax_3)^2+(Ax_2-Bx_1)^2}}\left[\left((Bx_3-Cx_2)(A'-Bk_1-Ck_2)\right.\right. \\ &\quad +(Cx_1-Ax_3)(Ak_1+B'-Ck_3)+(Ax_2-Bx_1)(Ak_2-Bk_3+C')) \\ &\quad -2\left(AB^2x_1x_3(A/B)^{\cdot}+A^3x_1x_2(C/A)^{\cdot}+AC^2x_1^2(B/C)^{\cdot}+B^3x_2x_3(A/B)^{\cdot}\right) \\ &\quad +2\left(BA^2x_2^2(C/A)^{\cdot}+BC^2x_1x_2(B/C)^{\cdot}+CB^2x_3^2(A/B)^{\cdot}+CA^2x_2x_3(C/A)\right. \\ &\quad \left.\left.+C^3x_1x_3(B/C)^{\cdot}\right)\right] \\ K_X &= -\frac{1}{W}\left(\frac{x_3B^2(A/B)^{\cdot}+x_2A^2(C/A)^{\cdot}+x_1C^2(B/C)^{\cdot}}{\sqrt{(Bx_3-Cx_2)^2+(Cx_1-Ax_3)^2+(Ax_2-Bx_1)^2}}\right)^2 \end{aligned}$$

where $A=w(x_1'-k_1x_2-k_2x_3)+1,\quad B=w(k_1x_1+x_2'-k_3x_3),\ C=w(k_2x_1+k_3x_2+x_3'),$ $W=A^2+B^2+C^2-A^2x_1^2+B^2x_2^2+2ABx_1x_2+C^2x_3^2+2ACx_1x_3+2BCx_2x_3,\ Y^{\cdot}=\dfrac{dY}{dw}$ *and* $Y'=\dfrac{dY}{ds}$.

Proof First and second partial derivatives of the surface given in (15) with respect to s and w are expressed as

$$\begin{aligned} \phi_s^X &= A\mathbf{t}+B\mathbf{n}_q+C\mathbf{b}_q \\ \phi_w^X &= x_1\mathbf{t}+x_2\mathbf{n}_q+x_3\mathbf{b}_q \end{aligned}$$

and

$$\begin{aligned} \phi_{ss}^X &= (A'-Bk_1-Ck_2)\,\mathbf{t}+(Ak_1+B'-Ck_3)\,\mathbf{n}_q+(Ak_2-Bk_3+C')\,\mathbf{b}_q \\ \phi_{sw}^X &= A^{\cdot}\mathbf{t}+B^{\cdot}\mathbf{n}_q+C^{\cdot}\mathbf{b}_q \\ \phi_{ww}^X &= 0, \end{aligned}$$

respectively. The coefficients of the first and second fundamental forms are calculated by $E=A^2+B^2+C^2,\ F=Ax_1+Bx_2+Cx_3,\ G=1$ and

$$\begin{aligned} e &= \frac{1}{\sqrt{(Bx_3-Cx_2)^2+(Cx_1-Ax_3)^2+(Ax_2-Bx_1)^2}}\left[(Bx_3-Cx_2)(A'-Bk_1-Ck_2)\right. \\ &\quad \left.+(Cx_1-Ax_3)(Ak_1+B'-Ck_3)+(Ax_2-Bx_1)(Ak_2-Bk_3+C')\right] \end{aligned}$$

$$\begin{aligned} f &= \frac{x_3B^2\left(\frac{A}{B}\right)^{\cdot}+x_2A^2\left(\frac{C}{A}\right)^{\cdot}+x_1C^2\left(\frac{B}{C}\right)^{\cdot}}{\sqrt{(Bx_3-Cx_2)^2+(Cx_1-Ax_3)^2+(Ax_2-Bx_1)^2}} \\ g &= 0 \end{aligned}$$

respectively. Using equations (13) and (14), the Gaussian and mean curvatures of ϕ^X are presented easily. This completes the proof. □

Corollary 3.7 *The ruled surface ϕ^t is developable and the ruled surfaces ϕ^{n_q} and ϕ^{b_q} are developable if and only if $k_3 = 0$.*

Corollary 3.8 *The ruled surface ϕ^t is minimal if and only if $k_3 = 0$, the ruled surface ϕ^{b_q} is minimal if and only if $k_1 = 0$ and the ruled surface ϕ^{n_q} is minimal if and only if either $k_2 = 0$ or $u = \frac{k_1(1\pm\sqrt{1-k_3^2})}{k_1^2+k_3^2}$.*

Corollary 3.9 *There is a relation between K_{b_q}, H_{b_q} and P_{b_q} as follows*

$$\frac{K_{b_q}}{H_{b_q}} = \frac{2k_3^2}{k_1(k_3^2z^2 + (1-k_2z)}$$

and

$$\frac{K_{b_q}}{P_{b_q}} = \frac{-k_3(k_2^2 + k_3^2)}{(k_3^2z^2 + (1-k_2z)^2)^2}.$$

§4. Examples

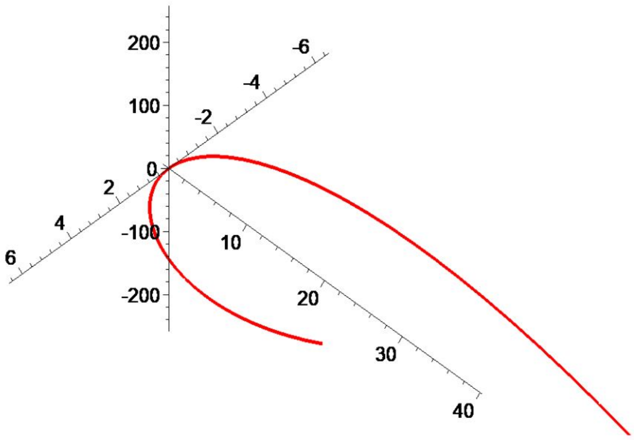

Figure 2 The curve $\alpha(s) = (s, s^2, s^3)$

Example 4.1 Consider the curve, shown in Figure 2,

$$\alpha(s) = (s, s^2, s^3)$$

with q-vectors and curvatures

$$\begin{aligned} t &= \frac{1}{\sqrt{1+4s^2+9s^4}}\left(1, 2s, 3s^2\right) \\ n_q &= \frac{1}{\sqrt{1+4s^2}}\left(2s, -1, 0\right) \\ b_q &= \frac{1}{\sqrt{1+4s^2+9s^4}\sqrt{1+4s^2}}\left(3s^2, 6s^3, -\sqrt{1+4s^2}\right) \end{aligned}$$

and

$$\begin{aligned} k_1 &= -\frac{2}{(1+4s^2+9s^4)\sqrt{1+4s^2}} \\ k_2 &= -\frac{6s(1+2s^2)}{(1+4s^2+9s^4)^{3/2}\sqrt{1+4s^2}} \\ k_3 &= \frac{6s^2}{(1+4s^2+9s^4)(1+4s^2)} \end{aligned}$$

respectively.

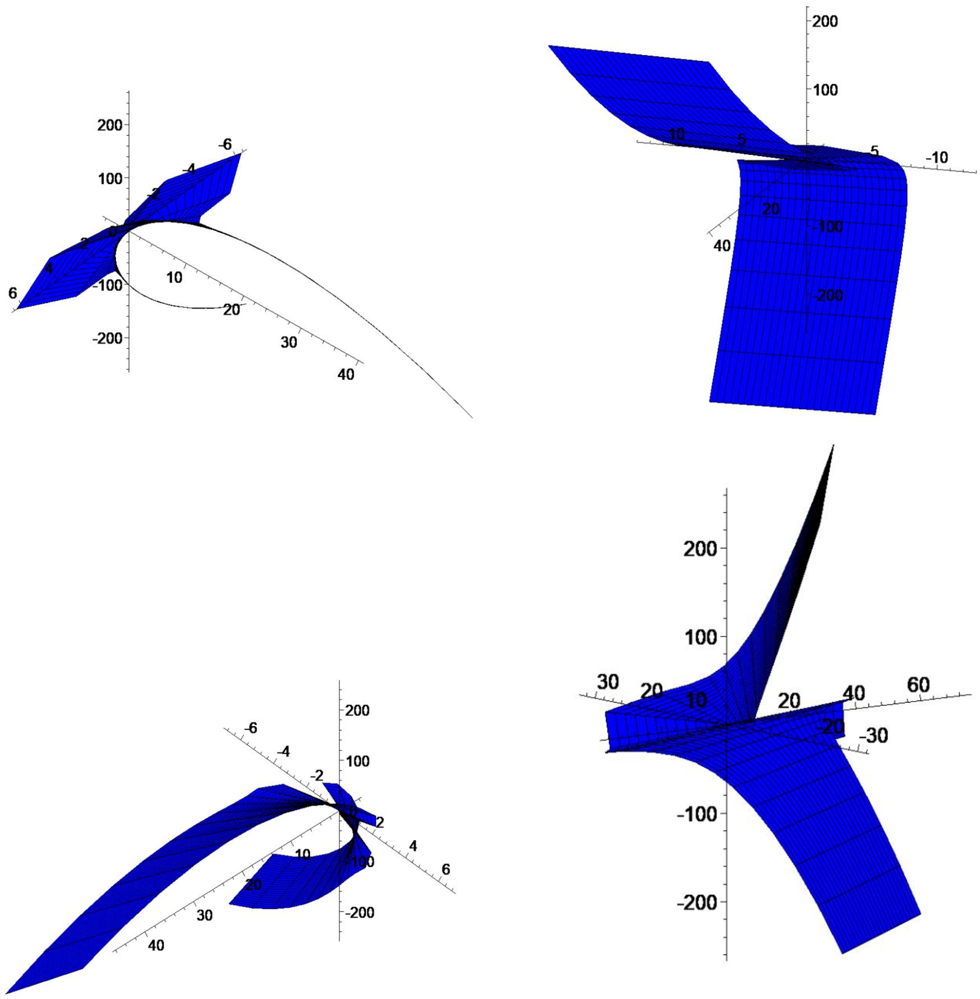

Figure 3 $\phi_1^t(s,v)$ (left top), $\phi_1^{n_q}(s,u)$ (right top), $\phi_1^{b_q}(s,z)$ (left bottom) and $\phi_1^X(s,w)$ for $x_1(s)=2,\ x_2(s)=3,\ x_3(s)=5$ (right bottom)

The ruled surfaces generated by q-frame vectors t, n_q, b_q and X shown in Figure 3, are given as

$$\begin{aligned} \phi_1^t(s,v) &= (s,s^2,s^3) + v\frac{1}{\sqrt{1+4s^2+9s^4}}\left(1,2s,3s^2\right), \\ \phi_1^{n_q}(s,u) &= (s,s^2,s^3) + u\frac{1}{\sqrt{1+4s^2}}\left(2s,-1,0\right), \end{aligned}$$

$$\phi_1^{b_q}(s,z) = (s,s^2,s^3) + z\frac{1}{\sqrt{1+4s^2+9s^4}\sqrt{1+4s^2}}\left(3s^2, 6s^3, -\sqrt{1+4s^2}\right),$$

and

$$\begin{aligned}\phi_1^{X}(s,w) &= (s,s^2,s^3) + w\Bigg(x_1(s)(\frac{1}{\sqrt{1+4s^2+9s^4}}\left(1,2s,3s^2\right))\\ &+x_2(s)(\frac{1}{\sqrt{1+4s^2}}\left(2s,-1,0\right))\\ &+x_3(s)(\frac{1}{\sqrt{1+4s^2+9s^4}\sqrt{1+4s^2}}\left(3s^2, 6s^3, -\sqrt{1+4s^2}\right))\Bigg),\end{aligned}$$

respectively.

Example 4.2 Consider the curve, shown in the Figure 4,

$$\alpha(s) = (\frac{5}{13}\sin\frac{s}{13}, -\frac{5}{13}\cos\frac{s}{13}, \frac{s}{169})$$

with q-vectors and curvatures

$$\begin{aligned} t &= \left(\frac{5}{\sqrt{26}}\cos\tfrac{s}{13}, \frac{5}{\sqrt{26}}\sin\tfrac{s}{13}, \frac{1}{\sqrt{26}}\right)\\ n_q &= \left(\sin\tfrac{s}{13}, -\cos\tfrac{s}{13}, 0\right)\\ b_q &= \left(\frac{1}{\sqrt{26}}\cos\tfrac{s}{13}, \frac{1}{\sqrt{26}}\sin\tfrac{s}{13}, -\frac{5}{\sqrt{26}}\right)\end{aligned}$$

and

$$\begin{aligned} k_1 &= -\frac{5}{2}\\ k_2 &= 0\\ k_3 &= \frac{1}{2}\end{aligned}$$

respectively.

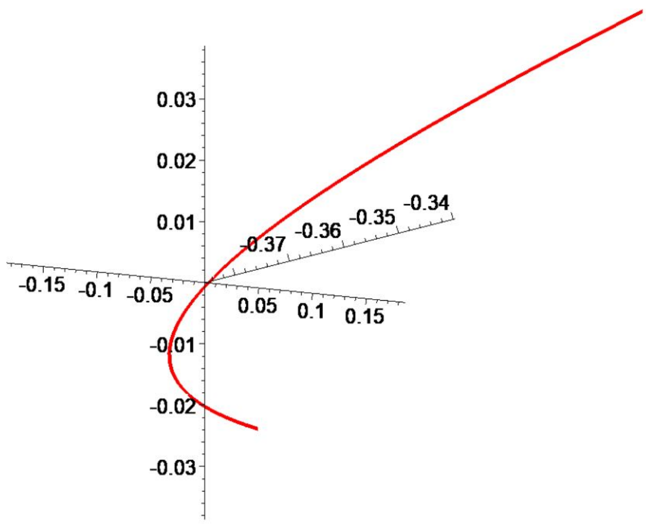

Figure 4 The curve $\alpha(s) = (\frac{5}{13}\sin\frac{s}{13}, -\frac{5}{13}\cos\frac{s}{13}, \frac{s}{169})$

The ruled surfaces generated by q-frame vectors t, n_q, b_q and X shown in Figure 5

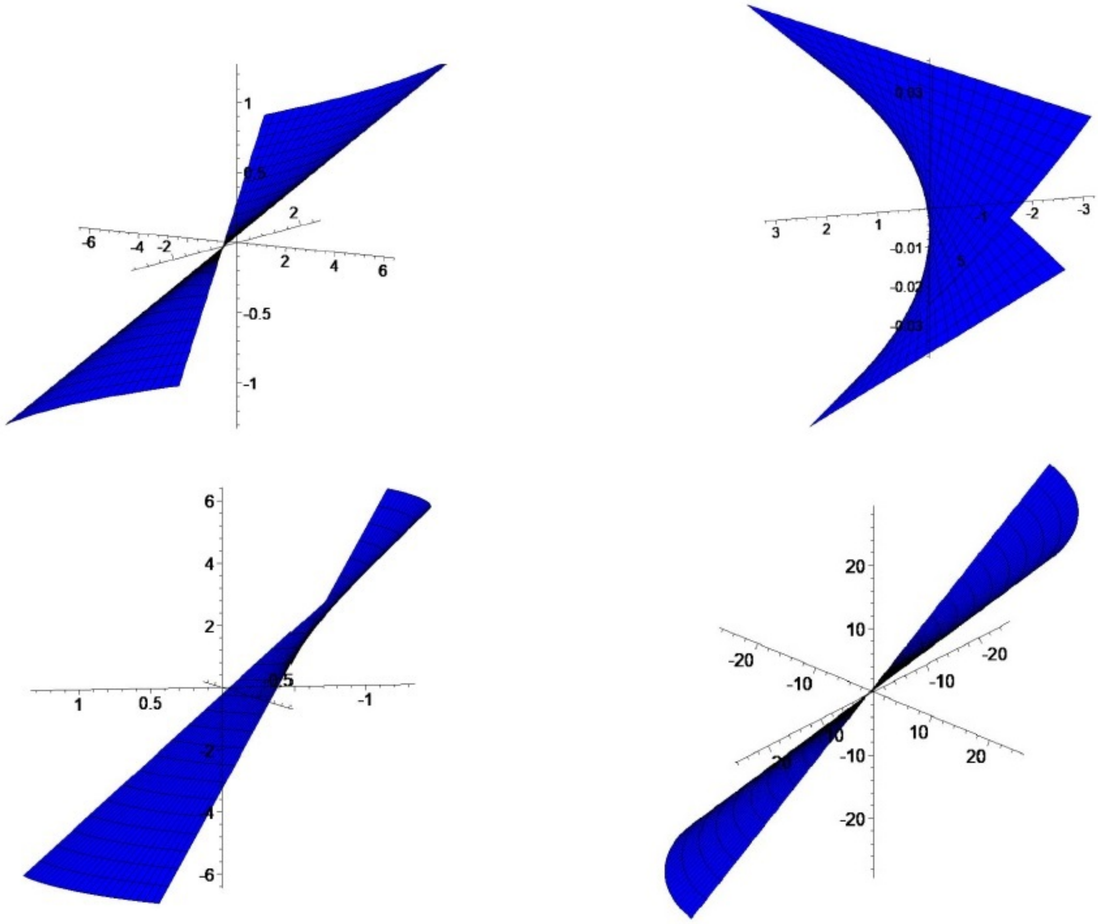

Figure 5 $\phi_2^t(s,v)$ (left top), $\phi_2^{n_q}(s,u)$ (right top), $\phi_2^{b_q}(s,z)$ (left bottom) and $\phi_2^X(s,w)$ for $x_1(s)=2,\ x_2(s)=3,\ x_3(s)=5$ (right bottom)

are given respectively by

$$\begin{aligned}
\phi_2^t(s,v) &= \left(\tfrac{5}{13}\sin\tfrac{s}{13}, -\tfrac{5}{13}\cos\tfrac{s}{13}, \tfrac{s}{169}\right) + v\left(\frac{5}{\sqrt{26}}\cos\tfrac{s}{13}, \frac{5}{\sqrt{26}}\sin\tfrac{s}{13}, \frac{1}{\sqrt{26}}\right),\\
\phi_2^{n_q}(s,u) &= \left(\tfrac{5}{13}\sin\tfrac{s}{13}, -\tfrac{5}{13}\cos\tfrac{s}{13}, \tfrac{s}{169}\right) + u\left(\sin\tfrac{s}{13}, -\cos\tfrac{s}{13}, 0\right),\\
\phi_2^{b_q}(s,z) &= \left(\tfrac{5}{13}\sin\tfrac{s}{13}, -\tfrac{5}{13}\cos\tfrac{s}{13}, \tfrac{s}{169}\right) + z\left(\frac{1}{\sqrt{26}}\cos\tfrac{s}{13}, \frac{1}{\sqrt{26}}\sin\tfrac{s}{13}, -\frac{5}{\sqrt{26}}\right),
\end{aligned}$$

and

$$\begin{aligned}
\phi_2^X(s,w) &= \left(\tfrac{5}{13}\sin\tfrac{s}{13}, -\tfrac{5}{13}\cos\tfrac{s}{13}, \tfrac{s}{169}\right) + w\left(x_1(s)\left(\frac{5}{\sqrt{26}}\cos\tfrac{s}{13}, \frac{5}{\sqrt{26}}\sin\tfrac{s}{13}, \frac{1}{\sqrt{26}}\right)\right.\\
&\quad \left. + x_2(s)\left(\sin\tfrac{s}{13}, -\cos\tfrac{s}{13}, 0\right) + x_3(s)\left(\frac{1}{\sqrt{26}}\cos\tfrac{s}{13}, \frac{1}{\sqrt{26}}\sin\tfrac{s}{13}, -\frac{5}{\sqrt{26}}\right)\right).
\end{aligned}$$

For $x_1(s) = 2,\ x_2(s) = 3,\ x_3(s) = 5$, the Gaussian curvature, mean curvature, distribution parameter and striction curve of the surface $\phi_2^X(s,w)$, generated by X are calculated as

$$\begin{aligned}
K_X &= -\frac{7056}{(68+1140w+8721w^2)^2},\\
H_X &= \frac{\sqrt{1228}(1228+26220w+200583w^2)}{4\sqrt{884+14820w+113373w^2}(68+1140w+8721w^2)},\\
P_X &= -\frac{56}{153}\\
\beta_X(s) &= \left(\frac{125}{663}\sin\frac{s}{13}-\frac{25\sqrt{26}}{663}\cos\frac{s}{13},-\frac{125}{663}\cos\frac{s}{13}-\frac{25\sqrt{26}}{663}\sin\frac{s}{13},\frac{s}{169}+\frac{115\sqrt{26}}{1989}\right),
\end{aligned}$$

respectively.

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On Sears's Basic Hypergeometric Series Transformation Formula

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Abstract: In this paper, we provide an alternative simple proof to Sears's ${}_3\varphi_2$ transformation formula using Gauss summation formula.

Key Words: Basic hypergeometric series, transformation formula, Gauss summation.

AMS(2010): 33D15.

§1. Introduction

For any complex number a and for any q with $|q| < 1$, $(a;q)_\infty$ or simply $(a)_\infty$ is defined as follows:

$$(a)_\infty = (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

For any integer n, we define $(a;q)_n$ or simply $(a)_n$ as

$$(a)_n = \frac{(a)_\infty}{(aq^n)_\infty}, \tag{1.1}$$

provided the denominator is well defined. For any two integers m and n, the following holds good

$$(a)_n(aq^n)_m = (a)_m(aq^m)_n. \tag{1.2}$$

The basic hypergeometric series ${}_{s+1}\varphi_s$ is defined by

$${}_{s+1}\varphi_s\left[\begin{matrix} a_1, a_2, \cdots, a_{s+1} \\ b_1, b_2, \cdots, b_s \end{matrix}; z\right] := \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_{s+1})_n}{(q)_n(b_1)_n(b_2)_n\cdots(b_s)_n}z^n,$$

where $a_1, a_2, \cdots, a_{s+1}, b_1, b_2, \cdots, b_s$ are any complex numbers, except that $(b_j)_n \neq 0$, $1 \leq j \leq s$, $0 \leq n \leq \infty$. This series converges for all z with $|z| < 1$.

In his paper [3], D. B. Sears deduced the following ${}_3\varphi_2$ transformation formula

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n(q)_n}\left(\frac{de}{abc}\right)^n = \frac{(b)_\infty\left(\frac{de}{ab}\right)_\infty\left(\frac{de}{bc}\right)_\infty}{(d)_\infty(e)_\infty\left(\frac{de}{abc}\right)_\infty}\sum_{n=0}^{\infty}\frac{\left(\frac{d}{b}\right)_n\left(\frac{e}{b}\right)_n\left(\frac{de}{abc}\right)_n}{\left(\frac{de}{ab}\right)_n\left(\frac{de}{bc}\right)_n(q)_n}b^n, \tag{1.3}$$

[1]Received June 19, 2021, Accepted August 28, 2021.

where, $\left|\frac{dc}{abc}\right| < 1$, $|b| < 1$, and $|q| < 1$. Sears deduced the above transformation by letting one of the parameters in his ${}_4\varphi_3$ transformation to infinity. This identity has been widely in areas of special functions and number theory. See for instance [1] and [2]. In this paper, we give a simple proof to (1.3) using q-analogous of Gauss summation formula. This proof seems to be new in the literature. We close this section by recalling the q-analogous of Gauss summation formula and prove (1.3) in Section 2 following.

$$\sum_{m=0}^{\infty} \frac{(a)_m\,(b)_m}{(c)_m\,(q)_m}\left(\frac{c}{ab}\right)^m = \frac{\left(\frac{c}{a}\right)_\infty\left(\frac{c}{b}\right)_\infty}{(c)_\infty\left(\frac{c}{ab}\right)_\infty}, \quad \left|\frac{c}{ab}\right| < 1. \tag{1.4}$$

§2. Proof of (1.3)

From (1.4), it follows that

$$\frac{(A)_\infty\,(B)_\infty}{(C)_\infty\left(\frac{AB}{C}\right)_\infty} = \sum_{n=0}^{\infty}\frac{\left(\frac{C}{A}\right)_n\left(\frac{C}{B}\right)_n}{(C)_n\,(q)_n}\left(\frac{AB}{C}\right)^n, \qquad \left|\frac{AB}{C}\right| < 1.$$

Taking $A = dq^m, B = eq^m$ and $C = \frac{deq^m}{b}$ in the above, we obtain

$$\frac{(dq^m)_\infty\,(eq^m)_\infty}{\left(\frac{deq^m}{b}\right)_\infty (bq^m)_\infty} = \sum_{n=0}^{\infty}\frac{\left(\frac{d}{b}\right)_n\left(\frac{e}{b}\right)_n}{(q)_n\left(\frac{deq^m}{b}\right)_n}(bq^m)^n. \tag{2.1}$$

Using (1.1), we can write

$$\sum_{m=0}^{\infty}\frac{(a)_m\,(b)_m\,(c)_m}{(d)_m\,(e)_n\,(q)_m}\left(\frac{de}{abc}\right)^m = \frac{(b)_\infty}{(d)_\infty\,(e)_\infty}\sum_{m=0}^{\infty}\frac{(a)_m\,(c)_m\left(\frac{deq^m}{b}\right)_\infty}{(q)_m}\left(\frac{de}{abc}\right)^m\frac{(dq^m)_\infty\,(eq^m)_\infty}{\left(\frac{deq^m}{b}\right)_\infty (bq^m)_\infty}.$$

Using (2.1) in the above, we obtain

$$\sum_{m=0}^{\infty}\frac{(a)_m\,(b)_m\,(c)_m}{(d)_m\,(e)_n\,(q)_m}\left(\frac{de}{abc}\right)^m = \frac{(b)_\infty\left(\frac{de}{b}\right)_\infty}{(d)_\infty\,(e)_\infty}\sum_{m=0}^{\infty}\frac{(a)_m\,(c)_m}{(q)_m\left(\frac{de}{b}\right)_m}\left(\frac{de}{abc}\right)^m\sum_{n=0}^{\infty}\frac{\left(\frac{d}{b}\right)_n\left(\frac{e}{b}\right)_n}{(q)_n\left(\frac{deq^m}{b}\right)_n}(bq^m)^n.$$

Interchanging the order of summation in the above and using (1.2), we obtain

$$\sum_{m=0}^{\infty}\frac{(a)_m\,(b)_m\,(c)_m}{(d)_m\,(e)_m\,(q)_m}\left(\frac{de}{abc}\right)^m = \frac{(b)_\infty\left(\frac{de}{b}\right)_\infty}{(d)_\infty\,(e)_\infty}\sum_{n=0}^{\infty}\frac{\left(\frac{d}{b}\right)_n\left(\frac{e}{b}\right)_n}{(q)_n\left(\frac{de}{b}\right)_n}(b)^n\sum_{m=0}^{\infty}\frac{(a)_m\,(c)_m}{(q)_m\left(\frac{deq^n}{b}\right)_m}\left(\frac{deq^n}{abc}\right)^m.$$

Now applying (1.4) to the inner series on the right hand side, we obtain

$$\sum_{m=0}^{\infty}\frac{(a)_m\,(b)_m\,(c)_m}{(d)_m\,(e)_n\,(q)_m}\left(\frac{de}{abc}\right)^m = \frac{(b)_\infty\left(\frac{de}{b}\right)_\infty}{(d)_\infty\,(e)_\infty}\sum_{n=0}^{\infty}\frac{\left(\frac{d}{b}\right)_n\left(\frac{e}{b}\right)_n}{(q)_n\left(\frac{de}{b}\right)_n}(b)^n\frac{\left(\frac{deq^n}{ab}\right)_\infty\left(\frac{deq^n}{bc}\right)_\infty}{\left(\frac{deq^n}{b}\right)_\infty\left(\frac{deq^n}{abc}\right)_\infty}.$$

Notice that the equation (1.3) follows directly from the above on using (1.1). This completes the proof. □

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On Laplacian of Product of Randić and Sum-Connectivity Energy of Graphs

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Abstract: In this paper, we define the Laplacian of product of Randić and sum-connectivity energy of a graph. Then, we compute the Laplacian of product of Randić and sum-connectivity energies of complete graph, star graph, complete bipartite graph, the crown graph, the $(S_m \wedge P_2)$ graph.

Key Words: Laplacian matrix, Laplacian of product of Randić and sum-connectivity energy, graph.

AMS(2010): 05C50.

§1. Introduction

In [6] we define product of Randić and sum-connectivity energy of a simple graph G as follows:

Let a and b be two nonnegative real number with $a \neq 0$. The product of Randić and sum-connectivity adjacency matrix of G is the $n \times n$ matrix $A_{prs} = (a_{ij})$ where

$$a_{ij} = \begin{cases} 0, & \text{if}\, i = j, \\ \dfrac{1}{\sqrt{\frac{1}{a(d_i+d_j)b(d_id_j)}}}, & \text{if the vertices}\, v_i \,\text{and}\, v_j \,\text{are adjacent}, \\ 0, & \text{if the vertices}\, v_i \,\text{and}\, v_j \,\text{are not adjacent}. \end{cases}$$

The eigenvalues of the graph G are the eigenvalues of A_{prs}. Since A_{prs} is real and symmetric, its eigenvalues are real numbers which are denoted by $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$, where

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n.$$

Then the product of Randić and sum-connectivity energy of G is defined as

$$E_{prs}(G) = \sum_{i=1}^{n} \mid \lambda_i \mid .$$

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In 2004, D. Vukičević and Gutman [5] defined the Laplacian matrix of the graph G, denoted by $L = (L_{ij})$, as a square matrix of order n whose elements are defined by

$$L_{ij} = \begin{cases} \delta_i, & \text{if } i = j, \\ -1, & \text{if } i \neq j \text{ and the vertices } v_i, v_j \text{ are adjacent,} \\ 0, & \text{if } i \neq j \text{ and the vertices } v_i, v_j \text{ are not adjacent,} \end{cases}$$

where δ_i is the degree of vertex v_i. The eigenvalues $\mu_1, \mu_2, \cdots, \mu_n$ of L, where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are called the Laplacian eigenvalues of G. In 2006, Gutman and B. Zhou [2] have defined the Laplacian energy of $LE(G)$ of G as

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|,$$

where m is number of edges and n is number of vertices of G.

Motivated by these works, we introduce the Laplacian of product of Randić and sum-connectivity energy of a simple graph G as follows. Let a and b be two nonnegative real number with $a \neq 0$. The Laplacian of product of Randić and sum-connectivity adjacency matrix of G is the $n \times n$ matrix $A_{lprs} = (a_{ij})$ where

$$a_{ij} = \begin{cases} \delta_i, & \text{if } i = j, \\ \frac{1}{\sqrt{\frac{1}{a(d_i+d_j)b(d_i d_j)}}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

where δ_i is the degree of vertex v_i. Where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are called the eigenvalues of A_{lprs}. Then Laplacian of product of Randić and sum-connectivity energy of G is

$$E_{lprs}(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|,$$

where m is number of edges and n is number of vertices of G.

§2. Laplacian of Product of Randić and Sum-Connectivity Energies of Some Families of Graphs

We begin with some basic definitions and notations.

Definition 2.1([3]) *A graph G is said to be complete if every pair of its distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n.*

Definition 2.2([1]) *The Crown graph S_n^0 for an integer $n \geq 3$ is the graph with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and edge set $\{u_i v_j; 1 \leq i, j \leq n, i \neq j\}$. S_n^0 is therefore equivalent* *to the complete bipartite graph $K_{n,n}$ from which the edges of perfect matching have been removed.*

Definition 2.3([3]) *A bigraph or bipartite graph G is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every line of G joins V_1 with V_2. (V_1, V_2) is a bipartition of G. If G contains every line joining V_1 and V_2, then G is a complete bigraph. If V_1 and V_2 have m and n points, we write $G = K_{m,n}$. A star is a complete bigraph $K_{1,n}$.*

Definition 2.4([4]) *The conjunction $(S_m \wedge P_2)$ of $S_m = \overline{K}_m + K_1$ and P_2 is the graph having the vertex set $V(S_m) \times V(P_2)$ and edge set $\{(v_i, v_j)(v_k, v_l) | v_i v_k \in E(S_m)$ and $v_j v_l \in E(P_2)$ and $1 \leq i, k \leq m+1, 1 \leq j, l \leq 2\}$.*

Now we compute Laplacian of product of Randić and sum-connectivity energies of complete graph, star graph, complete bipartite graph, the Crown graph, the $(S_m \wedge P_2)$ graph.

Theorem 2.5 *Let a and b be as defined above. Then the Laplacian of product of Randić and sum-connectivity energy of the complete bipartite graph $K_{n,n}$ is $2\sqrt{\frac{1}{2abn}}$.*

Proof Let the vertex set of the complete bipartite graph be $V(K_{n,n}) = \{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. Then, the Laplacian of product of Randić and sum-connectivity matrix of complete bipartite graph is given by

$$A_{lprs} = \begin{pmatrix} n & \cdots & 0 & \sqrt{\frac{1}{a(n+n)b(n^2)}} & \cdots & \sqrt{\frac{1}{a(n+n)b(n^2)}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & n & \sqrt{\frac{1}{a(n+n)b(n^2)}} & \cdots & \sqrt{\frac{1}{a(n+n)b(n^2)}} \\ \sqrt{\frac{1}{a(n+n)b(n^2)}} & \cdots & \sqrt{\frac{1}{a(n+n)b(n^2)}} & n & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{\frac{1}{a(n+n)b(n^2)}} & \cdots & \sqrt{\frac{1}{a(n+n)b(n^2)}} & & \cdots & n \end{pmatrix}.$$

Its characteristic polynomial is

$$|\lambda I - A_{lprs}| = \begin{vmatrix} (\lambda - n)I_n & -\sqrt{\frac{1}{b(n^2)a(n+n)}} J^T \\ -\sqrt{\frac{1}{b(n^2)a(n+n)}} J & (\lambda - n)I_n \end{vmatrix},$$

where J is an $n \times n$ matrix with all the entries are equal to 1. Hence, the characteristic equation is given by

$$\begin{vmatrix} \Lambda I_n & -\sqrt{\frac{1}{a(n+n)b(n^2)}} J^T \\ -\sqrt{\frac{1}{a(n+n)b(n^2)}} J & \Lambda I_n \end{vmatrix} = 0,$$

where $\Lambda = \lambda - n$ and which can be written as

$$|\Lambda I_n| \left| \Lambda I_n - \left(-\sqrt{\frac{1}{a(n+n)b(n^2)}} J \right) \frac{I_n}{\Lambda} \left(-\sqrt{\frac{1}{a(n+n)b(n^2)}} J^T \right) \right| = 0.$$

By simplification, we obtain

$$\frac{\Lambda^{n-n}}{((a(n+n)b(n^2))^n}\left|a(n+n)b(n^2)\Lambda^2 I_n - JJ^T\right| = 0,$$

which can be written as

$$\frac{\Lambda^{n-n}}{(a(n+n)b(n^2))^n}P_{JJ^T}(a(n+n)b(n^2))\Lambda^2) = 0,$$

where $P_{JJ^T}(\lambda)$ is the characteristic polynomial of the matrix ${}_nJ_n$. Thus, we have

$$\frac{\Lambda^{n-n}}{(a(n+n)bn^2)^n}((a(n+n)b(n^2))\Lambda^2 - n^2)(a(n+n)b(n^2)\Lambda^2)^{n-1} = 0,$$

which is the same as

$$\Lambda^{n+n-2}(\Lambda^2 - \frac{n^2}{b(n)^2 a(n+n)}) = 0.$$

Therefore, the spectrum of $K_{n,n}$ is given by

$$Spec\,(K_{n,n}) = \begin{pmatrix} n & n+\sqrt{\frac{1}{2abn}} & n-\sqrt{\frac{1}{2abn}} \\ n+n-2 & 1 & 1 \end{pmatrix}.$$

Hence, the Laplacian of product of Randić and sum-connectivity energy of the complete bipartite graph is

$$E_{lprs}(K_{n,n}) = \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|,$$

and

$$E_{lprs}(K_{n,n}) = 2\sqrt{\frac{1}{2abn}}$$

as desired. □

Theorem 2.6 *Let a and b be as defined above. Then the Laplacian of product of Randić and sum-connectivity energy of the S_n is*

$$\begin{aligned} E_{lprs}(S_n) &= \frac{(n-2)^2}{n} \\ &+ \left|\frac{n^2-2(n-1)}{n} + \sqrt{\frac{an^3b - 4(anb(n-1)-1)}{2nab}}\right| \\ &+ \left|\frac{n^2-2(n-1)}{n} - \sqrt{\frac{an^3b - 4(anb(n-1)-1)}{2nab}}\right|. \end{aligned}$$

Proof Let the vertex set of star graph be given by $V(S_n) = \{v_1, v_2, \cdots, v_n\}$. Then the Laplacian of product of Randić and sum-connectivity matrix of the star graph S_n is given by

$$A_{lprs} = \begin{pmatrix} n-1 & \sqrt{\frac{1}{anb(n-1)}} & \sqrt{\frac{1}{anb(n-1)}} & \cdots & \sqrt{\frac{1}{anb(n-1)}} & \sqrt{\frac{1}{anb(n-1)}} \\ \sqrt{\frac{1}{anb(n-1)}} & 1 & 0 & \cdots & 0 & 0 \\ \sqrt{\frac{1}{anb(n-1)}} & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sqrt{\frac{1}{anb(n-1)}} & 0 & 0 & \cdots & 1 & 0 \\ \sqrt{\frac{1}{anb(n-1)}} & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Hence, its characteristic polynomial is given by

$$\begin{aligned} |\lambda I - A_{lprs}| &= \begin{vmatrix} \lambda-(n-1) & -\sqrt{\frac{1}{anb(n-1)}} & -\sqrt{\frac{1}{anb(n-1)}} & \cdots & -\sqrt{\frac{1}{anb(n-1)}} \\ -\sqrt{\frac{1}{anb(n-1)}} & \lambda-1 & 0 & \cdots & 0 \\ -\sqrt{\frac{1}{anb(n-1)}} & 0 & \lambda-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sqrt{\frac{1}{anb(n-1)}} & 0 & 0 & \cdots & \lambda-1 \end{vmatrix} \\ &= \left(\sqrt{\frac{1}{anb(n-1)}}\right)^n \begin{vmatrix} \gamma & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & 0 & \cdots & 0 & 0 \\ -1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & \mu & 0 \\ -1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix}, \end{aligned}$$

where $\mu = (\lambda-1)\sqrt{anb(n-1)}$ and $\gamma = (\lambda-(n-1))\sqrt{anb(n-1)}$. Then,

$$|\lambda I - A_{lprs}| = \phi_n(\mu)\left(\sqrt{\frac{1}{a(n)b(n-1)}}\right)^n,$$

where

$$\phi_n(\mu) = \begin{vmatrix} \gamma & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & 0 & \cdots & 0 & 0 \\ -1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & \mu & 0 \\ -1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix}.$$

Applying the properties of determinant, we obtain after some simplifications $\phi_n(\mu) =$

$(\mu\phi_{n-1}(\mu) - \mu^{n-2})$ and iterating this formula, we obtain

$$\phi_n(\mu) = \mu^{n-2}(\mu\gamma - (n-1)).$$

Therefore,

$$\begin{aligned}|\lambda I - A_{lprs}| &= \left(\sqrt{\frac{1}{anb(n-1)}}\right)^n [((anb(n-1))(\lambda-1)(\lambda-(n-1)) \\ &\quad -(n-1))\,((\lambda-1)\sqrt{\frac{1}{anb(n-1)}})^{n-2}\Big].\end{aligned}$$

Thus, the characteristic equation is given by

$$(\lambda-1)^{n-2}\left((\lambda-1)(\lambda-(n-1)) - \frac{1}{anb}\right) = 0.$$

We know that

$$Spec\,(S_n) = \begin{pmatrix} 1 & n + \sqrt{\frac{an^3b-4(anb(n-1)-1)}{2nab}} & n - \sqrt{\frac{an^3b-4(anb(n-1)-1)}{2nab}} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Hence, the Laplacian of product of Randić and sum-connectivity energy of S_n is

$$\begin{aligned}E_{lprs}(S_n) &= \frac{(n-2)^2}{n} + \left|\frac{n^2-2(n-1)}{n} + \sqrt{\frac{an^3b-4(anb(n-1)-1)}{2nab}}\right| \\ &\quad + \left|\frac{n^2-2(n-1)}{n} - \sqrt{\frac{an^3b-4(anb(n-1)-1)}{2nab}}\right|.\end{aligned}$$

This completes the proof. □

Theorem 2.7 *Let a and b be as defined above. Then the Laplacian of product of Randić and sum-connectivity energy of K_n is* $2\sqrt{\frac{1}{a2b(n-1)}}$.

Proof Let the vertex set of Complete graph be given by $V(K_n) = \{v_1, v_2, \cdots, v_n\}$. Then the Laplacian of product of Randić and sum-connectivity energy of matrix of the complete graph K_n is given by

$$A_{lprs} = \begin{pmatrix} n-1 & \sqrt{\frac{1}{a2b(n-1)^3}} & \cdots & \sqrt{\frac{1}{a2b(n-1)^3}} \\ \sqrt{\frac{1}{a2b(n-1)^3}} & n-1 & \cdots & \sqrt{\frac{1}{a2b(n-1)^3}} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\frac{1}{a2b(n-1)^3}} & \sqrt{\frac{1}{a2b(n-1)^3}} & \cdots & n-1 \end{pmatrix}.$$

Hence, its characteristic polynomial is given by

$$|\lambda I - A_{lprs}| = \begin{vmatrix} \lambda-(n-1) & -\sqrt{\frac{1}{a2b(n-1)^3}} & \cdots & -\sqrt{\frac{1}{a2b(n-1)^3}} \\ -\sqrt{\frac{1}{a2b(n-1)^3}} & \lambda-(n-1) & \cdots & -\sqrt{\frac{1}{a2b(n-1)^3}} \\ \vdots & \vdots & \ddots & \vdots \\ -\sqrt{\frac{1}{a2b(n-1)^3}} & -\sqrt{\frac{1}{a2b(n-1)^3}} & \cdots & \lambda-(n-1) \end{vmatrix}$$

$$= \left(\sqrt{\frac{1}{a2b(n-1)^3}}\right)^n \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & \mu \end{vmatrix},$$

where,

$$\mu = (\lambda - (n-1))\sqrt{a2(n-1)(n-1)^2 b}.$$

Then

$$|\lambda I - A_{lprs}| = \phi_n(\mu)\left(\sqrt{\frac{1}{a2b(n-1)^3}}\right)^n$$

with

$$\phi_n(\mu) = \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & \mu \end{vmatrix}$$

$$= \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ 0 & 0 & 0 & \cdots & -1-\mu & \mu+1 \end{vmatrix}$$

$$= \quad (\mu+1)\begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & -1 \end{vmatrix} + (\mu+1)\begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & \mu \end{vmatrix}$$

and

$$\begin{aligned} \phi_n(\mu) &= -(\mu+1)^{n-1} + (\mu+1)\left[(\mu+1)^{n-2}(\mu-(n-2))\right] \\ &= -(\mu+1)^{n-1} + (\mu+1)^{n-1}(\mu-(n-2)). \end{aligned}$$

Iterating this formula, we obtain

$$\phi_n(\mu) = \mu^{n-2}(\mu^2-(n-1)),$$

thus the characteristic equation is given by

$$\left(\sqrt{\frac{1}{a2b(n-1)^3}}\right)^n (\mu+1)^{n-1}(\mu-(n-1)) = 0.$$

Hence, the Laplacian of product of Randić and sum-connectivity energy of K_n is

$$E_{lprs}(K_n) = \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|, \text{ i.e., } E_{lprs}(K_n) = 2\sqrt{\frac{1}{a2b(n-1)}}. \qquad \square$$

Theorem 2.8 *Let the vertex set $V(S_n^0)$ of the crown graph be given by $V(S_n^0) = \{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. Then, the Laplacian of product of Randić and sum-connectivity energy of the crown graph is $4\sqrt{\frac{1}{a2(n-1)b}}$.*

Proof The Laplacian of product of Randić and sum-connectivity energy matrix of crown graph is given by

$$A_{lprs} = \begin{pmatrix} n-1 & 0 & \cdots & 0 & 0 & \alpha & \cdots & \alpha \\ 0 & n-1 & \cdots & 0 & \alpha & 0 & \cdots & \alpha \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n-1 & \alpha & \alpha & \cdots & 0 \\ 0 & \alpha & \cdots & \alpha & n-1 & 0 & \cdots & 0 \\ \alpha & 0 & \cdots & \alpha & 0 & n-1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha & \alpha & \cdots & 0 & 0 & 0 & \cdots & n-1 \end{pmatrix},$$

where $\alpha = \sqrt{\frac{1}{ab(2(n-1)^3)}}$. Its characteristic polynomial is

$$|\lambda I - A_{lprs}| = \begin{vmatrix} (\lambda-(n-1))I_n & -\sqrt{\frac{1}{ab(2(n-1)^3)}}K^T \\ \sqrt{\frac{1}{ab(2(n-1)^3)}}K & (\lambda-(n-1)I_n \end{vmatrix},$$

where K is an $n \times n$ matrix. Hence the characteristic equation is given by

$$\begin{vmatrix} \Lambda I_n & -\sqrt{\frac{1}{ab(2(n-1)^3)}}K^T \\ \sqrt{\frac{1}{ab(2(n-1)^3)}}K & \Lambda I_n. \end{vmatrix} = 0,$$

where $\Lambda = (\lambda - (n-1))$. This is the same as

$$|\Lambda I_n| \left| \Lambda I_n - \left(-\sqrt{\frac{1}{ab(2(n-1)^3)}}K\right) \frac{I_n}{\Lambda} \left(-\sqrt{\frac{1}{ab(2(n-1)^3)}}K^T\right) \right| = 0,$$

which can be written as

$$\left(\frac{1}{ab(2(n-1)^3)}\right)^n P_{KK^T}((ab(2(n-1)^3))\Lambda^2) = 0,$$

where $P_{KK^T(\Lambda)}$ is the characteristic polynomial of the matrix KK^T. Thus we have

$$\left(\frac{1)}{ab(2(n-1)^3)}\right)^n [ab(2(n-1)^3))\Lambda^2 - (n-1)^2][ab(2(n-1)^3))\Lambda^2 - 1]^{n-1} = 0,$$

which is the same as

$$\left(\Lambda^2 - \frac{1}{2ab(n-1)}\right)\left(\Lambda^2 - \frac{1}{a2b(n-1)^3}\right)^{n-1} = 0.$$

Therefore

$$Spec\,(S_n^0) = \begin{pmatrix} \beta & -\beta & \beta & -\beta \\ 1 & 1 & n-1 & n-1 \end{pmatrix},$$

where, $\beta = \sqrt{\frac{1}{2ab(n-1)}} + (n-1)$. Hence, the Laplacian of product of Randić and sum-connectivity energy of crown graph is

$$E_{lprs}(S_n^0) = 4\sqrt{\frac{1}{a2b(n-1)}}.$$

□

Theorem 2.9 *Let a and b be as defined above. Then the Laplacian of product of Randić and sum-connectivity energy of $(S_m \wedge P_2)$ is*

$$\frac{(2n-4)(n-2)}{n} + 2\left|\frac{n^2-2(n-1)}{n} + \sqrt{\frac{an^3b - 4(anb(n-1)-1)}{2nab}}\right| + 2\left|\frac{n^2-2(n-1)}{n} - \sqrt{\frac{an^3b - 4(anb(n-1)-1)}{2nab}}\right|.$$

Proof Let the vertex set of $(S_m \wedge P_2)$ graph be given by $V(S_m \wedge P_2) = \{v_1, v_2, \cdots, v_{2m+2}\}$. Then the Laplacian of product of Randić and sum-connectivity matrix of $(S_m \wedge P_2)$ graph is given by

$$A_{lprs} = \begin{pmatrix} n-1 & 0 & \cdots & 0 & 0 & \omega & \cdots & \omega \\ 0 & 1 & \cdots & 0 & \omega & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \omega & 0 & \cdots & 0 \\ 0 & \omega & \cdots & \omega & n-1 & 0 & \cdots & 0 \\ \omega & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 \end{pmatrix}_{2n \times 2n},$$

where, $m+1=n$ and $\omega = \sqrt{\frac{1}{anb(n-1)}}$. Its characteristic polynomial is given by

$$|\lambda I - A_{lprs}| = \begin{vmatrix} \lambda-(n-1) & 0 & \cdots & 0 & 0 & -\omega & \cdots & -\omega \\ 0 & \lambda-1 & \cdots & 0 & -\omega & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda-1 & -\sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 \\ 0 & -\omega & \cdots & -\omega & \lambda-(n-1) & 0 & \cdots & 0 \\ -\omega & 0 & \cdots & 0 & 0 & \lambda-1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\omega & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda-1 \end{vmatrix}_{2n \times 2n}.$$

Hence, its characteristic equation is given by

$$(\omega)^{2n} \begin{vmatrix} \omega & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 \\ 0 & \Lambda & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda & -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & -1 & \omega & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n \times 2n} = 0,$$

where $\Lambda = \sqrt{na(n-1)b}(\lambda-1)$ and $\gamma = \sqrt{na(n-1)b}(\lambda-(n-1))$.

Let

$$\phi_{2n}(\Lambda) = \begin{vmatrix} \gamma & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \gamma & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n}.$$

$$= (-1)^{2n+2n}\Lambda \begin{vmatrix} \gamma & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \gamma & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{(2n-1)\times(2n-1)}$$

$$+(-1)^{2n+2} \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \gamma & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}$$

and let

$$\Psi_{2n-1}(\Lambda) = (-1)^{2n+2} \times \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \gamma & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}.$$

Applying the properties of determinant, we obtain

$$\Psi_{2n-1}(\Lambda) = -\Lambda^{n-2}\Theta_n(\Lambda),$$

after some simplifications, where

$$\Theta_n(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & -1 \\ 0 & 0 & \Lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \gamma \end{vmatrix}_{n\times n}.$$

Then,

$$\phi_{2n}(\Lambda) = -\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda\phi_{2n-1}(\Lambda).$$

Now, proceeding as the above we obtain

$$\begin{aligned} \phi_{2n-1}(\Lambda) &= (-1)^{(2n-1)+2}\Psi_{2n-2}(\Lambda) \\ &\quad +(-1)^{(2n-1)+(2n-1)}\Lambda\phi_{2n-2}(\Lambda) \\ &= -\Lambda^{n-3}\Theta_n(\Lambda) + \Lambda\phi_{2n-2}(\Lambda) \end{aligned}$$

and continuously like this, we obtain

$$\phi_{2n}(\Lambda) = -(n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{(n-1)}\xi_{n+1}(\Lambda)$$

at the $(n-1)^{th}$ step, where

$$\xi_{n+1}(\Lambda) = \begin{vmatrix} \gamma & 0 & 0 & \cdots & 0 \\ 0 & \Lambda & 0 & \cdots & -1 \\ 0 & 0 & \Lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{(n+1)\times(n+1)}.$$

$$\begin{aligned} \phi_{2n}(\Lambda) &= -(n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n-1}\gamma\Theta_n(\Lambda) \\ &= -(n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n-1}\gamma\Theta_n(\Lambda) \\ &= (\Lambda^{n-1}\gamma - (n-1)\Lambda^{n-2})\Theta_n(\Lambda). \end{aligned}$$

Applying the properties of determinants, we obtain

$$\Theta_n(\Lambda) = \Lambda^{n-1}\gamma - (n-1)\Lambda^{n-2}.$$

Therefore

$$\phi_{2n}(\Lambda) = (\Lambda^{n-1}\gamma - (n-1)\Lambda^{n-2})^2.$$

Hence, characteristic equation becomes

$$\left(\sqrt{\frac{1}{anb(n-1)}}\right)^{2n} \phi_{2n}(\Lambda) = 0,$$

which is the same as

$$\left(\sqrt{\frac{1}{anb(n-1)}}\right)^{2n} (\Lambda^{n-1}\gamma - (n-1)\Lambda^{n-2})^2 = 0,$$

which can be reduced to

$$(\lambda-1)^{2n-4}((nab(n-1)(\lambda-1)(\lambda-(n-1)) - (n-1))^2 = 0.$$

Therefore,

$$Spec\,((S_m \wedge P_2)) = \begin{pmatrix} 1 & n + \sqrt{\frac{an^3b-4(anb(n-1)-1)}{2nab}} & n - \sqrt{\frac{an^3b-4(anb(n-1)-1)}{2nab}} \\ 2n-4 & 2 & 2 \end{pmatrix}.$$

Hence, the Laplacian of product of Randić and sum-connectivity energy of $(S_m \wedge P_2)$ graph is

$$E_{lprs}((S_m \wedge P_2)) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$$

Whence,

$$E_{lqrs}((S_m \wedge P_2)) = \frac{(2n-4)(n-2)}{n} + 2\left|\frac{n^2-2(n-1)}{n} + \sqrt{\frac{an^3b-4(anb(n-1)-1)}{2nab}}\right| + 2\left|\frac{n^2-2(n-1)}{n} - \sqrt{\frac{an^3b-4(anb(n-1)-1)}{2nab}}\right|.$$

This completes the proof. □

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On Skew-Sum Eccentricity Energy of Digraphs

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Abstract: In this paper we introduce the concept of skew-sum eccentricity energy of directed graphs. We obtain upper and lower bounds for skew-sum eccentricity energy of digraphs. Then we compute the skew-sum eccentricity energy of some graphs such as star digraph, complete bipartite digraph, the $(S_m \wedge P_2)$ digraph, $(n, 2n-3)$ strong vertex graceful digraph and a crown digraph.

Key Words: Energy, eccentricity, skew-sum eccentricity energy, digraph.

AMS(2010): 05C50.

§1. Introduction

In 2018, B. Sharada and Mohammad Issa Ahmed Sowaity [5] introduced the sum eccentricity energy of a simple graph G as follows. The sum eccentricity adjacency matrix of G is the $n \times n$ matrix (a_{ij}), where

$$a_{ij} = \begin{cases} e(v_i) + e(v_j), & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

The sum eccentricity energy of G is defined as the sum of absolute values of the eigenvalues of the sum eccentricity adjacency matrix of G.

In 2010, Adiga, Balakrishnan and Wasin So [1] introduced the skew energy of a digraph as follows. Let D be a digraph of order n with vertex set $V(D) = \{v_1, v_2, \cdots, v_n\}$ and arc set $\Gamma(D) \subset V(D) \times V(D)$ where $(v_i, v_i) \notin \Gamma(D)$ for all i and $(v_i, v_j) \in \Gamma(D)$ implies $(v_j, v_i) \notin \Gamma(D)$. The skew-adjacency matrix of D is the $n \times n$ matrix $S(D) = (s_{ij})$ where $s_{ij} = 1$ whenever $(v_i, v_j) \in \Gamma(D)$, $s_{ij} = -1$ whenever $(v_j, v_i) \in \Gamma(D)$ and $s_{ij} = 0$ otherwise. Hence $S(D)$ is a skew symmetric matrix of order n and all its eigenvalues are of the form $i\lambda$ where $i = \sqrt{-1}$ and λ is a real number. The skew energy of G is the sum of the absolute values of eigenvalues of $S(D)$.

Motivated by these works, we introduce the concept of skew-sum eccentricity energy of a digraph as follows. Let D be a digraph of order n with vertex set $V(D) = \{v_1, v_2, \cdots, v_n\}$ and arc set $\Gamma(D) \subset V(D) \times V(D)$ where $(v_i, v_i) \notin \Gamma(D)$ for all i and $(v_i, v_j) \in \Gamma(D)$ implies $(v_j, v_i) \notin \Gamma(D)$. Then the skew-sum eccentricity adjacency matrix of D is the $n \times n$ matrix

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$A_{sse} = (a_{ij})$ where,

$$a_{ij} = \begin{cases} e(v_i) + e(v_j), & \text{if } (v_i, v_j) \in \Gamma(D), \\ -\left(e(v_i) + e(v_j)\right), & \text{if } (v_j, v_i) \in \Gamma(D), \\ 0, & \text{otherwise.} \end{cases}$$

Then the skew-sum eccentricity energy $E_{sse}(D)$ of D is defined as the sum of the absolute values of eigenvalues of A_{sse}. For example, let D be the directed circle on 4 vertices with the arc set $\{(1,2),(2,3),(3,4),(4,1)\}$. Then

$$A_{sse} = \begin{pmatrix} 0 & 4 & 0 & -4 \\ -4 & 0 & 4 & 0 \\ 0 & -4 & 0 & 4 \\ 4 & 0 & -4 & 0 \end{pmatrix}.$$

with the characteristic equation $\lambda^4 + 64\lambda^2 = 0$. Its eigenvalues are $8i$, 0, 0, $-8i$ and the skew-sum eccentricity energy of D is 16.

In section 2 of this paper we obtain the upper and lower bounds for skew-sum eccentricity energy of digraphs. In Section 3 we compute the skew-sum eccentricity energy of some directed graphs such as complete bipartite digraph, star digraph, the $(S_m \wedge P_2)$ digraph, $(n, 2n-3)$ strong vertex graceful digraph and a crown digraph.

§2. Upper and Lower Bounds for Skew-Sum Eccentricity Energy

Theorem 2.1 *Let D be a simple digraph of order n. Then*

$$E_{sse}(D) \le \sqrt{2n \sum_{j\sim k} \left(e(v_j) + e(v_k)\right)^2}.$$

Proof Let $i\lambda_1, i\lambda_2, i\lambda_3, \cdots, i\lambda_n$, be the eigenvalues of A_{sse}, where $\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \lambda_4 \ge \cdots \ge \lambda_n$. Since

$$\sum_{j=1}^{n} (i\lambda_j)^2 = tr(A_{sse}^2) = -\sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}^2 = -2\sum_{j\sim k} \left(e(v_j) + e(v_k)\right)^2,$$

we have

$$\sum_{j=1}^{n} |\lambda_j|^2 = 2\sum_{j\sim k} \left(e(v_j) + e(v_k)\right)^2. \tag{1}$$

Applying the Cauchy-Schwartz inequality

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 \le \left(\sum_{j=1}^{n} a_j^2\right) \cdot \left(\sum_{j=1}^{n} b_j^2\right)$$

with $a_j = 1$, $b_j =\mid \lambda_j \mid$, we obtain

$$\begin{aligned} E_{sse}(D) &= \sum_{j=1}^{n}|\lambda_j| = \sqrt{(\sum_{j=1}^{n}|\lambda_j|)^2} \leq \sqrt{n\sum_{j=1}^{n}|\lambda_j|^2} \\ &= \sqrt{2n\sum_{j\sim k}(e(v_j)+e(v_k))^2}. \end{aligned}$$

This completes the proof. □

Theorem 2.2 *Let D be a simple digraph of order n. Then*

$$E_{sse}(D) \geq \sqrt{2\sum_{j\sim k}(e(v_j)+e(v_k))^2 + n(n-1)p^{\frac{2}{n}}},. \quad (2)$$

where $p = |detA_{sg}| = \prod_{j=1}^{n}|\lambda_j|$.

Proof Notice that

$$(E_{sse}(D))^2 = \left(\sum_{j=1}^{n}|\lambda_j|\right)^2 = \sum_{j=1}^{n}|\lambda_j|^2 + \sum_{1\leq j\neq k\leq n}|\lambda_j||\lambda_k|.$$

By the arithmetic-geometric mean inequality, we get

$$\begin{aligned} \sum_{1\leq j\neq k\leq n}|\lambda_j||\lambda_k| &= |\lambda_1|(|\lambda_2|+|\lambda_3|+\cdots+|\lambda_n|) \\ &\quad +|\lambda_2|(|\lambda_1|+|\lambda_3|+\cdots+|\lambda_n|) \\ &\quad +\cdots+|\lambda_n|(|\lambda_1|+|\lambda_2|+\cdots+|\lambda_{n-1}|) \\ &\geq n(n-1)(|\lambda_1||\lambda_2|\cdots|\lambda_n|)^{\frac{1}{n}}(|\lambda_1|^{n-1}|\lambda_2|^{n-1}\cdots|\lambda_n|^{n-1})^{\frac{1}{n(n-1)}} \\ &= n(n-1)\left(\prod_{j=1}^{n}|\lambda_j|\right)^{\frac{1}{n}}\left(\prod_{j=1}^{n}|\lambda_j|\right)^{\frac{1}{n}} = n(n-1)\left(\prod_{j=1}^{n}|\lambda_j|\right)^{\frac{2}{n}}. \end{aligned}$$

Thus

$$(E_{sse}(D))^2 \geq \sum_{j=1}^{n}|\lambda_j|^2 + n(n-1)\left(\prod_{j=1}^{n}|\lambda_j|\right)^{\frac{2}{n}}.$$

From the equation (1), we get

$$(E_{sse}(D))^2 \geq 2\sum_{j\sim k}(e(v_j)+e(v_k))^2 + n(n-1)p^{\frac{2}{n}},$$

which gives the inequality (2). □

§3. Skew-Sum Eccentricity Energies of Some Graph Families

We begin with some basic definitions and notations.

Definition 3.1 *A graph G is said to be complete if every pair of its distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n.*

Definition 3.2 *A bigraph or bipartite graph G is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every line of G joins V_1 with V_2. (V_1, V_2) is a bipartition of G. If G contains every line joining V_1 and V_2, then G is a complete bigraph. If V_1 and V_2 have m and n points, we write $G = K_{m,n}$. A star is a complete bigraph $K_{1,n}$.*

Definition 3.3 *A crown graph S_n^0 for an integer $n \geq 3$ is the graph with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and edge set $\{u_iv_j; 1 \leq i, j \leq n, i \neq j\}$. S_n^0 is therefore S_n^0 coincides with complete bipartite graph $K_{n,n}$ with the horizontal edges removed.*

Definition 3.4 *The conjunction $(S_m \wedge P_2)$ of $S_m = \overline{K}_m + K_1$ and P_2 is the graph having the vertex set $V(S_m) \times V(P_2)$ and edge set $\{(v_i, v_j)(v_k, v_l) | v_iv_k \in E(S_m)$ and $v_jv_l \in E(P_2)$ and $1 \leq i, k \leq m+1, 1 \leq j, l \leq 2\}$.*

Definition 3.5 *A graph G is said to be strong vertex graceful if there exists a bijective mapping $f : V(G) \to \{1, 2, \cdots, n\}$ such that for the induced labeling $f^+ : E(G) \to \mathbb{N}$ defined by $f^+(e) = f(u) + f(v)$, where $e = uv$, the set $f^+(E(G))$ consists of consecutive integers.*

Now we compute skew-sum eccentricity energies of some directed graphs such as complete bipartite digraph, star digraph, the $(S_m \wedge P_2)$ digraph, $(n, 2n-3)$ strong vertex graceful digraph and a crown digraph.

Theorem 3.6 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of complete bipartite digraph $K_{m,n} (m > 1)$ be respectively given by $V(D) = \{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n\}$ and $\Gamma(D) = \{(u_i, v_j) \,|\, 1 \leq i \leq m, 1 \leq j \leq n\}$. Then, the skew-sum eccentricity energy of $K_{m,n}$ is $8\sqrt{mn}$.*

Proof The skew-sum eccentricity matrix of complete bipartite digraph is given by

$$A_{sse} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 4 & 4 & \cdots & 4 \\ 0 & 0 & \cdots & 0 & 4 & 4 & \cdots & 4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 4 & 4 & \cdots & 4 \\ -4 & -4 & \cdots & -4 & 0 & 0 & \cdots & 0 \\ -4 & -4 & \cdots & -4 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -4 & -4 & \cdots & -4 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then, its characteristic polynomial is

$$
\begin{aligned}
|\lambda I - A_{sse}| &= \begin{vmatrix}
\lambda & 0 & \cdots & 0 & -4 & -4 & \cdots & -4 \\
0 & \lambda & \cdots & 0 & -4 & -4 & \cdots & -4 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda & -4 & -4 & \cdots & -4 \\
4 & 4 & \cdots & 4 & \lambda & 0 & \cdots & 0 \\
4 & 4 & \cdots & 4 & 0 & \lambda & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
4 & 4 & \cdots & 4 & 0 & 0 & \cdots & \lambda
\end{vmatrix} \\
&= \begin{vmatrix} \lambda I_m & -4J^T \\ 4J & \lambda I_n \end{vmatrix},
\end{aligned}
$$

where, J is an $n \times m$ matrix with all the entries are equal to 1. Hence, its characteristic equation is given by

$$\begin{vmatrix} \lambda I_m & -4J^T \\ 4J & \lambda I_n \end{vmatrix} = 0,$$

which can be written as

$$|\lambda I_m| \left| \lambda I_n - (4J)\, \frac{I_m}{\lambda} \left(-4J^T\right) \right| = 0.$$

By simplification, we obtain

$$16\lambda^{m-n} \left| \frac{\lambda^2}{16} I_n + JJ^T \right| = 0,$$

which can be written as

$$16\lambda^{m-n} P_{JJ^T}\left(\frac{-\lambda^2}{16}\right) = 0,$$

where $P_{JJ^T}(\lambda)$ is the characteristic polynomial of the matrix ${}_mJ_n$. Thus, we have

$$16\lambda^{m-n} \left(\frac{\lambda^2}{16} + mn \right) \left(\frac{\lambda^2}{16} \right)^{n-1} = 0,$$

which is the same as

$$\lambda^{m+n-2}(\lambda^2 + 16mn) = 0.$$

Hence,

$$Spec\,(D) = \begin{pmatrix} 0 & i4\sqrt{mn} & -i4\sqrt{mn} \\ m+n-2 & 1 & 1 \end{pmatrix}.$$

Therefore, the skew-sum eccentricity energy of the complete bipartite digraph is

$$E_{sse}(D) = 8\sqrt{mn},$$

as desired. □

Theorem 3.7 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of star digraph S_n be respectively given by $V(D) = \{v_1, v_2, \cdots, v_n\}$ and $\Gamma(D) = \{(v_1, v_j) \,|\, 2 \le j \le n\}$. Then, the skew-sum eccentricity energy of D is $6\sqrt{n-1}$.*

Proof The skew-sum eccentricity matrix of the star digraph D is given by

$$A_{sse} = \begin{pmatrix} 0 & 3 & 3 & \cdots & 3 & 3 \\ -3 & 0 & 0 & \cdots & 0 & 0 \\ -3 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -3 & 0 & 0 & \cdots & 0 & 0 \\ -3 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Hence, its characteristic polynomial is given by

$$|\lambda I - A_{sse}| = \begin{vmatrix} \lambda & -3 & -3 & \cdots & -3 \\ 3 & \lambda & 0 & \cdots & 0 \\ 3 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 3 & 0 & 0 & \cdots & \lambda \end{vmatrix} = 3^n \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ 1 & \mu & 0 & \cdots & 0 & 0 \\ 1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & \mu & 0 \\ 1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix},$$

where $\mu = \frac{\lambda}{3}$. Then, $|\lambda I - A_{sse}| = 3^n \phi_n(\mu)$ with

$$\phi_n(\mu) = \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ 1 & \mu & 0 & \cdots & 0 & 0 \\ 1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & \mu & 0 \\ 1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix}.$$

Using the properties of the determinants, we obtain after some simplifications

$$\phi_n(\mu) = (\mu^{n-2} + \mu\phi_{n-1}(\mu)).$$

Iterating by this formula, we obtain

$$\phi_n(\mu) = \mu^{n-2}(\mu^2 + n - 1).$$

Therefore

$$|\lambda I - A_{sse}| = 3^n \left[\left(\frac{\lambda^2}{9} + (n-1) \right) \left(\frac{\lambda}{3} \right)^{n-2} \right].$$

Consequently, the characteristic equation is given by

$$\lambda^{n-2} \left(\lambda^2 + 9(n-1) \right) = 0.$$

Hence

$$Spec\,(D) = \begin{pmatrix} 0 & i3\sqrt{n-1} & -i3\sqrt{n-1} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Thus, the skew-sum eccentricity energy of D is $E_{sse}(D) = 6\sqrt{n-1}$. □

Theorem 3.8 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of crown digraph $S_n^0 (n > 2)$ be respectively given by $V(D) = \{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and $\Gamma(D) = \{(u_i, v_j) \,|\, 1 \leq i \leq n, 1 \leq j \leq n, i \neq j\}$. Then, the skew-sum eccentricity energy of the crown digraph is $16(n-1)$.*

Proof The skew-sum eccentricity matrix of crown digraph is given by

$$A_{sse} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 4 & \cdots & 4 \\ 0 & 0 & \cdots & 0 & 4 & 0 & \cdots & 4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 4 & 4 & \cdots & 0 \\ 0 & -4 & \cdots & -4 & 0 & 0 & \cdots & 0 \\ -4 & 0 & \cdots & -4 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -4 & -4 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then, its characteristic polynomial is

$$|\lambda I - A_{sse}| = \begin{vmatrix} \lambda I_n & -4K^T \\ 4K & \lambda I_n \end{vmatrix},$$

where K is an $n \times n$ matrix. Hence the characteristic equation is given by

$$\begin{vmatrix} \lambda I_n & -4K^T \\ 4K & \lambda I_n. \end{vmatrix} = 0.$$

This is same as

$$|\lambda I_n|\left|\lambda I_n-(4K)\frac{I_n}{\lambda}\left(-4K^T\right)\right|=0,$$

which can be written as

$$16^n P_{KK^T}(-\frac{\lambda^2}{16})=0,$$

where $P_{KK^T(\lambda)}$ is the characteristic polynomial of the matrix KK^T. Thus, we have

$$16^n\left[\frac{\lambda^2}{16}+(n-1)^2\right]\left[\frac{\lambda^2}{16}+1\right]^{n-1}=0,$$

which is same as

$$\left(\lambda^2+16(n-1)^2\right)\left(\lambda^2+16\right)^{n-1}=0.$$

Therefore

$$Spec\,(D)=\begin{pmatrix} i4(n-1) & -i4(n-1) & i4 & -i4 \\ 1 & 1 & n-1 & n-1 \end{pmatrix}$$

. Hence the skew-sum eccentricity energy of crown digraph is

$$E_{sse}(D)=16(n-1)$$

as desired. □

Theorem 3.9 *Let the vertex set* $V(D)$ *and arc set* $\Gamma(D)$ *of digraph* $(S_m\wedge P_2)(m>1)$ *be respectively given by*

$$V(D)=\{v_1,v_2,\cdots,v_{2m+2}\},$$

$$\Gamma(D)=\{(v_1,v_j),(v_{m+2},v_k)\,|\,2\le k\le m+1,m+3\le j\le 2m+2\}.$$

Then the skew-sum eccentricity energy of D *is* $12\sqrt{n-1}$.

Proof The skew-sum eccentricity matrix of $(S_m\wedge P_2)$ digraph is given by

$$A_{sse}=\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 3 & \cdots & 3 \\ 0 & 0 & \cdots & 0 & -3 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -3 & 0 & \cdots & 0 \\ 0 & 3 & \cdots & 3 & 0 & 0 & \cdots & 0 \\ -3 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -3 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{2n\times 2n},$$

where $m+1=n$. Then, its characteristic polynomial is given by

$$|\lambda I - A_{sse}| = \begin{vmatrix} \lambda & 0 & \cdots & 0 & 0 & -3 & \cdots & -3 \\ 0 & \lambda & \cdots & 0 & 3 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & 3 & 0 & \cdots & 0 \\ 0 & -3 & \cdots & -3 & \lambda & 0 & \cdots & 0 \\ 3 & 0 & \cdots & 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 3 & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{2n\times 2n}.$$

Hence, the characteristic equation is given by

$$3^{2n}\begin{vmatrix} \Lambda & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 \\ 0 & \Lambda & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda & 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n} = 0,$$

where $\Lambda = \frac{\lambda}{3}$. Let

$$\phi_{2n}(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \Lambda & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n}.$$

$$= (-1)^{2n+2n}\Lambda \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \Lambda & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{(2n-1)\times(2n-1)}$$

$$+(-1)^{2n+1} \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}$$

and let

$$\Psi_{2n-1}(\Lambda) = (-1)^{2n+1} \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}.$$

Applying the properties of the determinants, we obtain

$$\Psi_{2n-1}(\Lambda) = \Lambda^{n-2}\Theta_n(\Lambda),$$

after some simplifications, where

$$\Theta_n(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 1 \\ 0 & \Lambda & 0 & \cdots & 1 \\ 0 & 0 & \Lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{n\times n}.$$

Then, we have

$$\phi_{2n}(\Lambda) = \Lambda^{n-2}\Theta_n(\Lambda) + \Lambda\phi_{2n-1}(\Lambda).$$

Now, proceeding as the above, we obtain

$$\begin{aligned} \phi_{2n-1}(\Lambda) &= (-1)^{(2n-1)+1}\Psi_{2n-2}(\Lambda) + (-1)^{(2n-1)+(2n-1)}\Lambda\phi_{2n-2}(\Lambda) \\ &= \Lambda^{n-3}\Theta_n(\Lambda) + \Lambda\phi_{2n-2}(\Lambda). \end{aligned}$$

and continuous like this, we finally obtain

$$\phi_{2n}(\Lambda) = (n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{(n-1)}\xi_{n+1}(\Lambda)$$

at the $(n-1)^{th}$ step, where

$$\xi_{n+1}(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 \\ 0 & \Lambda & 0 & \cdots & 1 \\ 0 & 0 & \Lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{(n+1)\times(n+1)}.$$

$$\begin{aligned} \phi_{2n}(\Lambda) &= (n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n-1}\Lambda\Theta_n(\Lambda) \\ &= (n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n}\Theta_n(\Lambda) \\ &= ((n-1)\Lambda^{n-2} + \Lambda^n)\Theta_n(\Lambda). \end{aligned}$$

Using the properties of the determinants, we get that

$$\Theta_n(\Lambda) = (n-1)\Lambda^{n-2} + \Lambda^n.$$

Therefore

$$\phi_{2n}(\Lambda) = ((n-1)\Lambda^{n-2} + \Lambda^n)^2.$$

Hence, the characteristic equation becomes

$$3^{2n}\phi_{2n}(\Lambda) = 0,$$

which is the same as

$$3^{2n}((n-1)\Lambda^{n-2} + \Lambda^n)^2 = 0$$

and can be reduced to

$$\lambda^{2n-4}((n-1) + \frac{\lambda}{9}^2)^2 = 0.$$

Therefore

$$Spec\,(D) = \begin{pmatrix} 0 & i3\sqrt{n-1} & -i3\sqrt{n-1} \\ 2n-4 & 2 & 2 \end{pmatrix}.$$

Hence the skew-sum eccentricity energy of $(S_m \wedge P_2)$ digraph is

$$E_{sse}(D) = 12\sqrt{n-1}.$$

□

Theorem 3.10 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of strong vertex graceful digraph $(n, 2n-3)$ $D = K_2 + \overline{K}_{n-2} (n > 3)$ be respectively given by $V(D) = \{v_1, v_2, \cdots, v_n\}$ and $\Gamma(D) = \{(v_1, v_j) \,|\, 2 \le j \le n\} \cup \{(v_j, v_n) \,|\, 2 \le j \le n-1\}$. Then, the skew-sum eccentricity energy of D is $2\sqrt{4 + 18(n-2)}$.*

Proof The skew-sum eccentricity matrix of the graph is given by

$$A_{sse} = \begin{pmatrix} 0 & 3 & 3 & \cdots & 2 \\ -3 & 0 & 0 & \cdots & 3 \\ -3 & 0 & 0 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -2 & -3 & -3 & \cdots & 0 \end{pmatrix}$$

and its characteristics polynomial is

$$|\lambda I - A_{sse}| = 3^n \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -\gamma \\ 1 & \mu & 0 & \cdots & 0 & -1 \\ 1 & 0 & \mu & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & \mu & -1 \\ \gamma & 1 & 1 & \cdots & 1 & \mu \end{vmatrix},$$

where $\mu = \frac{\lambda}{3}$ and $\gamma = \frac{2}{3}$. Using the properties of the determinants, we obtain

$$|\lambda I - A_{sse}| = 3^n \left[(-1)^{2n+1}\left(\mu^2 - \gamma^2\right)\mu^{n-2} + (-1)^{2n} 2\mu\phi_{n-1}(\mu)\right] \tag{3}$$

after some simplifications, where

$$\phi_{n-1}(\mu) = \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 \\ 1 & \mu & 0 & \cdots & 0 \\ 1 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & \mu \end{vmatrix}_{(n-1)\times(n-1)}.$$

Now, as in the proof of the Theorem 3.7, we obtain

$$\phi_{n-1}(\mu) = \mu^{n-3} + \mu\phi_{n-2}(\mu).$$

Iterating with this formula, we obtain

$$\phi_{n-1}(\mu) = \mu^{n-3}(\mu^2 + n - 2). \tag{4}$$

Substituting (4) in (3) and using $\mu = \frac{\lambda}{3}$, we obtain

$$\begin{aligned} |\lambda I - A_{sse}| &= 3^n \left[-\left(\mu^2 - \gamma^2\right)(\mu)^{n-2} + 2(\mu)^{n-2}(\mu^2 + n - 2)\right] \\ &= 9\lambda^{n-2}\left(\mu^2 + \gamma^2 + 2(n-2)\right). \end{aligned}$$

Thus, the characteristic equation is given by

$$\lambda^{n-2}\left(\frac{\lambda^2}{9} + \frac{4 + 18(n-2)}{9}\right) = 0.$$

Hence

$$Spec\,(D) = \begin{pmatrix} 0 & i\sqrt{4 + 18(n-2)} & -i\sqrt{4 + 18(n-2)} \\ n-2 & 1 & 1 \end{pmatrix}.$$

So, the skew-sum eccentricity energy of D is $E_{sse}(D) = 2\sqrt{4 + 18(n-2)}$ as desired. □

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Computation of (a,b)-KA Indices of Some Special Graphs

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Abstract: The significance of the (a, b) -KA indices lies on the fact that their special cases, for pertinently chosen values of the parameters a and b, coincide with the vast majority of previously considered vertex based topological indices. In this paper, we computed the (a, b) - KA indices of some special class of graphs such as regular graph (hyper cube graph and generalized petersen graph), cartesian product graph (grid graph, torus graph and cylinder graph), lollipop graph and Harary graph.

Key Words: Molecular descriptors, (a, b)-KA indices, (a, b)-KA coindices.

AMS(2010): 05C05, 05C07, 05C35.

§1. Introduction

All the graphs considered here are finite, undirected, connected with no loops and multiple edges. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph $G = (V, E)$, respectively. The edge connecting the vertices u and v will be denoted by uv. Let $d_G(e)$ denote the degree of an edge e in G, which is defined by $d_G(e) = d_G(u) + d_G(v) - 2$ with $e = uv$. For graph theoretic terminology and notation not given here we refer the reader to Kulli [15].

Chemical graph theory is a branch of the mathematical chemistry which has an important effect on the development of the chemical sciences. A single number that can be used to characterize some property of the graph of a molecular structure is called a topological index for that graph. There are numerous molecular descriptors, which are also referred to as topological indices, that have found some applications in theoretical chemistry, especially in QSPR/QSAR/QSTR research. For the historical milestones, some applications and mathematical properties of graph theory, we refer to [4, 6, 8, 10, 20, 32].

The first (a, b)-KA index $KA^1_{(a,b)}(G)$, the second (a, b)-KA index $KA^2_{(a,b)}(G)$ and the third (a, b)-KA index $KA^3_{(a,b)}(G)$ of a graph G are defined as

$$KA^1_{(a,b)}(G) = \sum_{uv\in E(G)} [d_G(u)^a + d_G(v)^a]^b,$$

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$$KA^2_{(a,b)}(G) = \sum_{uv\in E(G)} [d_G(u)^a.d_G(v)^a]^b$$

and

$$KA^3_{(a,b)}(G) = \sum_{uv\in E(G)} \mid d_G(u)^a - d_G(v)^a \mid^b,$$

where a and b are real numbers.

These (a,b)-KA indices were introduced by Kulli [21] and elaborated in [24].

§2. The Particular Values of a and b in (a,b)-KA Indices

The majority of hitherto studied vertex degree based topological indices are special cases of (a,b)-KA indices, for particular values of real numbers a and b as follows:

(1) $KA^1_{(1,1)}(G) = M_1(G)$, the first Zagreb index, [12].

(2) $KA^2_{(1,1)}(G) = M_2(G)$, the second Zagreb index, [12].

(3) $KA^1_{(-1,1)}(G) = RM_1(G)$, the redefined first Zagreb index, [29].

(4) $KA^1_{(-1,-1)}(G) = RM_2(G)$, the redefined second Zagreb index, [29].

(5) $KA^1_{(2,1)}(G) = F_1(G)$, the first Forgotten index, [5].

(6) $KA^2_{(2,1)}(G) = F_2(G)$, the second Forgotten index, [16].

(7) $KA^1_{(2,2)}(G) = HF_1(G)$, the first hyper Forgotten index, [9].

(8) $KA^1_{(1,2)}(G) = HM_1(G)$, the first hyper Zagreb index, [31].

(9) $KA^2_{(1,2)}(G) = HM_2(G)$, the second hyper Zagreb index, [34].

(10) $KA^1_{(1,-\frac{1}{2})}(G) = \chi(G)$, the sum connectivity index, [35].

(11) $KA^2_{(1,-\frac{1}{2})}(G) = R(G)$, the Randic index, [28].

(12) $KA^2_{(1,\frac{1}{2})}(G) = RR(G)$, the reciprocal Randić index, [15].

(13) $\frac{1}{2}KA^1_{(1,1)}(G) = SK(G)$, the SK-index, [30].

(14) $KA^1_{(2,\frac{1}{2})}(G) = SO(G)$, the Sombor index, [7].

(15) $KA^3_{(1,1)}(G) = Alb(G)$, the Albertson ($M_i(G)$ Minus) index, [1].

(16) $KA^3_{(-1,1)}(G) = M_{in}(G)$, the misbalance indeg index, [33].

(17) $KA^3_{(\frac{1}{2},1)}(G) = M_{ro}(G)$, the misbalance rodeg index, [33].

(18) $KA^3_{(-\frac{1}{2},1)}(G) = M_{ir}(G)$, the misbalance irdeg index, [33].

(19) $KA^3_{(1,2)}(G) = \sigma(G)$, the Sigma index, [11].

(20) $KA^3_{(-2,1)}(G) = M_s(G)$, the misbalance sdeg index, [33].

(21) $KA^3_{(2,1)}(G) = MF(G)$, the minus F-index, [17].

(22) $KA^3_{(2,2)}(G) = \sigma F(G)$, the square F-index (sigma F-index), [17].

(23) $KA^3_{(2,\frac{1}{2})}(G) = RMF_c(G)$, the receiprocal minus F- index, [18].

(24) $KA^3_{(1,b)}(G) = M_i^b(G)$, the general minus index, [19].

(25) $KA^3_{(a,1)}(G) = M_i^a(G)$, the general misbalance deg index, [25].

(26) $KA^3_{(2,b)}(G) = MF^b(G)$, the general minus F- index, [18].

(27) $KA^1_{(1,b)}(G) = M_1^b(G)$, the first general Zagreb index, [26].

(29) $KA^2_{(1,b)}(G) = R^b(G)$, the general Randic index, [3].

(30) $KA^1_{(a,1)}(G) = M_a(G)$, An edge a- Zagreb index, [27].

(31) $KA^1_{(3,1)}(G) = Y(G)$, the Y-index, [2].

(32) $KA^1_{(1,\frac{1}{2})}(G) = N(G)$, the Nirmala index, [22].

(33) $KA^1_{(3,\frac{1}{2})}(G) = D(G)$, the Dharwad index, [23].

§3. Regular Graphs

An r-regular graph is a graph where each vertex has the same degree r. i.e., $d_G(u) = d_G(v) = r$.

Theorem 3.1 *Let G be an r- regular graph with $p \geq 2$ vertices. Then*

(*i*) $KA^1_{(a,b)}(G) = 2^b qr^{ab} = 2^{b-1}pr^{ab+1}$;

(*ii*) $KA^2_{(a,b)}(G) = qr^{2ab} = \frac{1}{2}pr^{2ab+1}$;

(*iii*) $KA^3_{(a,b)}(G) = 0$.

Proof Let G be an r-regular graph with $p \geq 2$ vertices. Since every vertex is of same degree r, we have $2q = pr$.

(*i*) The first (a,b)-KA index is

$$\begin{aligned} KA^1_{(a,b)}(G) &= \sum_{uv \in E(G)} [d_G(u)^a + d_G(v)^a]^b \\ &= \sum_{uv \in E(G)} [r^a + r^a]^b = 2^b qr^{ab} = 2^{b-1}pr^{ab+1}. \end{aligned}$$

(ii) The second (a,b)-KA index is

$$\begin{aligned} KA^2_{(a,b)}(G) &= \sum_{uv\in E(G)} [d_G(u)^a.d_G(v)^a]^b \\ &= \sum_{uv\in E(G)} [r^a.r^a]^b = qr^{2ab} = \frac{1}{2}pr^{2ab+1}. \end{aligned}$$

(iii) The third (a,b)-KA index is

$$\begin{aligned} KA^3_{(a,b)}(G) &= \sum_{uv\in E(G)} |d_G(u)^a - d_G(v)^a|^b \\ &= \sum_{uv\in E(G)} |r^a - r^a|^b = 0. \end{aligned}$$

Thus, the result follows. □

Corollary 3.1 *If cycle C_p with $p \geq 3$ is a 2-regular graph, then*

(i) $KA^1_{(a,b)}(C_p) = 2^{b(1+a)}q$;
(ii) $KA^2_{(a,b)}(C_p) = 4^{ab}q$.

Corollary 3.2 *If complete graph K_p with $p \geq 2$ is a $(p-1)$-regular graph, then*

(i) $KA^1_{(a,b)}(K_p) = 2^b q(p-1)^{ab}$;
(ii) $KA^2_{(a,b)}(K_p) = q(p-1)^{2ab}$.

Corollary 3.3 *If complete regular bipartite graph $K_{s,s}$ with $s \geq 1$ is a s-regular graph, then*

(i) $KA^1_{(a,b)}(K_{s,s}) = 2^b qs^{ab}$;
(ii) $KA^2_{(a,b)}(K_{s,s}) = qs^{2ab}$.

Corollary 3.4 *If n-hypercube graph Q_n is n-regular with 2^n vertices and $n2^{n-1}$ edges, then*

(i) $KA^1_{(a,b)}(Q_n) = 2^{b+n-1}n^{ab+1}$;
(ii) $KA^2_{(a,b)}(Q_n) = 2^{n-1}n^{2ab+1}$,

where the hypercube Q_n is the simple graph whose vertices are the n-tuples with entries in $0, 1$ and whose edges are the pairs of n-tuples that differ in exactly one position.

Corollary 3.5 *If G is generalized Petersen graph $GP(n,k)$ which is 3-regular with $2n$ vertices and $3n$ edges, then*

(i) $KA^1_{(a,b)}(G) = 2^b n3^{ab+1}$;
(ii) $KA^2_{(a,b)}(G) = n3^{2ab+1}$,

where the generalized Petersen graph denoted by $GP(n,k)$ for $n \geq 3$ and $1 \leq k \leq \lfloor (n-1)/2 \rfloor$ is a connected cubic graph consisting of an inner star polygon $\{n,k\}$ (circulant graph $Ci_n(k)$) and an outer regular polygon n (cycle graph C_n) with corresponding vertices in the inner and outer polygons connected with edges.

§4. Cartesian Product Graphs

According to Hammack et al. [14], the Cartesian products of graphs were defined in 1912 by Whitehead and Russel. They were repeatedly rediscovered later, notably by Sabidussi (1960) and independently by Vizing (1963). The cartesian product of two graphs G and H, denoted by $G\Box H$, is a graph with vertex set $V(G\Box H) = V(G) \times V(H)$, that is, the set $\{(g,h)/g \in G, h \in H\}$. The edge set of $G\Box H$ consists of all pairs $[(g_1,h_1),(g_2,h_2)]$ of vertices with $[g_1,g_2] \in E(G)$ and $h_1 = h_2$, or $g_1 = g_2$ and $[h_1,h_2] \in E(H)$.

4.1. Grid Graph

The $m \times n$ grid graph can be represented as a cartesian product of $P_m\Box P_n$ of a path of length $m-1$ and a path of length $n-1$. This grid graph is denoted by $G_{m,n}$ with (mn)-vertices and $(2mn-m-n)$-edges.

Theorem 4.1 *Let $G = G_{m,n}$ be an $m \times n$ grid graph. Then*

$$KA^1_{(a,b)}(G) = \begin{cases} 4\,(2^b\ 2^{ab}), & \text{if } m=n=2 \\ 2^{b(a+1)+1} & \\ +4(2^a+3^a)^b & \\ +(3n-8)2^b3^{ab}, & \text{if } m=2, n\geq 3 \\ 8(2^a+3^a)^b & \\ +2^{b+1}(m+n-6)3^{ab} & \\ +2(m+n-4)(3^a+4^a)^b & \\ +(2mn-5(m+n)+12)2^b4^{ab} & \text{if } m\geq 3, m\leq n. \end{cases}$$

$$KA^2_{(a,b)}(G) = \begin{cases} 4^{ab+1}, & \text{if } m=n=2 \\ 2.4^{ab}+4.6^{ab} & \\ +(3n-8)9^{ab}, & \text{if } m=2, n\geq 3 \\ 8.6^{ab}+2(m+n-6)9^{ab} & \\ +2(m+n-4)12^{ab} & \\ +(2mn-5(m+n)+12)16^{ab} & \text{if } m\geq 3, m\leq n. \end{cases}$$

and

$$KA^3_{(a,b)}(G) = \begin{cases} 0, & \text{if } m=n=2 \\ 4.|2^a-3^a|^b, & \text{if } m=2, n\geq 3 \\ 8.|2^a-3^a|^b & \\ +2(m+n-4)|3^a-4^a|^b & \text{if } m\geq 3, m\leq n. \end{cases}$$

Proof Let $G = G_{m,n}$ be an $m\times n$ grid graph with (mn)-vertices and $(2mn-m-n)$-edges. By algebraic method, we have

$m\times n$ grid	$(d_G(u), d_G(v))$	Number of edges
2×2	$(2,2)$	4
$2\times n$, $n\geq 3$	$(2,2)$	2
	$(2,3)$	4
	$(3,3)$	$3n-8$
$m\times n, m\leq n, m\geq 3$	$(2,3)$	8
	$(3,3)$	$2(m+n-6)$
	$(3,4)$	$2(m+n-4)$
	$(4,4)$	$2mn-5(m+n)+12$

Table 1 Degree partition of $G_{m,n}$.

By the definitions of (a,b)-KA indices and Table 1, on simplification, we have

Case 1. If $m=n=2$, then

(i) $KA^1_{(a,b)}(G)=4(2^b2^{ab})$;
(ii) $KA^2_{(a,b)}(G)=4^{ab+1}$;
(iii) $KA^3_{(a,b)}(G)=0$.

Case 2. If $m=2$ and $n\geq 3$, then

(i) $KA^1_{(a,b)}(G)=2^{b(a+1)+1}+4(2^a+3^a)^b+(3n-8)2^b3^{ab}$;
(ii) $KA^2_{(a,b)}(G)=2.4^{ab}+4.6^{ab}+(3n-8)9^{ab}$;
(iii) $KA^3_{(a,b)}(G)=4.|2^a-3^a|^b$.

Case 3. If $m\geq 3$ and $m\leq n$, then

(i) $KA^1_{(a,b)}(G)=8(2^a+3^a)^b+2^{b+1}(m+n-6)3^{ab}+2(m+n-4)(3^a+4^a)^b+(2mn-5(m+n)+12)2^b4^{ab}$;
(ii) $KA^2_{a,b}(G)=8.6^{ab}+2(m+n-6)9^{ab}+2(m+n-4)12^{ab}+(2mn-5(m+n)+12)16^{ab}$;
(iii) $KA^3_{a,b}(G)=8.|2^a-3^a|^b+2(m+n-4)|3^a-4^a|^b$.

Thus, the result follows. □

4.2. Torus Grid Graph

The Torus grid graph $T_{m,n}$ is the graph formed from the graph cartesian product $C_m\Box C_n$ of the cycle graphs C_m and C_n. $C_m\Box C_n$ is isomorphic to $C_n\Box C_m$.

Theorem 4.2 *Let $G=T_{m,n}$ be a Torus grid graph which is 4- regular graph with mn vertices and $2mn$ edges. Then*

(i) $KA^1_{(a,b)}(G)=mn.2^{b+1}4^{ab}$;
(ii) $KA^2_{(a,b)}(G)=2mn.16^{ab}$;
(iii) $KA^3_{(a,b)}(G)=0$.

Proof Let $G=T_{m,n}$ be a Torus grid graph which is 4- regular graph with mn vertices and

$2mn$ edges.

(i) The first (a, b)-KA index is

$$\begin{aligned} KA^1_{(a,b)}(G) &= \sum_{uv \in E(G)} [d_G(u)^a + d_G(v)^a]^b \\ &= \sum_{uv \in E(G)} [4^a + 4^a]^b = q2^b 4^{ab} = mn.2^{b+1}4^{ab}. \end{aligned}$$

(ii) The second (a, b)-KA index is

$$\begin{aligned} KA^2_{(a,b)}(G) &= \sum_{uv \in E(G)} [d_G(u)^a . d_G(v)^a]^b \\ &= \sum_{uv \in E(G)} [4^a . 4^a]^b = q4^{2ab} = 2mn.16^{ab}. \end{aligned}$$

(iii) The third (a, b)-KA index is

$$\begin{aligned} KA^3_{(a,b)}(G) &= \sum_{uv \in E(G)} |d_G(u)^a - d_G(v)^a|^b \\ &= \sum_{uv \in E(G)} |4^a - 4^a|^b = 0. \end{aligned}$$

Thus, the result follows. □

4.3. Cylinder Grid Graph

The cylinder grid graph $C_{m,n}$ is the graph formed from the cartesian product $P_m \times C_n$ of the path graph P_m and the cycle graph C_n. That is, the cylinder grid graph consists of m copies of C_n represented by circles and n copies of P_m represented by paths transverse the m circles.

Theorem 4.3 *Let $G = C_{m,n}$ be a cylinder grid graph with $n \geq 3$. Then,*

$$KA^1_{(a,b)}(G) = \begin{cases} n2^b 3^{ab+1}, & \text{if } m = 2 \\ n2^{b+1}3^{ab} + 2n(3^a + 4^a)^b & \\ +(2mn - 5n)2^b 4^{ab}, & \text{if } m \geq 3. \end{cases}$$

$$KA^2_{(a,b)}(G) = \begin{cases} n3^{2ab+1}, & \text{if } m = 2 \\ 2n3^{2ab} + 2n3^{ab}4^{ab} & \\ +(2mn - 5n)4^{2ab}, & \text{if } m \geq 3. \end{cases}$$

and

$$KA^3_{(a,b)}(G) = \begin{cases} 0, & \text{if } m = 2 \\ 2n|3^a - 4^a|^b, & \text{if } m \geq 3. \end{cases}$$

Proof Let $G = C_{m,n}$ be a cylinder grid graph with mn vertices, where $n \geq 3$.

Case 1. If $m = 2$, then the graph G is 3-regular with $3n$ edges. Thus

(i) The first (a, b)-KA index is

$$KA^1_{(a,b)}(G) = \sum_{uv\in E(G)} [d_G(u)^a + d_G(v)^a]^b = \sum_{uv\in E(G)} [3^a + 3^a]^b = q2^b3^{ab} = n2^b3^{ab+1}.$$

(ii) The second (a, b)-KA index is

$$KA^2_{a,b}(G) = \sum_{uv\in E(G)} [d_G(u)^a.d_G(v)^a]^b = \sum_{uv\in E(G)} [3^a.3^a]^b = q3^{2ab} = n3^{2ab+1}.$$

(iii) The third (a, b)-KA index is

$$KA^3_{(a,b)}(G) = \sum_{uv\in E(G)} |d_G(u)^a - d_G(v)^a|^b = \sum_{uv\in E(G)} |3^a - 3^a|^b = 0.$$

Case 2. If $m \geq 3$, then by algebraic method, we have Table 2.

$(d_G(u), d_G(v))$	Number of edges
$(3, 3)$	$2n$
$(3, 4)$	$2n$
$(4, 4)$	$2mn - 5n$

Table 2 Degree partition of $C_{m,n}$ with $m \geq 3$.

By the definitions of (a, b)-KA indices and Table-2, on simplification, we have

(i) $KA^1_{(a,b)}(G) = n2^{b+1}3^{ab} + 2n(3^a + 4^a)^b + (2mn - 5n)2^b4^{ab}$.

(ii) $KA^2_{(a,b)}(G) = 2n3^{2ab} + 2n3^{ab}4^{ab} + (2mn - 5n)4^{2ab}$.

(iii) $KA^3_{(a,b)}(G) = 2n|3^a - 4^a|^b$.

Thus, the result follows. □

§5. Lollipop Graph

The (m, n)-lollipop graph is the graph obtained by joining a complete graph K_m to a path P_n with a bridge.

Theorem 5.1 *Let $G = LP(m,n)$ be a Lollipop graph. Then,*

$$KA^1_{(a,b)}(G) = \begin{cases} (1+m^a)^b \\ +(m-1)[(m-1)^a+m^a]^b \\ +2^b(m-1)^{ab}\Big(\dfrac{m^2-3m}{2}+1\Big), & \text{if } n=1,\ m\geq 3 \\ (1+2^a)^b+(n-1)2^{b(1+a)} \\ +3(2^a+3^a)^b, & \text{if } m=3, n\geq 2 \\ (1+2^a)^b+(n-2)2^{b(1+a)} \\ +(2^a+m^a)^b \\ +2^b(m-1)^{ab}\Big(\dfrac{m^2-3m}{2}+1\Big) \\ +(m-1)[(m-1)^a+m^a]^b, & \text{if } m\geq 4, n\geq 2. \end{cases}$$

$$KA^2_{(a,b)}(G) = \begin{cases} m^{ab}+m^{ab}(m-1)^{ab+1} \\ +(m-1)^{2ab}\Big(\dfrac{m^2-3m}{2}+1\Big), & \text{if } n=1,\ m\geq 3 \\ 2^{ab}+(n-1)4^{ab}+3.6^{ab}, & \text{if } m=3, n\geq 2 \\ 2^{ab}+(n-2)4^{ab}+2^{ab}m^{ab} \\ +(m-1)^{ab+1}m^{ab} \\ +(m-1)^{2ab}\Big(\dfrac{m^2-3m}{2}+1\Big), & \text{if } m\geq 4, n\geq 2. \end{cases}$$

and

$$KA^3_{(a,b)}(G) = \begin{cases} |1-m^a|^b \\ +(m-1)|(m-1)^a-m^a|^b, & \text{if } n=1,\ m\geq 3 \\ |1-2^a|^b+3|2^a-3^a|^b, & \text{if } m=3, n\geq 2 \\ |1-2^a|^b+|2^a-m^a|^b \\ +(m-1)|(m-1)^a-m^a|^b, & \text{if } m\geq 4, n\geq 2. \end{cases}$$

Proof Let $G = LP(m,n)$ be a Lollipop graph.

Case 1. If $n = 1, m \geq 3$, then by algebraic computations, we have

$(d(u), d(v))$	Number of edges
$(1, m)$	1
$(m-1, m-1)$	$\frac{m^2-3m}{2}+1$
$(m-1, m)$	$m-1$

Table 3 Degree partition of $L(m,n)$ with $n = 1, m \geq 3$.

By the definitions of (a,b)-KA indices and Table-3, on simplification, we have

(i) $KA^1_{(a,b)}(G) = (1+m^a)^b + (m-1)[(m-1)^a + m^a]^b + 2^b(m-1)^{ab}\Big(\dfrac{m^2-3m}{2}+1\Big).$

(ii) $KA^2_{(a,b)}(G) = m^{ab} + m^{ab}(m-1)^{ab+1} + (m-1)^{2ab}\Big(\frac{m^2-3m}{2}+1\Big).$

(iii) $KA^3_{(a,b)}(G) = |1-m^a|^b + (m-1)|(m-1)^a - m^a|^b.$

Case 2. If $m = 3, n \geq 2$, then by algebraic computations, we have

$(d(u), d(v))$	Number of edges
$(1, 2)$	1
$(2, 2)$	$n-1$
$(2, 3)$	3

Table 4 Degree partition of $L(m, n)$ with $m = 3, n \geq 2$.

By the definitions of (a, b)-KA indices and Table-4, on simplification, we have

(i) $KA^1_{(a,b)}(G) = (1+2^a)^b + (n-1)2^{b(1+a)} + 3(2^a+3^a)^b.$

(ii) $KA^2_{(a,b)}(G) = 2^{ab} + (n-1)4^{ab} + 3.6^{ab}.$

(iii) $KA^3_{(a,b)}(G) = |1-2^a|^b + 3|2^a - 3^a|^b.$

Case 3. If $m \geq 4, n \geq 2$, then by algebraic computations.

$(d(u), d(v))$	Number of edges
$(1, 2)$	1
$(2, 2)$	$n-2$
$(2, m)$	1
$(m-1, m-1)$	$\frac{m^2-3m}{2}+1$
$(m-1, m)$	$m-1$

Table 5 Degree partition of $L(m, n)$ with $m \geq 4, n \geq 2$.

By the definitions of (a, b)-KA indices and Table-5, on simplification, we have

(i) $KA^1_{(a,b)}(G) = (1+2^a)^b + (n-2)2^{b(1+a)} + (2^a+m^a)^b + 2^b(m-1)^{ab}\Big(\frac{m^2-3m}{2}+1\Big) +$ $(m-1)[(m-1)^a + m^a]^b.$

(ii) $KA^2_{(a,b)}(G) = 2^{ab} + (n-2)4^{ab} + 2^{ab}m^{ab} + (m-1)^{ab+1}m^{ab} + (m-1)^{2ab}\Big(\frac{m^2-3m}{2}+1\Big).$

(iii) $KA^3_{(a,b)}(G) = |1-2^a|^b + |2^a - m^a|^b + (m-1)|(m-1)^a - m^a|^b.$

Thus the result follows. □

§6. Harary Graph

In 1962, Harary [13] introduced the Harary graph, which has maximum connectivity; therefore, it plays an important role in designing networks. This is a k-connected graph on p vertices of degree at least k with $\left\lceil \frac{kp}{2} \right\rceil$ edges. The Harary graph $H_{k,p}$ is constructed as follows:

1. *k even:* Let $k = 2r$. Then $H_{2r,p}$ is constructed by the vertices $0, 1, \ldots, p-1$ and two vertices i and j are joined if $i - r \leq j \leq i + r$.

2. *k odd, p even.* Let $k = 2r + 1$. Then $H_{2r+1,p}$ is constructed by first drawing $H_{2r,p}$ and then adding edges joining vertex i to vertex $i + (\frac{p}{2})$ for $1 \leq i \leq \frac{p}{2}$.

3. *k odd, p odd :* Let $k = 2r+1$. Then $H_{2r+1,p}$ is constructed by first drawing $H_{2r,p}$ and then adding edges joining vertex 0 to vertices $\frac{(p-1)}{2}$ and $\frac{(p+1)}{2}$ and vertex i to vertex $i + \frac{(p+1)}{2}$ for $1 \leq i \leq \frac{(p-1)}{2}$.

Theorem 6.1 *Let $G = H_{k,p}$ be a Harary graph.*

(1) *If k is even, then*

(*i*) $KA^1_{(a,b)}(G) = \left\lceil \frac{kp}{2} \right\rceil 2^b k^{ab}$;

(*ii*) $KA^2_{(a,b)}(G) = \left\lceil \frac{kp}{2} \right\rceil k^{2ab}$;

(*iii*) $KA^3_{(a,b)}(G) = 0$.

(2) *If k is odd and p is even, then*

(*i*) $KA^1_{(a,b)}(G) = \left\lceil \frac{kp}{2} \right\rceil 2^b k^{ab}$;

(*ii*) $KA^2_{(a,b)}(G) = \left\lceil \frac{kp}{2} \right\rceil k^{2ab}$;

(*iii*) $KA^3_{(a,b)}(G) = 0$.

(3) *If k is odd and p is odd, then*

(*i*) $KA^1_{(a,b)}(G) = \left\{ \left\lceil \frac{kp}{2} \right\rceil - (k+1) \right\} (2^b k^{ab}) + (k+1)(k^a + (k+1)^a)^b$;

(*ii*) $KA^2_{(a,b)}(G) = \left\{ \left\lceil \frac{kp}{2} \right\rceil - (k+1) \right\} (k^{2ab}) + (k+1)(k^{ab}(k+1)^{ab})$;

(*iii*) $KA^3_{(a,b)}(G) = (k+1)|k^a - (k+1)^a|^b$.

Proof Let $G = H_{k,p}$ be a Harary graph. We have

(1) If k is even, then

(*i*) The first (a,b)-KA index is

$$\begin{aligned} KA^1_{(a,b)}(G) &= \sum_{uv \in E(G)} [d_G(u)^a + d_G(v)^a]^b \\ &= \sum_{uv \in E(G)} [k^a + k^a]^b = q 2^b k^{ab} = \left\lceil \frac{kn}{2} \right\rceil 2^b k^{ab}. \end{aligned}$$

(*ii*) The second (a,b)-KA index is

$$\begin{aligned} KA^2_{(a,b)}(G) &= \sum_{uv \in E(G)} [d_G(u)^a . d_G(v)^a]^b \\ &= \sum_{uv \in E(G)} [k^a . k^a]^b = q k^{2ab} = \left\lceil \frac{kn}{2} \right\rceil k^{2ab}. \end{aligned}$$

(iii) The third (a,b)-KA index is

$$\begin{aligned} KA^3_{(a,b)}(G) &= \sum_{uv\in E(G)} |d_G(u)^a - d_G(v)^a|^b \\ &= \sum_{uv\in E(G)} |k^a - k^a|^b = 0. \end{aligned}$$

(2) If k is odd and p is even, then

(i) The first (a,b)-KA index is

$$\begin{aligned} KA^1_{(a,b)}(G) &= \sum_{uv\in E(G)} [d_G(u)^a + d_G(v)^a]^b \\ &= q2^bk^{ab} = \left\lceil \frac{kn}{2} \right\rceil 2^bk^{ab}. \end{aligned}$$

(ii) The second (a,b)-KA index is

$$\begin{aligned} KA^2_{(a,b)}(G) &= \sum_{uv\in E(G)} [d_G(u)^a . d_G(v)^a]^b \\ &= qk^{2ab} = \left\lceil \frac{kn}{2} \right\rceil k^{2ab}. \end{aligned}$$

(iii) The third (a,b)-KA index is

$$KA^3_{(a,b)}(G) = \sum_{uv\in E(G)} |d_G(u)^a - d_G(v)^a|^b = \sum_{uv\in E(G)} |k^a - k^a|^b = 0.$$

(3) If k is odd and p is odd, then

(i) The first (a,b)-KA index is

$$\begin{aligned} KA^1_{(a,b)}(G) &= \sum_{uv\in E(G)} [d_G(u)^a + d_G(v)^a]^b \\ &= \left\{ \left\lceil \frac{kp}{2} \right\rceil - (k+1) \right\} (k^a + k^a)^b + (k+1)[k^a + (k+1)^a]^b \\ &= \left\{ \left\lceil \frac{kp}{2} \right\rceil - (k+1) \right\} (2^bk^{ab}) + (k+1)[k^a + (k+1)^a]^b. \end{aligned}$$

(ii) The second (a,b)-KA index is

$$\begin{aligned} KA^2_{(a,b)}(G) &= \sum_{uv\in E(G)} [d_G(u)^a . d_G(v)^a]^b \\ &= \left\{ \left\lceil \frac{kp}{2} \right\rceil - (k+1) \right\} (k^a . k^a)^b + (k+1)[k^a . (k+1)^a]^b \\ &= \left\{ \left\lceil \frac{kp}{2} \right\rceil - (k+1) \right\} (k^{2ab}) + (k+1)[k^{ab}(k+1)^{ab}]. \end{aligned}$$

(iii) The third (a, b)-KA index is

$$\begin{aligned} KA^3_{(a,b)}(G) &= \sum_{uv\in E(G)} |d_G(u)^a - d_G(v)^a|^b \\ &= \left\{ \left\lceil \frac{kp}{2} \right\rceil - (k+1) \right\} |k^a - k^a|^b + (k+1)|k^a - (k+1)^a|^b \\ &= (k+1)|k^a - (k+1)^a|^b. \end{aligned}$$

Thus, the result follows. □

§7. Conclusions

Being new generalized version of vertex degree based topological indices, the (a, b) KA-indices lies on the fact that their special cases, for pertinently chosen values of the parameters a and b, coincide with the vast majority of previously considered topological indices. For the comparative advantages, applications and in mathematical point of view, few problems are suggested by this research, among them are the following.

Problem 7.1 *Find the extremal values and extremal graphs of the (a, b)-KA indices.*

Problem 7.2 *Find the values of (a, b)-KA indices for certain classes of chemical graphs and explore some results towards QSPR / QSAR / QSTR Model.*

Problem 7.3 *Obtain the relationship between (a, b)-KA indices in terms of other degree/ distance/spectral based topological indices.*

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The Signed Product Cordial for Corona Between Paths and Fourth Power of Cycles

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Abstract: A graph $G = (V, E)$ is called signed product cordial if it is possible to label the vertex by the function $f : V \to \{-1, 1\}$ and label the edges by $f^* : E \to \{-1, 1\}$, where $f^*(uv) = f(u).f(v)$, $u, v \in V$ so that $|v_{-1} - v_1| \leqslant 1$ and $|e_{-1} - e_1| \leqslant 1$. In this paper,some new results on signed product cordial labeling are proposed. The necessary and sufficient conditions of signed product cordial for corona between paths and fourth power of cycles are presented.

Key Words: Corona graph, fourth power, second power, signed product cordial graph, Smarandachely signed product cordial labeling.

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§1. Introduction

The graph theory is a mathematical subfield of discrete mathematics. One area of graph theory of considerable recent research is that of graph labeling. The concept of graph labeling was introduced during the sixties' of the last century by Rosa [14]. Labeling methods are used for a wide range of applications in different subjects including coding theory, computer science and communication networks. Yegnanarayanan [16] explores some of the interesting applications of graph labeling With advancement in technology the occurrence of more complex networking system is seen to have emerged. Many researches have been working with different types of labeling graphs [5,8,11]. In 1954 Harray introduced S-cordiality [12]. An excellent reference for this purpose is the survey written by Gallian [9]. The original concept of cordial graphs is due to Cahit[6]. Cordial Labeling finds its application in Automated Routing algorithms, Communications relevant Adhoc Networks and many others[10].

For more details about the graph labeling, the reader can refer to [2-4,13]. We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1([15]) *If the vertices of the graph are assigned values subject to certain conditions is known as graph labeling. Following three are the common features of any graph labeling*

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problem:

(1) *a set of numbers from which vertex labels are assigned;*

(2) *a rule that assigns a value to each edge;*

(3) *a condition that these values must satisfy.*

The present work is targeted to discuss one such labeling known as signed product cordial labeling defined as follows:

Definition 1.2 *Let $G = (V, E)$ be a graph and let $f : V \to \{-1, 1\}$ be a labeling of its vertices, and let the induced edge labeling $f^* : E \to \{-1, 1\}$ be given by $f^*(e) = (f(u).f(v))$, where $e = uv$ and $u, v \in V$. Let v_{-1} and v_1 be the numbers of vertices that are labeled by -1 and 1, respectively, and let e_{-1} and e_1 be the corresponding numbers of edges. Such a labeling is called signed product cordial if both $|v_{-1} - v_1| \leqslant 1$ and $|e_{-1} - e_1| \leqslant 1$ hold. On the other hand, if $|v_{-1} - v_1| \geq 2$ or $|e_{-1} - e_1| \geq 2$, such a labeling is called Smarandachely signed product cordial labeling.*

The concept of signed product cordial labeling was introduced by Baskar Babujee [1].

Definition 1.3 *The corona $G_1 \bigodot G_2$ of two graphs G_1 (with n_1 vertices, m_1 edges) and G_2 (with n_2 vertices, m_2 edges) is defined as the graph obtained by taking one copy of G_1 and copies of G_2, and then joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2.*

Definition 1.4 *The fourth power of a paths P_n, denoted by P_n^4, is $P_n \bigcup J$, where J is the set of all edges of the form edges v_iv_j such that $2 \leq d(v_iv_j) \leq 4$ and $i < j$ where $d(v_iv_j)$ is the shortest distance from v_i to v_j.*

§2. Terminologies and Notations

A path with m vertices and $m - 1$ edges is denoted by P_m. Also, a cycle with n vertices and n edges, denoted by C_n, and its fourth power C_n^4 has n vertices and $4n - 9$ edges. We let L_{4r} denote the labeling $(-1)_2 11$ $(-1)_2 11 \cdots (-1)_2 11$ (repeated r-times), Let L'_{4r} denote the labeling $(-1)11(-1)$ $(-1)11(-1) \cdots (-1)11(-1)$ (repeated r-times). The labeling $11(-1)_2$ $11(-1)_2 \cdots 11(-1)_2$ (repeated r-times) and labeling $1(-1)_2 1$ $1(-1)_2 1 \cdots 1(-1)_2 1$ (repeated r-times) are written S_{4r} and S'_{4r}. Let M_r denote the labeling $(-1)1$ $(-1)1 \cdots (-1)1$, zero-one repeated $r$times if r is even and $(-1)1$ $(-1)1 \cdots (-1)1(-1)$ if r is odd; for example, $M_6 = (-1)1(-1)1(-1)1$ and $M_5 = (-1)1(-1)1(-1)$. We let M'_r denote the labeling $1(-1)1(-1) \cdots 1(-1)$.

Sometimes, we modify the labeling M_r or M'_r by adding symbols at one end or the other (or both). Also, L_{4r} (or L'_{4r}) with extra labeling from right or left (or both sides). If L is a labeling for a path p_m and M is a labeling for fourth power of path C_n, then we use the notation $[L; M]$ to represent the labeling of the corona $P_m \bigodot C_n^4$. Additional notation that we use is the following: for a given labeling of the corona $P_m \bigodot C_n^4$, we let v_i and e_i (for $i = -1, 1$) be the numbers of vertices and edges, respectively, that are labeled by i of the corona $P_m \bigodot C_n^4$, and let x_i and a_i be the corresponding quantities for p_m, and we let y_i and b_i be those for C_n^4, which are connected with vertices labeled (-1) of P_m.

Similarly, let y'_i and b'_i for C_n^4 which are connected with vertices labeled 1 of P_m. It is easy to verify that $v_{-1} = x_{-1} + x_{-1}y_{-1} + x_1y'_{-1}$, $v_1 = x_1 + x_{-1}y_1 + x_1y'_1$, $e_{-1} = a_{-1} + x_{-1}b_{-1} + x_1b'_{-1} + x_{-1}y_{-1} + x_1y'_1$ and $e_1 = a_1 + x_{-1}b_1 + x_1b'_1 + x_{-1}(x_{-1}y_1) + x_1y'_{-1}$. Thus, $v_{-1} - v_1 = (x_{-1} - x_1) + x_{-1}(y_{-1} - y_1) + x_1(y'_{-1} - y'_1)$ and $e_{-1} - e_1 = (a_{-1} - a_1) + x_{-1}(b_{-1} - b_1) + x_1(b'_{-1} - b'_1) + x_{-1}(y_{-1} - y_1) - x_1(y'_{-1} - y'_1)$. When it comes to the proof, we only need to show that, for each specified combination of labeling, $|v_{-1} - v_1| \leq 1$ and $|e_{-1} - e_1| \leq 1$.

§3. Main Results

In this section, we study The signed product cordial for corona between paths and fourth power of cycles.

Lemma 3.1 *The corona $P_m \odot c_3^4$ is signed product cordial if and only if $m \neq 1$.*

Proof Since $C_3^4 \equiv P_3^4$, $P_m \odot c_3^4$ is signed product cordial [9]. □

Lemma 3.2 *If $n \equiv 0(mod\ 4)$, then $P_m \odot c_n^4$ is signed product cordial for all $m \geq 1$.*

Proof Suppose that $n = 4s$, where $s \geq 2$. The following cases will be examined.

Case 1. Suppose that $m = 1$. Then we label the vertices of $P_1 \odot c_{4s}^4$ by $[(-1); S'_{4s}]$. Therefore $x_{-1} = 1, x_1 = 0, a_{-1} = a_1 = 0, y_{-1} = y_1 = 2s, b_{-1} = 8s - 4, b_1 = 8s - 5$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 1$. As an example, Figure 1 illustrates $P_1 \odot c_8^4$. Hence, $P_1 \odot c_{4s}^4$ is signed product cordial.

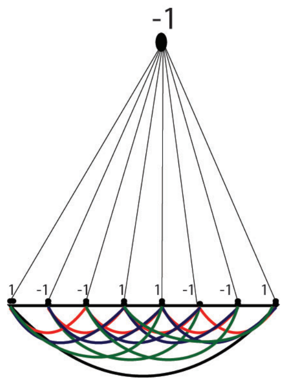

Figure 1 $P_1 \odot c_8^4$

Case 2. Suppose that $m = 2$. Then we label the vertices of $P_2 \odot c_{4s}^4$ by $[(-1)1; S'_{4s}, (-1)_3 1_3 M_{4S-6}]$. Therefore $x_{-1} = x_1 = 1, a_{-1} = 1, a_1 = 0, y_{-1} = y_1 = 2s, b_{-1} = 8s-4, b_1 = 8s-5, y'_{-1} = y'_1 = 2s$ and $b'_{-1} = 8s - 5, b'_1 = 8s - 4$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. As an example, Figure (2) illustrates $P_2 \odot c_8^4$. Hence, $P_2 \odot c_{4s}^4$ is signed product cordial.

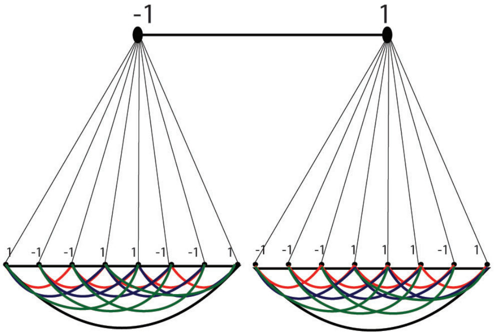

Figure 2 $P_2 \bigodot c_8^4$

Case 3. Suppose that $m = 3$. Then we label the vertices of $P_3 \bigodot c_{4s}^4$ by $[(-1)(-1)1; S'_{4S}, S'_{4S}, (-1)_3 1_3 M_{4S-6}]$. Therefore $x_{-1} = 2, x_1 = 1, a_{-1} = a_1 = 1, y_{-1} = y_1 = 2s, b_{-1} = 8s-4, b_1 = 8s-5, y'_{-1} = y'_1 = 2s$ and $b'_{-1} = 8s-5, b'_1 = 8s-4$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 1$. As an example, Figure (3) illustrates $P_3 \bigodot c_8^4$. Hence, $P_3 \bigodot c_{4s}^4$ is signed product cordial.

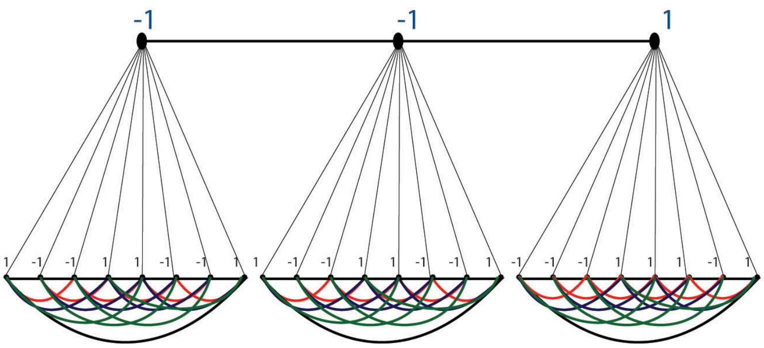

Figure 3 $P_3 \bigodot c_8^4$

Case 4. $m = 0(mod\ 4)$. Suppose that $m = 4r$, where $r \geq 1$. Then we label the vertices of $P_{4r} \bigodot c_{4s}^4$ by $[L_{4r}; S'_{4s}, S'_{4s}, 1_3 M_{4s-6}(-1)_3, 1_3 M_{4s-6}(-1)_3, \cdots, (r - time)]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = 2r-1, a_1 = 2r, y_{-1} = y_1 = 2s, b_{-1} = 8s-4, b_1 = 8s-5, y'_{-1} = y'_1 = 2s$ and $b'_{-1} = 8s-5, b'_1 = 8s-4$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = -1$. As an example, Figure (4) illustrates $P_4 \bigodot c_8^4$. Hence, $P_{4r} \bigodot c_{4s}^4$ is signed product cordial.

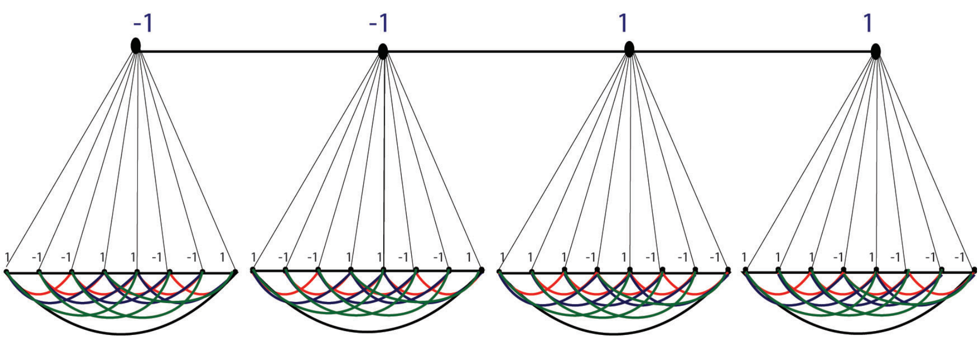

Figure 4 $P_4 \bigodot c_8^4$

Case 5. Suppose that $m = 4r + 1$, where $r \geq 1$. Then we label the vertices of $P_{4r+1} \bigodot c^4_{4s}$ by $[L_{4r}(-1); S'_{4s}, S'_{4s}, 1_3M_{4s-6}(-1)_3, 1_3M_{4s-6}(-1)_3, \cdots, (r-time)s'_{4s}$. Therefore $x_{-1} = 2r + 1, x_1 = 2r, a_{-1} = a_1 = 2r, y_{-1} = y_1 = 2s, b_{-1} = 8s-4, b_1 = 8s-5, y'_{-1} = y'_1 = 2s$ and $b'_{-1} = 8s-5, b'_1 = 8s-4$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 1$. Hence, $P_{4r+1} \bigodot c^4_{4s}$ is signed product cordial.

Case 6. Suppose that $m = 4r+2$, where $r \geq 1$. Then we label the vertices of $P_{4r+2} \bigodot C^4_{4s}$ by $[L_{4r}(-1)1; S'_{4s}, S'_{4s}, 1_3M_{4s-6}(-1)_3, 1_3M_{4s-6}(-1)_3, \cdots, (r-time)S'_{4s}, 1_3M_{4s-6}(-1)_3]$. Therefore $x_{-1} = x_1 = 2r+1, a_{-1} = 2r+1, a_1 = 2r, y_{-1} = y_1 = 2s, b_{-1} = 8s-4, b_1 = 8s-5, y'_{-1} = y'_1 = 2s$ and $b'_{-1} = 8s-5, b'_1 = 8s-4$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. Hence, $P_{4r+2} \bigodot c^4_{4s}$ is signed product cordial.

Case 7. Suppose that $m = 4r+3$, where $r \geq 1$. Then we label the vertices of $P_{4r+3} \bigodot C^4_{4s}$ by $[L_{4r}1(-1)(-1); S'_{4s}, S'_{4s}, 1_3M_{4s-6}(-1)_3, 1_3M_{4s-6}(-1)_3, \cdots, (r-time)\ 1_3M_{4s-6}(-1)_3, S'_{4s}, S'_{4s}]$. Therefore $x_{-1} = 2r+2, x_1 = 2r+1, a_{-1} = 2r, a_1 = 2r+2, y_{-1} = y_1 = 2s, b_{-1} = 8s-4, b_1 = 8s-5, y'_{-1} = y'_1 = 2s$ and $b'_{-1} = 8s-5, b'_1 = 8s-4$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = -1$. Hence, $P_{4r+3} \bigodot c^4_{4s}$ is signed product cordial. □

Lemma 3.3 *If $n \equiv 1(mod\ 4)$, then $P_m \bigodot c^4_n$signed product cordial for all $m \geq 1$.*

Proof Suppose that $n = 4s+1$, where $s \geq 2$. The following cases will be examined.

Case 1. Suppose that $m = 1$. Then we label the vertices of $P_1 \bigodot c^4_{4s+1}$ by $[(-1); 1_3S_{4s-4}(-1)_2]$. Therefore $x_{-1} = 1, x_1 = 0, a_{-1} = a_1 = 0, y_{-1} = 2s, y_1 = 2s+1, b_{-1} = 8s-3, b_1 = 8s-2$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Hence, $P_1 \bigodot c^4_{4s+1}$ is signed product cordial.

Case 2. Suppose that $m = 2$. We label the vertices of $P_2 \bigodot c^4_{4s+1}$ by $[(-1)1; L_{4s}, 1_3S_{4s-4}(-1)_2]$. Therefore $x_{-1} = x_1 = 1, a_{-1} = 1, a_1 = 0, y_{-1} = 2s+1, y_1 = 2s, b_{-1} = 8s-2, b_1 = 8s-3, y'_{-1} = 2s, y'_1 = 2s+1$ and $b'_{-1} = 8s-3, b'_1 = 8s-2$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = -1$. Hence, $P_2 \bigodot c^4_{4s+1}$ is signed product cordial.

Case 3. Suppose that $m = 3$. Then we label the vertices of $P_3 \bigodot c^4_{4s+1}$ by

$$[(-1)(-1)1; 1(-1)_3S'_{4s-4}1, 1(-1)_3S'_{4s-4}1, 1_3S_{4s-4}(-1)_2].$$

Therefore $x_{-1} = 2, x_1 = 1, a_{-1} = a_1 = 1, y_{-1} = 2s, y_1 = 2s+1, b_{-1} = 8s-3, b_1 = 8s-2, y'_{-1} = 2s+1, y'_1 = 2s$ and $b'_{-1} = 8s-3, b'_1 = 8s-2$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Hence, $P_3 \bigodot c^4_{4s+1}$ is signed product cordial.

Case 4. Suppose that $m = 4r$, where $r \geq 1$. Then we label the vertices of $P_{4r} \bigodot C^4_{4s+1}$ by $[L_{4r}; L_{4s}(-1), L_{4s}(-1), S'_{4s}1, S'_{4s}1, ..., (r-time)]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = 2r-1, a_1 = 2r, y_{-1} = 2S+1, y_1 = 2S, b_{-1} = 8S-2, b_1 = 8S-3, y'_{-1} = 2S, y'_1 = 2S+1$ and $b'_{-1} = 8s-2, b'_1 = 8s-3$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = -1$. Hence, $P_{4r} \bigodot c^4_{4s+1}$ is signed product cordial.

Case 5. Suppose that $m = 4r+1$, where $r \geq 1$. Then we label the vertices of $P_{4r+1} \bigodot c^4_{4s+1}$ by $[S_{4r}(-1); S'_{4s}1, S'_{4s}1, L_{4s}(-1), L_{4s}(-1), \cdots, (r-time)\ s_{4s}1$. Therefore $x_{-1} = 2r+1, x_1 =$

$2r, a_{-1} = 2r-1, a_1 = 2r+1, y_{-1} = 2s+1, y_1 = 2s, b_{-1} = 8s-2, b_1 = 8s-3, y'_{-1} = 2s, y'_1 = 2s+1, b'_{-1} = 8s-2, b'_1 = 8s-3, y''_{-1} = 2s, y''_1 = 2s+1$ and $b''_{-1} = 8s-2, b''_1 = 8s-3$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r+1} \bigodot c^4_{4s+1}$ is signed product cordial.

Case 6. Suppose that $m = 4r+2$, where $r \geq 1$. Then we label the vertices of $P_{4r+2} \bigodot c^4_{4s+1}$ by $[L_{4r}(-1)1; L_{4s}(-1), L_{4s}(-1), S'_{4s}1, S'_{4s}1, \cdots, (r-time), L_{4s}(-1), S'_{4s}1]$. Therefore $x_{-1} = x_1 = 2r+1, a_{-1} = 2r+1, a_1 = 2r, y_{-1} = 2s+1, y_1 = 2s, b_{-1} = 8s-2, b_1 = 8s-3, y'_{-1} = 2s, y'_1 = 2s+1$ and $b'_{-1} = 8s-2, b'_1 = 8s-3$. It follows that $v_{-1}-v_1 = 0$ and $e_{-1}-e_1 = 1$. Hence, $P_{4r+2} \bigodot c^4_{4s+1}$ is signed product cordial.

Case 7. Suppose that $m = 4r+3$, where $r \geq 1$. Then we label the vertices of $P_{4r+3} \bigodot C^4_{4s+1}$ by $[L_{4r}1(-1)(-1); L_{4s}(-1), L_{4s}(-1), S'_{4s}1, S'_{4s}1, \cdots, (r-time)\ S'_{4s}1, L_{4s}(-1), s_{4s}1]$. Therefore $x_{-1} = 2r+2, x_1 = 2r+1, a_{-1} = 2r, a_1 = 2r+2, y_{-1} = 2s+1, y_1 = 2s, b_{-1} = 8s-2, b_1 = 8s-3, y'_{-1} = 2s, y'_1 = 2s+1, b'_{-1} = 8s-2, b_1 = 8s-3, y''_{-1} = 2s, y''_1 = 2s+1$ and $b''_{-1} = 8s-2, b''_1 = 8s-3$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r+3} \bigodot c^4_{4s+1}$ is signed product cordial. □

Lemma 3.4 *If $n \equiv 2(mod\ 4)$, then $P_m \bigodot c^4_n$signed product cordial for all $m \geq 1$.*

Proof Suppose that $n = 4s+2$, where $s \geq 2$. The following cases will be examined.

Case 1. Suppose that $m = 1$. Then we label the vertices of $P_1 \bigodot C^4_{4s+2}$ by $[(-1); (-1)_3 1_3 S_{4s-4}]$. Therefore $x_{-1} = 1, x_1 = 0, a_{-1} = a_1 = 0, y_{-1} = y_1 = 2s+1, b_{-1} = 8s, b_1 = 8s-1$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 1$. Hence, $P_1 \bigodot c^4_{4s+2}$ is signed product cordial.

Case 2. Suppose that $m = 2$. Then we label the vertices of $P_2 \bigodot c^4_{4s+2}$ by

$$[(-1)1; (-1)_3 1_3 L'_{4s-4}, (-1)_3 M'_2 1_3 M_{4s-6}].$$

Therefore $x_{-1} = x_1 = 1, a_{-1} = 1, a_1 = 0, y_{-1} = y_1 = 2s+1, b_{-1} = 8s, b_1 = 8s-1, y'_{-1} = y'_1 = 2s+1$ and $b'_{-1} = 8s-1, b'_1 = 8s$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. Hence, $P_2 \bigodot c^4_{4s+2}$ is signed product cordial.

Case 3. Suppose that $m = 3$. Then we label the vertices of $P_3 \bigodot C^4_{4s+2}$ by

$$[(-1)(-1)1; (-1)_3 1_3 L'_{4s-4}, (-1)_3 1_3 L'_{4s-4}, (-1)_3 M'_2 1_3 M_{4s-6}].$$

Therefore $x_{-1} = 2, x_1 = 1, a_{-1} = a_1 = 1, y_{-1} = y_1 = 2s+1, b_{-1} = 8s, b_1 = 8s-1, y'_{-1} = y'_1 = 2s+1$ and $b'_{-1} = 8s-1, b'_1 = 8s$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 1$. Hence, $P_3 \bigodot c^4_{4s+2}$ is signed product cordial.

Case 4. Suppose that $m = 4r$, where $r \geq 1$. Then we label the vertices of $P_{4r} \bigodot c^4_{4s+2}$ by $[L_{4r}; (-1)_3 1_3 L'_{4s-4}, (-1)_3 1_3 L'_{4s-4}, (-1)_3 M'_2 1_3 M_{4s-6},$
$(-1)_3 M'_2 1_3 M_{4s-6}, \cdots, (r-time)]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = 2r-1, a_1 = 2r, y_{-1} = y_1 = 2s+1, b_{-1} = 8s, b_1 = 8s-1, y'_{-1} = y'_1 = 2s+1$ and $b'_{-1} = 8s-1, b'_1 = 8s$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = -1$. Hence, $P_{4r} \bigodot c^4_{4s+2}$ is signed product cordial.

Case 5. Suppose that $m = 4r+1$, where $r \geq 1$. Then we label the vertices of $P_{4r+1} \bigodot c^4_{4s+2}$

by $[L_{4r}(-1);(-1)_3 1_3 L'_{4s-4},(-1)_3 1_3 L'_{4s-4},(-1)_3 M'_2 1_3\ M_{4s-6},(-1)_3 M'_2 1_3 M_{4s-6},\cdots,(r-time)$ $(-1)_3 1_3 L'_{4s-4}$. Therefore $x_{-1}=2r+1, x_1=2r, a_{-1}=a_1=2r, y_{-1}=y_1=2s+1, b_{-1}=8s, b_1=8s-1, y'_{-1}=y'_1=2s+1$ and $b'_{-1}=8s-1, b'_1=8s$. It follows that $v_{-1}-v_1=1$ and $e_{-1}-e_1=1$. Hence, $P_{4r+1}\bigodot c^4_{4s+2}$ is signed product cordial.

Case 6. Suppose that $m=4r+2$, where $r\geq 1$. Then we label the vertices of $P_{4r+2}\bigodot c^4_{4s+2}$ by

$$[L_{4r}(-1)1;(-1)_3 1_3 L'_{4s-4},(-1)_3 1_3 L'_{4s-4},(-1)_3 M'_2 1_3 M_{4s-6},(-1)_3 M'_2 1_3 M_{4s-6},$$
$$\cdots,(r-time)(-1)_3 1_3 L'_{4s-4},(-1)_3 M'_2 1_3 M_{4s-6}].$$

Therefore $x_{-1}=x_1=2r+1, a_{-1}=2r+1, a_1=2r, y_{-1}=y_1=2s+1, b_{-1}=8s, b_1=8s-1, y'_{-1}=y'_1=2s+1$ and $b'_{-1}=8s-1, b'_1=8s$. It follows that $v_{-1}-v_1=0$ and $e_{-1}-e_1=1$. Hence, $P_{4r+2}\bigodot c^4_{4s+2}$ is signed product cordial.

Case 7. Suppose that $m=4r+3$, where $r\geq 1$. Then we label the vertices of $P_{4r+3}\bigodot c^4_{4s+2}$ by $[L_{4r}1(-1)(-1);(-1)_3 1_3 L'_{4s-4},(-1)_3 1_3 L'_{4s-4},(-1)_3 M'_2 1_3\ M_{4s-6},(-1)_3 M'_2 1_3 M_{4s-6},\cdots,(r-time)$, $(-1)_3 M'_2 1_3 M_{4s-6},(-1)_3 1_3 L'_{4s-4},\ (-1)_3 1_3 L'_{4s-4}]$. Therefore $x_{-1}=2r+2, x_1=2r+1, a_{-1}=2r, a_1=2r+2, y_{-1}=y_1=2s+1, b_{-1}=8s, b_1=8s-1, y'_{-1}=y'_1=2s+1$ and $b'_{-1}=8s-1, b'_1=8s$. It follows that $v_{-1}-v_1=1$ and $e_{-1}-e_1=-1$. Hence, $P_{4r+3}\bigodot c^4_{4s+2}$ is signed product cordial. □

Lemma 3.5 *If $n\equiv 3(mod\ 4)$, then $P_m\bigodot c^4_n$ signed product cordial for all $m\geq 1$.*

Proof Suppose that $n=4s+3$, where $s\geq 2$. The following cases will be examined.

Case 1. Suppose that $m=2$. We label the vertices of $P_2\bigodot c^4_{4s+3}$ by $[(-1)1;M_{4s+3},S'_{4s}1(-1)1]$. Therefore $x_{-1}=x_1=1, a_{-1}=1, a_1=0, y_{-1}=2s+2, y_1=2s+1, b_{-1}=8s+2, b_1=8s+1, y'_{-1}=2s+1, y'_1=2s+2$ and $b'_{-1}=8s+2, b'_1=8s+1$. It follows that $v_{-1}-v_1=0$ and $e_{-1}-e_1=1$. Hence, $P_2\bigodot c^4_{4s+3}$ is signed product cordial.

Case 2. Suppose that $m=4r$, where $r\geq 1$. Then we label the vertices of $P_{4r}\bigodot C^4_{4s+3}$ by $[L_{4r};M_{4s+3},M_{4s+3},S'_{4s}1(-1)1,S'_{4s}1(-1)1,...,(r-time)]$. Therefore $x_{-1}=x_1=2r, a_{-1}=2r-1, a_1=2r, y_{-1}=2s+2, y_1=2s+1, b_{-1}=8s+2, b_1=8s+1, y'_{-1}=2s+1, y'_1=2s+2$ and $b'_{-1}=8s+2, b'_1=8s+1$. It follows that $v_{-1}-v_1=0$ and $e_{-1}-e_1=-1$. Hence, $P_{4r}\bigodot c^4_{4s+3}$ is signed product cordial.

Case 3. Suppose that $m=4r+1$, where $r\geq 1$. Then we label the vertices of $P_{4r+1}\bigodot c^4_{4s+3}$ by $[S_{4r}(-1);S'_{4s}1(-1)1,S'_{4s}1(-1)1,M_{4s+3},M_{4s+3},,\cdots,(r-time)$,$s'_{4s}1(-1)1$. Therefore $x_{-1}=2r+1, x_1=2r, a_{-1}=2r-1, a_1=2r+1, y_{-1}=2s+2, y_1=2s+1, b_{-1}=8s+2, b_1=8s+1, y'_{-1}=2s+1, y'_1=2s+2, b'_{-1}=8s+2, b'_1=8s+1, y''_{-1}=2s+1, y'_1=2s+2$ and $b''_{-1}=8s+2, b''_1=8s+1$. It follows that $v_{-1}-v_1=0$ and $e_{-1}-e_1=0$. Hence, $P_{4r+1}\bigodot c^4_{4s+3}$ is signed product cordial.

Case 4. Suppose that $m=4r+2$, where $r\geq 1$. Then we label the vertices of $P_{4r+2}\bigodot c^4_{4s+3}$ by $[L_{4r}(-1)1;M_{4s+3},M_{4s+3},S'_{4s}1(-1)1,S'_{4s}1(-1)1,\cdots,(r-time),\ M_{4s+3},S'_{4s}1(-1)1]$. Therefore $x_{-1}=x_1=2r+1, a_{-1}=2r+1, a_1=2r, y_{-1}=2s+2, y_1=2s+1, b_{-1}=8s+2, b_1=8s+1, y'_{-1}=2s+1, y'_1=2s+2$ and $b'_{-1}=8s+2, b'_1=8s+1$. It follows that $v_{-1}-v_1=0$ and

$e_{-1} - e_1 = 1$. Hence, $P_{4r+2} \bigodot c^4_{4s+3}$ is signed product cordial.

Case 5. Suppose that $m = 4r + 3$, where $r \geq 1$. Then we label the vertices of $P_{4r+3} \bigodot c^4_{4s+3}$ by

$$[L_{4r}1(-1)(-1); M_{4s+3}, M_{4s+3}, S'_{4s}1(-1)1, S'_{4s}1(-1)1, \cdots, (r - time), S'_{4s}1(-1)1, M_{4s+3}, S_{4s}1(-1)1].$$

Therefore $x_{-1} = 2r + 2, x_1 = 2r + 1, a_{-1} = 2r, a_1 = 2r + 2, y_{-1} = 2s + 2, y_1 = 2s + 1, b_{-1} = 8s+2, b_1 = 8s+1, y'_{-1} = 2s+1, y'_1 = 2s+2, b'_{-1} = 8s+2, b_1 = 8s+1, y''_{-1} = 2s+1, y''_1 = 2s+2$ and $b''_{-1} = 8s + 2, b''_1 = 8s + 1$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r+3} \bigodot c^4_{4s+3}$ is signed product cordial. □

As a consequence of all lemmas mentioned above we conclude that the corona between paths and fourth power of cycles is signed product cordial for all m, n if and only if $m \geq 1$, $n \geq 7$ except $(m, n) = (1, 7)$ or $(3, 7)$ and also if $n = 3$ for all $m \neq 1$, i.e., the conclusion following.

Theorem 3.5 *$P_m \bigodot c^4_n$ is signed product cordial for all m, n if and only if $m \geq 1$, $n \geq 7$ except $(m, n) = (1, 7)$ or $(3, 7)$.*

In [5] it is proved that the corona $P_m \bigodot c^4_3$ is signed product cordial if and only if $m \neq 1$. Certainly, we get a general result on the signed product cordial of the corona between paths and fourth power of cycles in this paper.

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Roman Domination Polynomial of Cycles

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Abstract: A Roman dominating function on a graph $G = (V, E)$ is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value

$$W(f(V)) = \sum_{u \in V(G)} f(u).$$

The minimum weight of a Roman dominating function on a graph G is called the Roman domination number of G and is denoted by $\gamma_R(G)$. In [9], we have introduced and established the study of the Roman domination polynomial of graphs and obtained some important properties about the polynomial and we have computed the polynomial for some specific graphs and graph operations. In this paper, we study the Roman domination polynomial of a cycle C_n on n vertices. Exact formula for the polynomial, important properties of its coefficients and interesting results have obtained.

Key Words: Domination polynomial of cycles, Roman domination polynomial of graphs, Roman domination polynomial of cycles, Smarandachely dominating set.

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§1. Introduction

Let $G = (V, E)$ be a simple graph, where V and E are the set of vertices and edges of G, respectively. The open neighborhood and the closed neighborhood of a vertex $v \in V(G)$ are defined by $N(v) = \{u \in V(G) : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$, respectively. The cardinality of $N(v)$ is called the degree of the vertex v and denoted by $deg(v)$ in G. For more terminology and notations about graph, the reader is referred to [6,10].

A subset D of $V(G)$ is a *dominating set* of G, if for every vertex $v \in V - D$, there exists a vertex $u \in D$ such that v is adjacent to u and a subset D is *Smarandachely dominating set* of G on a complete subgraph $K_s \prec G, s \geq 0$ if for every vertex $v \in V - V(H) - D$ there exists a vertex $u \in D$ such that v is adjacent to u but for vertices $w \in V(K_s)$ there are at least 2 vertices $u_1, u_2 \in D$ such that $wu_1, wu_2 \in E(G)$. Clearly, if $s = 0$, a Smarandachely dominating set of G is nothing else but the usual dominating set. A dominating set of G of cardinality

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$\gamma(G)$ is called the domination number of G. For more details about domination of graphs, we refer to [11].

The domination polynomial $D(G,x)$ of a graph G is defined by

$$D(G,x) = \sum_{i=\gamma(G)}^{n} d(G,i)x^i,$$

where $d(G,i)$ is the number of all the dominating sets of G of size i [5]. The dominating sets and the domination polynomial of graphs have been studied extensively in [5, 3, 4, 2]. Recently, the injective domination polynomial of graphs has been studied in [1].

A Roman dominating function of a graph $G = (V,E)$ (or in brief RDF of G) is a function $f : V(G) \to \{0,1,2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is

$$W(f(V)) = \sum_{u\in V(G)} f(u).$$

A Roman dominating function of a graph G with weight $\gamma_R(G)$ is called the Roman domination number of G. For more details about Roman domination and its properties, the reader is referred to [7]. The next proposition showed that the exact value of γ_R of a path P_n and a cycle C_n on n vertices is $\left\lceil \frac{2n}{3} \right\rceil$.

Proposition 1.1([7]) *For the classes of paths P_n and cycles C_n,*

$$\gamma_R(P_n) = \gamma_R(C_n) = \left\lceil \frac{2n}{3} \right\rceil.$$

In [8], we have introduced the Roman domination polynomial of graph as

$$R(G,x) = \sum_{j=\gamma_R(G)}^{2n} r(G,j)x^j,$$

where $r(G,j)$ is the number of Roman dominating functions of G of weight j. We have established this study by obtaining some important properties of the polynomial and its coefficients, and determining the exact formula of the polynomial for some families of graphs and graph operations.

In the next proposition, we obtain some important properties of $R(G,x)$ of a graph G which we need to use in this paper.

Proposition 1.2([9]) *Let G be a non trivial graph on n vertices. Then,*

(*i*) *$R(G,x)$ has no constant term;*

(*ii*) *$R(G,x)$ has no term of degree one;*

(*iii*) *Zero is a root of $R(G,x)$, with multiplicity $\gamma_R(G)$;*

(iv) *$R(G,x)$ never equal x^p for any $2 \le p \le 2n$;*

(v) For any graph G, $r(G,2n)=1$ and $r(G,2n-1)=n$;

(vi) *$r(G,j)=0$ if and only if $j<\gamma_R(G)$ or $j>2n$;*

(vii) *$R(G,x)$ is a strictly increasing function in $[0,\infty)$;*

(viii) The only polynomial of degree two can $R(G,x)$ be equal is x^2+x if and only if $G \cong K_1$;

(ix) *Let H be any induced subgraph of G. Then*

$$deg\big(R(G,x)\big) \ge deg\big(R(H,x)\big).$$

In this paper, we study the Roman domination polynomial of a cycle C_n on n vertices. Exact formula for $R(C_n,x)$, important properties and relations between the coefficient of $R(C_n,x)$ are obtained.

§2. Roman Domination Polynomial of a Cycle

In [3], Alikhani and Peng have showed that the number of all dominating sets with cardinality i of a cycle C_n equal to the sum of the number of all dominating sets of the cycle C_{n-1} with cardinality $i-1$, the cycle C_{n-2} with cardinality $i-1$ and the cycle C_{n-3} with cardinality $i-1$, and then they have found the exact formula of the domination polynomial of cycles, as follows:

Theorem 2.1([3]) *(i) If $\mathcal{C}_n^i$ is the family of all dominating sets with cardinality i of a cycle C_n, then*

$$\left|\mathcal{C}_n^i\right| = \left|\mathcal{C}_{n-1}^{i-1}\right| + \left|\mathcal{C}_{n-2}^{i-1}\right| + \left|\mathcal{C}_{n-3}^{i-1}\right|;$$

(ii) For every $n \ge 4$,

$$D(C_n,x) = x\big[D(C_{n-1},x)+D(C_{n-2},x)+D(C_{n-3},x)\big],$$

with initial values $D(C_1,x)=x$, $D(C_2,x)=x^2+2x$ and $D(C_3,x)=x^3+3x^2+3x$.

In this section, we find the Roman domination polynomial of a cycle C_n on n vertices, and then we study some of its properties, and finally, we illustrate in a table the coefficients of all Roman domination polynomials of cycles C_n with $n \le 10$.

Let $\mathbb{C}_n^j$ be the set of all RDFs of C_n with weight j. Actually, to find a RDF of C_n, we do not need to consider RDFs of C_{n-4} with weight $j-2$ (weight $j-1$ is not possible here), we will show this in the next lemma. Note that, when we talk about a RDF f with weight $j-1$ or $j-2$ in C_{n-r}, where $r=1,2,3$ such that $f \in \mathbb{C}_n^j$, we mean a RDF f of C_n minus only one vertex $v \in C_n \setminus C_{n-r}$ taking a value $f(v)=1$ or $f(v)=2$, respectively.

Lemma 2.2 *Let $f \in \mathbb{C}_n^j$. Then, if $f \in \mathbb{C}_{n-4}^{j-2}$, this implies that $f \in \mathbb{C}_{n-3}^{j-2}$.*

Proof Consider $V(C_n)=\{v_1,v_2,\ldots,v_n\}$. Suppose $f \in \mathbb{C}_{n-4}^{j-2}$. Then we have two cases following.

Case 1. If $f(v_1) = 0$ or 1, then the vertex v_{n-4} must take the value 2 under the function f (shortly $f(v_{n-4}) = 2$) because otherwise $f \notin \mathbb{C}_n^j$, a contradiction. Therefore, $f \in \mathbb{C}_{n-3}^{j-2}$.

Case 2. If $f(v_1) = 2$, then whatever the vertex v_{n-4} taking under the function f ($f(v_{n-4}) = 0$ or 1 or 2) $\Rightarrow f \in \mathbb{C}_{n-3}^{j-2}$. □

In the following theorem, according to Theorem 2.1 part (i) (since every RDF of a graph G it just a labeling on some dominating set of the graph G itself) and Lemma 2.2, we state the Roman domination polynomial of C_n in terms of the Roman domination polynomial of C_{n-1}, C_{n-2} and C_{n-3}.

Theorem 2.3 *Let C_n be a cycle on $n \geq 4$ vertices. Then*

$$R(C_n, x) = (x^2 + x)R(C_{n-1}, x) + x^2 R(C_{n-2}, x) + (x^3 + x^2)R(C_{n-3}, x),$$

with initial values $R(C_3, x) = x^6 + 3x^5 + 6x^4 + 7x^3 + 3x^2$, $R(C_2, x) = x^4 + 2x^3 + 3x^2$ *and* $R(C_1, x) = x^2 + x$.

Proof Consider $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Let $f \in \mathbb{C}_n^j$. Then we have the following cases.

Case 1. Suppose that $f \in \mathbb{C}_{n-1}^{j-1}$ or $f \in \mathbb{C}_{n-1}^{j-2}$ (this means that, for the last vertex v_n either $f(v_n) = 1$ or $f(v_n) = 2$, respectively). Then we get the term $(x^2 + x)R(C_{n-1}, x)$.

Case 2. Suppose that $f \in \mathbb{C}_{n-2}^{j-1}$ or $f \in \mathbb{C}_{n-2}^{j-2}$. Thus, we have the following subcases.

Subcase 2.1 Suppose $f(v_{n-1}) = 1$ and $f(v_n) = 0$. Then the vertex v_1 should take the value 2. Hence, we get the term $x^3 R(C_{n-3}, x)$.

Subcase 2.2 Suppose $f(v_{n-1}) = 0$ and $f(v_n) = 1$. This case included in Case 1.

Subcase 2.3 Suppose $f(v_{n-1}) = 2$ and $f(v_n) = 0$. Then we get the term $x^2 R(C_{n-2}, x)$.

Subcase 2.4 Suppose $f(v_{n-1}) = 0$ and $f(v_n) = 2$. This situation has some connection with Case 1, so to avoid the repetition, we will take only the situations when $\mathbb{C}_{n-1}^{j-2} = \phi$. Therefore, we will choose $f(v_{n-2}) = 0$. Then the vertex v_{n-3} should take the value 2 ($f(v_{n-3}) = 2$), but if the vertex v_1 take the value 2 also, we get a repetition with Case 1. Hence, we get the term $x^4 R(C_{n-4}, x) - x^6 R(C_{n-5}, x)$.

Case 3. Suppose now $f \in \mathbb{C}_{n-3}^{j-2}$, where $f(v_{n-2}) = 2$, $f(v_{n-1}) = 0$ and $f(v_n) = 0$. Then we obtain the term $x^4 R(C_{n-4}, x)$.

Case 4. In this case, we have remaining the situation when $f(v_{n-2}) = 0$, $f(v_{n-1}) = 2$ and $f(v_n) = 0$ such that $\mathbb{C}_{n-2}^{j-2} = \phi$. Therefore, we have the term $x^2 R(C_{n-3}, x)$ but there are two possibilities of repetition, when $f(v_{n-3}) = 2$ or $f(v_n) = 2$. Thus, we have remove the term $2x^4 R(C_{n-4}, x)$. But while we remove the term $2x^4 R(C_{n-4}, x)$ we will miss the situation when $f(v_{n-3}) = 2$ and $f(v_n) = 2$ which give us the term $x^6 R(C_{n-5}, x)$. Hence in the end of this case we obtain the term

$$x^2 R(C_{n-3}, x) - 2x^4 R(C_{n-4}, x) + x^6 R(C_{n-5}, x).$$

The proof is completed. □

Using Theorem 2.3, we obtain $r(C_n, j)$ for $1 \leq n \leq 10$ as shown in Table 1.

j	1	2	3	4	5	6	7	8	9	10
n										
1	1	1								
2	0	3	2	1						
3	0	3	7	6	3	1				
4	0	0	4	15	16	10	4	1		
5	0	0	0	10	31	40	30	15	5	1
6	0	0	0	3	24	69	96	84	50	21
7	0	0	0	0	7	56	155	231	224	154
8	0	0	0	0	0	20	128	351	552	584
9	0	0	0	0	0	3	54	297	799	1314
10	0	0	0	0	0	0	10	140	690	1833
j	11	12	13	14	15	16	17	18	19	20
n										
6	6	1								
7	77	28	7	1						
8	448	258	112	36	8	1				
9	1494	1257	810	405	156	45	9	1		
10	3120	3770	3430	2430	1362	605	210	55	10	1

Table 1 $r(C_n, j)$, the number of Roman dominating functions of C_n with cardinality j.

In the following theorem, we obtain some important properties about the coefficients of the Roman domination polynomial of a cycle C_n.

Theorem 2.4 *The following properties are satisfied for the Roman domination polynomial $R(C_n, x)$ of a cycle C_n:*

(*i*) $r(C_n, j) = r(C_{n-1}, j-1) + r(C_{n-1}, j-2) + r(C_{n-2}, j-2) + r(C_{n-3}, j-2) + r(C_{n-3}, j-3)$;

(*ii*) $r(C_{3k}, 2k) = 3$, *where* $n = 3k$ *for some* $k \in \mathbb{N}$;

(*iii*) *If* $n = 3k+1$ *for some* $k \in \mathbb{N}$, *then* $r(C_{3k+1}, 2k+1) = 3k+1$;

(*iv*) *If* $n = 3k+2$ *for some* $k \in \mathbb{N}$, *then* $r(C_{3k+2}, 2k+2) = \dfrac{(3k+2)(k+3)}{2}$;

(*v*) *If* $n = 3k$ *for some* $k \in \mathbb{N}$, *then* $r(C_{3k}, 2k+1) = \dfrac{k(k+1)(k+6)}{2}$;

(*vi*) *If* $n = 3k+1$ *for some* $k \in \mathbb{N}$, *then* $r(C_{3k+1}, 2k+2) = \dfrac{(3k+1)(k+4)(k^2+11k+6)}{24}$;

(vii) If $n = 3k + 2$ *for some* $k \in \mathbb{N}$*, then*

$$r(C_{3k+2}, 2k+3) = \frac{(3k+2)(k+3)(k^3+23k^2+122k+40)}{120};$$

(viii) If $n = 3k$ *for some* $k \in \mathbb{N}$*, then*

$$r(C_{3k}, 2k+2) = \frac{k(k^5+35k^4+365k^3+1165k^2+234k-360)}{240};$$

(ix) $r(C_n, 2n) = 1$;

(x) $r(C_n, 2n-1) = n$;

(xi) $r(C_n, 2n-2) = \dfrac{n(n+1)}{2}$;

(xii) $r(C_n, 2n-3) = \dfrac{n(n-1)(n+4)}{6}$;

(xiii) $r(C_n, 2n-4) = \dfrac{n(n+1)(n^2+5n-18)}{24}$;

(xiv) $r(C_n, 2n-5) = \dfrac{n(n-1)(n^3+11n^2-14n-144)}{120}$;

(xv) For every $k \in \mathbb{N}$,

$$1 = r(C_k, 2k) < r(C_{k+1}, 2k) < r(C_{k+2}, 2k) < \cdots < r(C_{2k}, 2k) > \cdots > r(C_{3k-1}, 2k) > r(C_{3k}, 2k) = 3;$$

(xvi) For every $k \in \mathbb{N}$,

$$k+1 = r(C_{k+1}, 2k+1) < r(C_{k+2}, 2k+1) < r(C_{k+3}, 2k+1) < \cdots < r(C_{2k+1}, 2k+1) > \cdots > r(C_{3k}, 2k+1) > r(C_{3k+1}, 2k+1) = 3k+1;$$

(xvii) If $\alpha_n = \sum_{j=\lceil \frac{2n}{3} \rceil}^{2n} r(C_n, j)$*, then for every* $n \geq 4$, $\alpha_n = 2\alpha_{n-1} + \alpha_{n-2} + 2\alpha_{n-3}$*, with initial values* $\alpha_1 = 2$, $\alpha_2 = 6$ *and* $\alpha_3 = 20$.

(xviii) For $j \geq 2$,

$$\sum_{i=j}^{3j} r(C_i, 2j) = \sum_{i=j}^{3j-2} r(C_i, 2j-1) + 3\sum_{i=j-1}^{3j-3} r(C_i, 2j-2) + \sum_{i=j-1}^{3j-5} r(C_i, 2j-3);$$

(xix) For $j \geq 3$,

$$\sum_{i=j}^{3j-2} r(C_i, 2j-1) = \sum_{i=j-1}^{3j-3} r(C_i, 2j-2) + 3\sum_{i=j-1}^{3j-5} r(C_i, 2j-3) + \sum_{i=j-2}^{3j-6} r(C_i, 2j-4).$$

Proof Let C_n be a path on n vertices with $V(C_n) = \{v_1, v_2, \cdots, v_n\}$.

(i) The proof of this result is straightforward from Theorem 2.3.

(ii) Let $n = 3k$ for some $k \in \mathbb{N}$. Since $\mathcal{C}_{3k}^k = \{\{v_1, v_4, \cdots, v_{3k-5}, v_{3k-2}\}, \{v_2, v_5, \cdots, v_{3k-4},$

$v_{3k-1}\}, \{v_3, v_6, \cdots, v_{3k-3}, v_{3k}\}\}$, then we have only three RDF of C_n, in this case, such that each vertex taking the value 2. Hence, $r(P_{3k}, 2k) = 3$.

(iii) Proof by induction on k. If $k = 1$, then $r(C_4, 3) = 4$ (see Table 1). Therefore, the result is true for $k = 1$. Now, suppose the result is true for all natural numbers less than or equal $k - 1$. We will prove that the result still true for k. By parts (i) and (ii), the induction hypothesis and Proposition 1.2 part (vi), we get.

$$\begin{aligned} r(C_{3k+1}, 2k+1) &= r(C_{3k}, 2k) + r(C_{3k}, 2k-1) + r(C_{3k-1}, 2k-1) \\ &\quad + r(C_{3k-2}, 2k-1) + r(C_{3k-2}, 2k-2) \\ &= 3 + 0 + 0 + r\big(C_{3(k-1)+1}, 2(k-1)+1\big) + 0 \\ &= 3k+1. \end{aligned}$$

(iv) By induction on k. If $k = 1$, then $r(C_5, 4) = 10 = \frac{(3+2)(1+3)}{2}$ (see Table 1). Suppose now the result is true for all natural numbers less than or equal $k - 1$. Then by parts (i), (ii) and (iii) and Proposition 1.2 part (vi), we have.

$$\begin{aligned} r(C_{3k+2}, 2k+2) &= r(C_{3k+1}, 2k+1) + r(C_{3k+1}, 2k) + r(C_{3k}, 2k) \\ &\quad + r(C_{3k-1}, 2k) + r(C_{3k-1}, 2k-1) \\ &= 3k + 1 + 0 + 3 + r\big(C_{3(k-1)+2}, 2(k-1)+2\big) + 0 \\ &= 3k + 4 + \frac{(3k-1)(k+2)}{2} = \frac{(3k+2)(k+3)}{2}. \end{aligned}$$

(v) Proof by induction on k. If $k = 1$, then $r(C_3, 3) = 7 = \frac{1(1+1)(1+6)}{2}$ (see Table 1). Suppose the result is true for all natural numbers less than k. Then by using parts (i), (ii), (iii) and (iv) and Proposition 1.2 part (vi), we obtain.

$$\begin{aligned} r(C_{3k}, 2k+1) &= r(C_{3(k-1)+2}, 2(k-1)+2) + r(C_{3(k-1)+2}, 2(k-1)+1) \\ &\quad + r(C_{3(k-1)+1}, 2(k-1)+1) + r(C_{3(k-1)}, 2(k-1)+1) + r(C_{3(k-1)}, 2(k-1)) \\ &= \frac{(3k-1)(k+2)}{2} + 0 + 3k - 2 + \frac{(k-1)(k)(k+5)}{2} + 3 = \frac{k(k+1)(k+6)}{2}. \end{aligned}$$

(vi) By induction on k. When $k = 1$, $r(C_4, 4) = 15 = \frac{(3+1)(1+4)(1+11+6)}{24}$ (see Table 1). Suppose the result is true for all natural numbers less than k. Then by using parts (i), (ii), (iii), (iv) and (v), we get.

$$\begin{aligned} r(C_{3k+1}, 2k+2) &= r(C_{3k}, 2k+1) + r(C_{3k}, 2k) + r(C_{3(k-1)+2}, 2(k-1)+2) \\ &\quad + r(C_{3(k-1)+1}, 2(k-1)+2) + r(C_{3(k-1)+1}, 2(k-1)+1) \\ &= \frac{k(k+1)(k+6)}{2} + 3 + \frac{(3k-1)(k+2)}{2} \\ &\quad + \frac{(3k-2)(k+3)\big((k-1)^2 + 11(k-1) + 6\big)}{24} + 3k - 2 \\ &= \frac{(3k+1)(k+4)(k^2+11k+6)}{24}. \end{aligned}$$

(vii) By induction on k. If $k=1$, then $r(C_5,5)=31=\frac{(3+2)(1+3)(1+23+122+40)}{120}$ (see Table 1). Now, suppose the result is true for all natural numbers less than k. Then by using parts (i), (iii), (iv), (v) and (vi), we get.

$$\begin{aligned}
r(C_{3k+2},2k+3) &= r(C_{3k+1},2k+2)+r(C_{3k+1},2k+1)+r(C_{3k},2k+1)\\
&\quad + r(C_{3(k-1)+2},2(k-1)+3)+r(C_{3(k-1)+2},2(k-1)+2)\\
&= \frac{(3k+1)(k+4)(k^2+11k+6)}{24}+3k+1+\frac{k(k+1)(k+6)}{2}\\
&\quad + \frac{(3k-1)(k+2)\big((k-1)^3+23(k-1)^2+122(k-1)+40\big)}{120}+\frac{(3k-1)(k+2)}{2}\\
&= \frac{(3k+2)(k+3)(k^3+23k^2+122k+40)}{120}.
\end{aligned}$$

$(viii)$ By induction on k. If $k=1$, then $r(C_3,4)=6=\frac{1(1+35+365+1165+234-360)}{240}$ (see Table 1). Now, suppose the result is true for all natural numbers less than k. Then by using parts (i), (iv), (v), (vi) and (vii), we get.

$$\begin{aligned}
r(C_{3k},2k+2) &= r(C_{3(k-1)+2},2(k-1)+3)+r(C_{3(k-1)+2},2(k-1)+2)\\
&\quad + r(C_{3(k-1)+1},2(k-1)+2)+r(C_{3(k-1)},2(k-1)+2)\\
&\quad + r(C_{3(k-1)},2(k-1)+1)\\
&= \frac{(3k-1)(k+2)\big((k-1)^3+23(k-1)^2+122(k-1)+40\big)}{120}+\frac{(3k-1)(k+2)}{2}\\
&\quad + \frac{(3k-2)(k+3)\big((k-1)^2+11(k-1)+6\big)}{24}+\frac{(k-1)(k)(k+5)}{2}\\
&\quad + \frac{(k-1)\Big[(k-1)^5+35(k-1)^4+365(k-1)^3+1165(k-1)^2+234(k-1)-360\Big]}{240}\\
&= \frac{k(k^5+35k^4+365k^3+1165k^2+234k-360)}{240}.
\end{aligned}$$

(ix) The proof is clear.

(x) Clearly, for every vertex $v\in V(C_n$ the function $f:V(C_n)\to\{0,1,2\}$ with $f(v)=1$ and weight $W(f(V))=2n-1$ is a Roman dominating function of G. Hence, $r(C_n,2n-1)=\binom{n}{1}=n$.

(xi) By induction on n. The result is true for $n=2$, since $r(C_2,2)=3$ (see Table 1). Suppose the result is true for every natural number less than n. Then by parts (i), (ix) and (x) and Proposition 1.2 part (vi), we have.

$$\begin{aligned}
r(C_n,2n-2) &= r\big(C_{n-1},2(n-1)-1\big)+r\big(C_{n-1},2(n-1)-2\big)+r\big(C_{n-2},2(n-2)\big)\\
&\quad + r\big(C_{n-3},2(n-3)+2\big)+r\big(C_{n-3},2(n-3)+1\big)\\
&= n-1+\frac{n(n-1)}{2}+1+0+0=\frac{n(n+1)}{2}.
\end{aligned}$$

(xii) By induction on n. The result is true for $n=3$, since $r(C_3,3)=7$ (see Table 1).

Suppose the result is true for every natural number less than n. Then by parts (i), (ix), (x) and (xi) and Proposition 1.2 part (vi), we have.

$$\begin{aligned}
r(C_n, 2n-3) &= r\big(C_{n-1}, 2(n-1)-2\big) + r\big(C_{n-1}, 2(n-1)-3\big) + r\big(C_{n-2}, 2(n-2)-1\big) \\
&\quad + r\big(C_{n-3}, 2(n-3)+1\big) + r\big(C_{n-3}, 2(n-3)\big) \\
&= \frac{n(n-1)}{2} + \frac{(n-1)(n-2)(n+3)}{6} + n - 2 + 0 + 1 \\
&= \frac{n(n-1)(n+4)}{6}.
\end{aligned}$$

$(xiii)$ By induction on n. If $n = 5$, then $r(C_5, 6) = 40$. Therefore, the result is true for $n = 4$(see Table 1). Suppose now the result is true for every natural number less than n. Then by parts (i), (ix), (x), (xi) and (xii), we have.

$$\begin{aligned}
r(C_n, 2n-4) &= r\big(C_{n-1}, 2(n-1)-3\big) + r\big(C_{n-1}, 2(n-1)-4\big) + r\big(C_{n-2}, 2(n-2)-2\big) \\
&\quad + r\big(C_{n-3}, 2(n-3)\big) + r\big(C_{n-3}, 2(n-3)-1\big) \\
&= \frac{(n-1)(n-2)(n+3)}{6} + \frac{(n-1)(n)\big[(n-1)^2 + 5(n-1) - 18\big]}{24} \\
&\quad + \frac{(n-1)(n-2)}{2} + 1 + n - 3 \\
&= \frac{n(n+1)(n^2+5n-18)}{24}.
\end{aligned}$$

(xiv) By induction on n. The result is true for $n = 5$, since $r(C_5, 5) = 31$ (see Table 1). Suppose now the result is true for every natural number less than n. Then by parts (i), (x), (xi), (xii) and $(xiii)$, we have.

$$\begin{aligned}
r(C_n, 2n-5) &= r\big(C_{n-1}, 2(n-1)-4\big) + r\big(C_{n-1}, 2(n-1)-5\big) + r\big(C_{n-2}, 2(n-2)-3\big) \\
&\quad + r\big(C_{n-3}, 2(n-3)-1\big) + r\big(C_{n-3}, 2(n-3)-2\big) \\
&= \frac{(n-1)(n)\big[(n-1)^2 + 5(n-1) - 18\big]}{24} \\
&\quad + \frac{(n-1)(n-2)\big[(n-1)^3 + 11(n-1)^2 - 14(n-1) - 144\big]}{120} \\
&\quad + \frac{(n-2)(n-3)(n+2)}{6} + n - 3 + \frac{(n-3)(n-2)}{2} \\
&= \frac{n(n-1)(n^3+11n^2-14n-144)}{120}.
\end{aligned}$$

(xv) We need to prove that for every $k \in \mathbb{N}$, $r(C_i, 2k) < r(C_i, 2k)$ for $k \leq i \leq 2k-1$ and $r(C_i, 2k) > r(C_i, 2k)$ for $2k \leq i \leq 3k$. By induction on k, the result is true for $k = 1$. Now, suppose that the result is true for every i less than or equal k. We will prove it for $i = k+1$ which means $r(C_i, 2k+2) < r(C_{i+1}, 2k+2)$ for $k+1 \leq i \leq 2k+1$. By part (i) and the induction hypothesis, we have

$$\begin{aligned} r(C_i, 2k+2) = & r(C_{i-1}, 2k+1) + r(C_{i-1}, 2k) + r(C_{i-2}, 2k) \\ & + r(C_{i-3}, 2k) + r(C_{i-3}, 2k-1) \\ < & r(C_i, 2k+1) + r(C_i, 2k) + r(C_{i-1}, 2k) \\ & + r(C_{i-2}, 2k) + r(C_{i-2}, 2k-1) \\ = & r(C_{i+1}, 2k+2). \end{aligned}$$

Similarly, we prove for the other inequality.

(xvi) Similar to the prove of part (xv), we will prove that for every $k \in \mathbb{N}$, $r(C_i, 2k+1) < r(C_i, 2k+1)$ for $k+1 \leq i \leq 2k$ and $r(C_i, 2k+1) > r(C_i, 2k+1)$ for $2k+1 \leq i \leq 3k+1$. By induction on k, the result is true for $k = 1$. Now, suppose that the result is true for every i less than or equal $k+1$. We will prove it for $i = k+2$ which means $r(C_i, 2k+3) < r(C_{i+1}, 2k+3)$ for $k+2 \leq i \leq 2k+2$. By part (i) and the induction hypothesis, we have

$$\begin{aligned} r(C_i, 2k+3) = & r(C_{i-1}, 2k+2) + r(C_{i-1}, 2k+1) + r(C_{i-2}, 2k+1) \\ & + r(C_{i-3}, 2k+1) + r(C_{i-3}, 2k) \\ < & r(C_i, 2k+2) + r(C_i, 2k+1) + r(C_{i-1}, 2k+1) \\ & + r(C_{i-2}, 2k+1) + r(C_{i-2}, 2k) = r(C_{i+1}, 2k+3). \end{aligned}$$

Similarly, we prove for the other inequality.

$(xvii)$ By Theorem 2.3, we have

$$\begin{aligned} R(C_n, x) = & \sum_{j=\gamma_R(C_n)}^{2n} r(C_n, j)\, x^j = (x^2+x)R(C_{n-1}, x) + x^2 R(C_{n-2}, x) \\ & + (x^3+x^2)R(C_{n-3}, x) \\ = & \sum_{j=\lceil \frac{2n-2}{3} \rceil}^{2n-2} r(C_{n-1}, j)\big[x^{j+2} + x^{j+1}\big] + \sum_{j=\lceil \frac{2n-4}{3} \rceil}^{2n-4} r(C_{n-2}, j) x^{j+2} \\ & + \sum_{j=\lceil \frac{2n-6}{3} \rceil}^{2n-6} r(C_{n-3}, j)\big[x^{j+3} + x^{j+2}\big]. \end{aligned}$$

Now, if $\alpha_n = \sum_{j=\lceil \frac{2n}{3} \rceil}^{2n} r(C_n, j)$, we can see that all the coefficients of $R(C_{n-1}, x)$ and $R(C_{n-3}, x)$ counted twice and all the coefficients of $R(C_{n-2}, x)$ counted once in α_n. Hence, $\alpha_n = 2\alpha_{n-1} + \alpha_{n-2} + 2\alpha_{n-3}$.

$(xviii)$ If $j = 2$, then

$$\begin{aligned} \sum_{i=2}^{6} r(C_i, 4) = & \sum_{i=2}^{4} r(C_i, 3) + 3\sum_{i=1}^{3} r(C_i, 2) + \sum_{i=1}^{1} r(C_i, 1) \\ 35 = & 13 + 3(7) + 1 = 35. \end{aligned}$$

By part (i), we have

$$\sum_{i=j}^{3j} r(C_i, 2j) = \sum_{i=j}^{3j} r(C_{i-1}, 2j-1) + \sum_{i=j}^{3j} r(C_{i-1}, 2j-2) + \sum_{i=j}^{3j} r(C_{i-2}, 2j-2) + \sum_{i=j}^{3j} r(C_{i-3}, 2j-2) + \sum_{i=j}^{3j} r(C_{i-3}, 2j-3).$$

Now, by Proposition 1.2 part (vi), we have

$$\begin{aligned}
\sum_{i=j}^{3j} r(C_{i-1}, 2j-1) &= \sum_{i=j-1}^{3j} r(C_i, 2j-1) = \sum_{i=j}^{3j-2} r(C_i, 2j-1),\\
\sum_{i=j}^{3j} r(C_{i-1}, 2j-2) &= \sum_{i=j-1}^{3j} r(C_i, 2j-2) = \sum_{i=j-1}^{3j-3} r(C_i, 2j-2),\\
\sum_{i=j}^{3j} r(C_{i-2}, 2j-2) &= \sum_{i=j-2}^{3j} r(C_i, 2j-2) = \sum_{i=j-1}^{3j-3} r(C_i, 2j-2),\\
\sum_{i=j}^{3j} r(C_{i-3}, 2j-2) &= \sum_{i=j-3}^{3j} r(C_i, 2j-2) = \sum_{i=j-1}^{3j-3} r(C_i, 2j-2)
\end{aligned}$$

and

$$\sum_{i=j}^{3j} r(C_{i-3}, 2j-3) = \sum_{i=j-3}^{3j} r(C_i, 2j-3) = \sum_{i=j-1}^{3j-5} r(C_i, 2j-3).$$

(xix) The proof is similar to the proof of part $(xviii)$. □

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Application of Abstract Algebra to Musical Notes and Indian Music

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Abstract: This paper critically analysis the behavior and the relationship that exist between musical notes and abstract algebra. The musical notes form additive Abelian group modulo 12. Finally, the work come up with some propositions due to the musical notes behavior and their proofs, one of which was name Dido's Theorem.

Key Words: Abstract algebra, Abelian group modulo 12, Abelian group modulo 7, musical notes, Indian sargam, Dido's Theorem, transposition, inversion, Smarandache multigroup.

AMS(2010): 30K20.

§1. Introduction

In mathematics and group theory, abstract algebra, studies the algebraic structures known as groups. The concept of a group is central to abstract algebra. Other well-known algebraic structures, such as rings, fields, and vectors space can all be seen as groups endowed with additional operations and axioms. Various physical systems, such as crystals and the hydrogen atom, can be modeled by symmetry groups. Thus, abstract algebra has many important applications in physics, chemistry, and materials science. Abstract algebra is also central to public key cryptography. The modern concept of abstract group developed out of several fields of mathematics (Wussing, 2007). The idea of group theory although developed from the concept of abstract algebra, yet can be applied in many other areas of mathematical areas and other field in sciences and as well as in music. Music theory is a big field within mathematics and lots of different people have taken it in different directions. Music is that one of the fine arts which is concerned with the combination of sounds with a view to beauty of form and the expression of thought or feeling. Music itself is not complete without musical notes. And the musical notes and Indian music are: $\mathbf{C}, \mathbf{C\#}, \mathbf{D}, \mathbf{D\#}, \mathbf{E}, \mathbf{F}, \mathbf{F\#}, \mathbf{G}, \mathbf{G\#}, \mathbf{A}, \mathbf{A\#}, \mathbf{B}$. Now in India, Indian seven sargam are: Sa. Re, Ga, Ma, Pa, Dha, Ni. For years, many people find it difficult to comprehend some concept in group theory satisfactorily, but with the behavior of these musical notes, group theory can be studied. "All is number" (only musical notes) is the motto of the Pythagorean School. This school was founded by the Greek mathematician and philosopher Pythagoras (ca. 580-500 B.C.). The members of the school pursued the study of mathematics, philosophy, astronomy and music.

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§2. Group Theory

Group theory is the branch of pure mathematics which is emanated from abstract algebra. Due to its abstract nature, it was seeming to be an arts subject rather than a science subject. In fact, was considered pure abstract and not practical. Even students of group theory after being introduced to the course seems not to believe as to whether the subject has any practical application in real life, because of its abstract nature The problem prompts the researchers to study the different ways in which group can be express concretely both from theoretical and practical point of view, with intention of bringing its real- life application in musical notes. This paper aim at taking some concepts of group theory to study and understand musical notes in relation to the group's axioms. The main objective is to see these musical notes interpretation algebraically as regard to their behavior. This work focus on the behavior of musical notes which largely depend on groups axioms, theorems such as two left cosets, cyclic groups, Langrange's and Sylow's first theorem.

§3. Definition of Terms

We present here some few definitions that will help us to be familiar with concepts in music and abstract algebra.

3.1 Musical Notes

Musical notes are the following notes:

$\mathrm{C, C\#, D, D\#, E, F, F\#, G, G\#,\ A,\ A\#,\ B.}$

When logically combined, give out pleasant sound to the ear. The first note which is C, is called the root note; C# is called the 2^{nd} note; D is called the 3^{rd} note; $D\#$ is called the 4^{th} note; E is called the 5^{th} note; F is called the 5^{th} note; F# is called the 6^{th} note; G is called the 7^{th} note; G is called the 8^{th} note; A is called the 10^{th} note; A# is called the 11^{th} note and B is called the 12^{th} note.

3.2 Musical Flat b

Musical flats can be defined as the movement of sound from one pitch to the one lower, and it is donated to b. For example, movement from F to any other not

3.3 Musical Sharp

This can be considered as the movement of sound from a pitch (note) to another pitch higher, and it bis denoted by #. For example, movement from F to any other note to the right on the musical notes. Tone This simply meant any movement from a musical note to the next note two steps forward or backward on the musical notes. For example, movement from F to G or to D#. Semitone This can be defined as any movement from a musical note to the next note a step forward or backward on the musical notes (Scales). For example, movement from F to F# or F to E. Now in India Indian seven sargam are:

Sa (For Agni Devta);

Re means Rishabh (For Brahamma Devta);

Ga means Gandhar (For Goddess Saraswati);

Ma means Madhyam (For God Mahadev or Shiv);

Pa means Pancham (For Goddess Laxmi);

Dha means Dhaivata (For Lord Ganesha) and
Ni means Nishad (For Sun God).

3.4 Chord

A chord is produced when two, three or more notes are sounded together.

3.5 Transposition

Transposition involves playing or writing a given melody at a different pitch higher or lower other than the original.

3.6 Abstract Group

A group is a non-empty set $(G,*)$ together with an operation $(*)$ on it which satisfies the following axioms:

C_1 : $\forall \mathrm{a}, \mathrm{b} \in \mathrm{G}, \quad \mathrm{a} * \mathrm{b} \in G$ (Closure);
C_2 : $\forall \mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathrm{G}, \quad (\mathrm{a} * \mathrm{b}) * \mathrm{c} = \mathrm{a} * (\mathrm{b} * \mathrm{c})$ (Associative);
C_3 : $\forall \mathrm{a}, \mathrm{b} \in G, \exists e \in G$, then $\mathrm{a} * e = e * \mathrm{a} = \mathrm{a}$ (Identity);
C_4 : $\forall \mathrm{a} \in G, \exists \mathrm{a}^{-1} \in G$, then $\mathrm{a} * \mathrm{a}^{-1} = \mathrm{a}^{-1} * \mathrm{a} = e$ (Inverse);
C_5 : $\forall \mathrm{a}, \mathrm{b} \in G, \mathrm{a} * \mathrm{b} = \mathrm{b} * \mathrm{a} \in G$ (commutative).

A generalization of group is the multigroup ([7]). Usually, a *Smarandache multigroup*

$$\widetilde{G} = \left(\bigcup_{i=1}^{m} G_i; \bigcup_{i=1}^{m} \{\cdot_i\} \right)$$

is the union of m groups, i.e., $(G_i; \cdot_i)$ is a group for integers $1 \le i \le m$ constraint with conditions on their intersection, for instance their intersection $\bigcap_{i=1}^{m} G_i = \{\mathrm{e}\}$, the identity of all G_i, i.e., $e_i = e$ for integers $1 \le i \le m$, which can be also applied to characterize the musical notes.

3.7 Integers Modulo m

This is a finite group that is called the additive group of the residue class of integers modulo m. it is denoted by Z_m.

3.8 p-Group

Let p be an arbitrary but fixed prime number. A finite group G is said to be a p-group if its order is power of p.

If $T \le G$ and $|T| = p^r$ for an integer $r \ge 0$ then T is called p-subgroup of G.

§4. Theoretical Underpinning

Notice that the musical notes is obviously a multigroup consists of 12 trivial groups. However, several authors worked on the application of group theory to many fields in sciences, games and many more other fields but only few have ventured the field of music. Pythagoras ($428-347$ B.C.), who is considered as founder of the first school of mathematics as a purely deductive science is also the founder of a theoretical music. He used to say that "all is number "and musical notes are not exceptional, that is C, C#, D, D#, E, F, F#, G, G#, A, A#, B.

But why "*all is number*"? The Pythagoreans associated certain meanings and characters to numbers. They considered odd numbers as males and even numbers as females. To the Pythagoreans, one is the number of reason, two is the number of opinion, three is the number of harmony, four is the number of justice, five is the number of marriage, six is the number of creation, seven is the number of awe, and ten is the number of the universe. A couple of possible reasons were given. The first one is the Eastern influence.

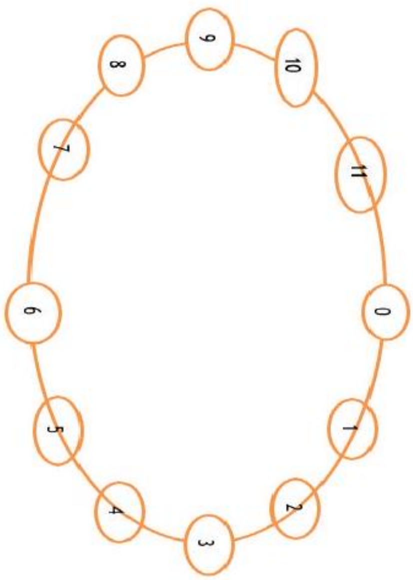

Figure 1 Musical clock

Having traveled to Egypt and Babylon, Pythagoras might have been influenced by numerology, which deals with numbers and mystical relations among them, that was common in these two regions. A second possible reason is to give an alternative view to the contemporary belief in Greek concerning the principles of things. At the time, it was believed that earth, air, fire and water are the four basics principles of things. This did not convince Pythagoras in explaining the principles of immaterial things. A third possibility comes from astronomy, a subject that was studied by Pythagoras. In studying stars, one observes that each constellation can be characterized by the number of stars composing it and the geometrical figure that they form. The fourth possible reason comes from music. The members of the school practiced music. Pythagoras observed that musical notes produced from a vibrating string of some length could be characterized by (ratios of) numbers. Dividing a vibrating string by some movable object into two different lengths produced different types of musical notes. These notes are then described by the ratios of the lengths of the parts of the vibrating string. Explaining musical notes and describing stars by numbers may have then led the Pythagoreans to think that numbers can also be used to explain other phenomena (Heath [6] and Thomas M. Flore [9]). He referred to C, C#, D, D#, E, F, F#, G, G#, A, A#, B. As the Z_{12} Model of pitch class. He constructed a musical clock shown in Figure 1.

He also said that there is a bijection between the set of pitch classes and Z_{12}. He defined transposition as: $T_n : Z_{12} \to Z_{12}$ then their exist $T_n(X) : X + n$ and inversion was also defined as $I_n : Z_{12} \to Z_{12}$ then their exist $I_n(X) : -x + n$, where n is in mod 12.

Ada Zhang [1] considered possibly musical notes with corresponding integers as

C	$C\#$	D	$D\#$	E	F	$F\#$	G	$G\#$	A	$A\#$
0	1	2	3	4	5	6	7	8	9	10

He defined transposition T_n as that moves a pitch-class or pitch-class set up by n (mod12) [1].

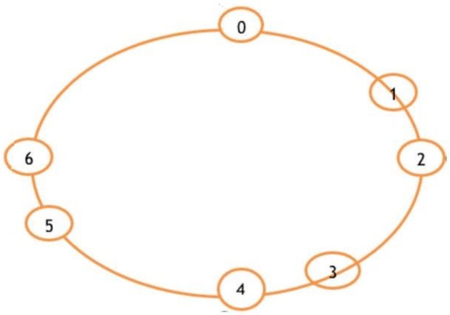

Figure 2 Indian musical clock

In my ideas considered possibly indian musical sargam with corresponding integers as

Sa	Re	Ga	Ma	Pa	Dha	Ni
0	1	2	3	4	5	6

We defined transposition, T_n as that moves a sargam class by $n(\text{mod } 7)$ And inversion was also defined here as T_nI as the pitch (A) about $C(0)$ and then transposes it by n. that is, $T_nI(a) = -a + n(\text{mod}12)$. Then further, laid out all the pitches in a circular pattern on a 12 -sided polygon. That is, consider the transposition T_{11}. It sends C to B, C to C, Alissa [3] assert that the musical actions of the dihedral groups. This paper considers two ways in which the dihedral groups act on the set of major and minor triads.

According to David Wright [4], referred to the musical notes with their corresponding integers as in Ada Zhang [1] as M_{12}, that is the Mathieu group. He asserts that this can be generated by just two permutations Expressed below in both two-line notation and cycle notation. We denote these generating permutations as P_1 and P_0

Adam ([2] defined transposition and inversion as: Transposition is define as $T_n : Z_{12} \to Z_{12}$ then their exist $(X) : x + n \bmod 12$ and he also define Inversion as $I_n(X) : Z_{12} \to Z_{12}$ then their exist $I_n(X) : -x + n$ where n is in mod12.

§5. Methodology

We need a few of conclusions in group theory following ([7]).

Lemma 5.1 *Let $H \leq G$ be groups and $g \in G$. Then,*

(*i*) $g \in gH$;

(*ii*) *Two left cosets of H in G are either identical or disjoint;*

(*iii*) *The number of elements in gH is $|H|$.*

Lemma 5.2(Langrange's Theorem) *The order of a subgroup of a finite group is a factor of the order of the group.*

Lemma 5.3 *Every subgroup of a cyclic group is cyclic.*

Lemma 5.4(First Sylow's Theorem) *Let G be a finite group, p a prime and p^r the highest power of P diving the order of G. Then there is a subgroup of G of order of G. Then there is a subgroup of G of order p^r.*

§6. Results and Discussion

The numbering of the musical notes are listed following:

C	C#	D	D#	E	F	F#	G	G#	A	A#	B
0	1	2	3	4	5	6	7	8	9	10	11

Note that $B\# = C$. It shows that the musical notes form a group of integers of Modulo 12. That is $Z_{12} = \{C, C\#, D, D\#, E, F, F\#, G, G\#, A, A\#, B\}$. Let the operation be $* = \# = +$. The behavior of the musical note on groups musical notes related with groups axiom is shown in the following:

(*i*) Closure $E, F \in Z_{12}$. Hence $E * F = A \in Z_{12}$.

(*ii*) Associative E, F and $F\# \in Z_{12}$. Hence, $(E * F) * F\# = E * (F * F\#) = A * F\# = E * B = D\# = D\#$.

With the behavior of the musical notes of Indian sargam we have just seen, we personally suggest for the root note of musical scales (notes) to be algebraically named as the identity note. Table 1 lists the musical notes and their inverse.

Sargam	Inverse
Re	Ni
Ga	Dha
Ma	Pa

Table 1

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Famous Words

What we know is very slight, what we don't know is immense. Man follows only phantoms.

By Pierre-Simon Laplace, a French mathematician and also a physicist

Author Information

Submission: Papers only in electronic form are considered for possible publication. Papers prepared in formats, viz., .tex, .dvi, .pdf, or.ps may be submitted electronically to one member of the Editorial Board for consideration in the **International Journal of Mathematical Combinatorics** (*ISSN 1937-1055*). An effort is made to publish a paper duly recommended by a referee within a period of $3-4$ months. Articles received are immediately put the referees/members of the Editorial Board for their opinion who generally pass on the same in six week's time or less. In case of clear recommendation for publication, the paper is accommodated in an issue to appear next. Each submitted paper is not returned, hence we advise the authors to keep a copy of their submitted papers for further processing.

Abstract: Authors are requested to provide an abstract of not more than 250 words, latest Mathematics Subject Classification of the American Mathematical Society, Keywords and phrases. Statements of Lemmas, Propositions and Theorems should be set in italics and references should be arranged in alphabetical order by the surname of the first author in the following style:

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