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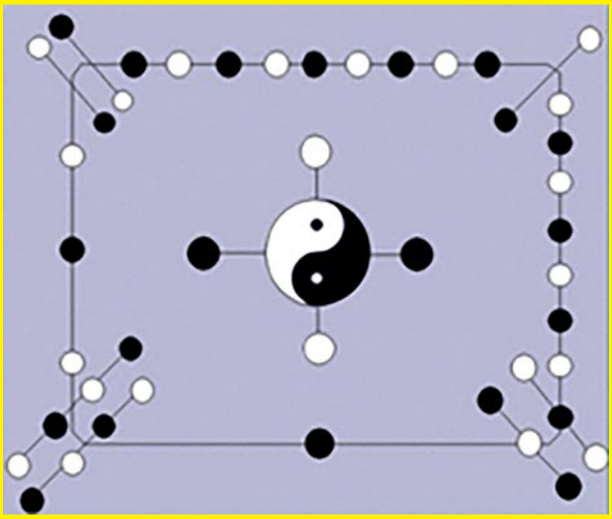

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If you want to love your own value, you have to create value for the world.

By Johann Wolfgang von Goethe, a German thinker,writer and scientist

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Quadratic Symmetry Algebra and Spectrum of the 3D Nondegenerate Quantum Superintegrable System

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Abstract: In this paper, we present the quadratic associative symmetry algebra of the 3D nondegenerate maximally quantum superintegrable system. This is the complete symmetry algebra of the system. It is demonstrated that the symmetry algebra contains suitable quadratic subalgebras, each of which is generated by three generators with relevant structure constants, which may depend on central elements. We construct corresponding Casimir operators and present finite-dimensional unirreps and structure functions via the realizations of these subalgebras in the context of deformed oscillators. By imposing constraints on the structure functions, we obtain the spectrum of the 3D nondegenerate superintegrable system. We also show that this model is multiseparable and admits separation of variables in cylindrical polar and paraboloidal coordinates. We derive the physical spectrum by solving the Schrödinger equation of the system and compare the result with those obtained from algebraic derivations.

Key Words: Quantum superintegrable system, symmetry algebra, quadratic subalgebras, Schrödinger equation

AMS(2010): 81Q60, 81T60.

§1. Introduction

Superintegrable Hamiltonian systems are a very exclusive family of physical systems as they are exactly solvable systems and their symmetries are, in many cases, generated by a nonlinear generalization of Lie algebras [1]. In a classical system, a d-dimensional dynamical system with Hamiltonian $\mathcal{H} = \frac{1}{2}g^{jk}p_jp_k + V(x)$ and constants of motion $\mathcal{A}_l = f_l(x,p)$, $l = 1,\ldots,d-$

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1 is (Liouville) completely integrable if the system allows d integrals including $\mathcal{H}$ that are functionally independent on the phase space, and are in involution $\{\mathcal{H}, \mathcal{A}_l\} = 0$, $\{\mathcal{A}_l, \mathcal{A}_m\} = 0$, $l, m = 1, \ldots, d-1$. The integrable system is known as superintegrable if it allows additional well-defined constants of motion $\mathcal{B}_m$ on the phase space and they are in involution $\{\mathcal{H}, \mathcal{B}_m\} = 0$, $m = 1, \cdots, k$. It is assumed that the set of integrals $\{\mathcal{H}, \mathcal{A}_1, \cdots, \mathcal{A}_{d-1}, \mathcal{B}_1, \ldots, \mathcal{B}_k\}$ are functionally independent. The system will be maximally superintegrable if the integrals set has $(2d-1)$ integrals and minimally superintegrable if the set has $d+1$ such integrals. There is a remark that the additional integrals $\mathcal{B}_m$ need not be in involution with $\mathcal{A}_1, \cdots, \mathcal{A}_{d-1}$ as well as not with each other.

In quantum mechanics, similar definitions apply with the coordinates x_i and momenta p_k, in which they represent as hermitian operators in the Hilbert space satisfying the Heisenberg algebra. Thus, the quantum counterpart of the system is integrable, if there exist $d-1$ well-defined algebraically independent quantum integrals of motion $A_1, \cdots, A_{d-1}$ on the Hilbert space that commute with the Hamiltonian operator H and pair-wise with each other, that is, $[H, A_l] = 0$ and $[A_l, A_m] = 0$ for $1 \leq l, m \leq d-1$. The system is superintegrable for the existence of the additional algebraically independent quantum integrals of motion B_m such that $[H, B_m] = 0$ for $m = 1, \cdots, k$. Moreover, the system is known as the quasi-maximally superintegrable system which has $2d-2$ independent constants of motion including the Hamiltonian H. The maximally superintegrable system is more special for the existence of a large number of symmetries and for arising many unique properties such as periodic motions and finite closed trajectories or accidental degeneracies of the energy spectrum in classical/(or) quantum mechanics. More exhaustive algebraic descriptions of superintegrable systems in classical and quantum mechanics, symmetry algebras, and their connections to special functions can be found in the review paper [1]. Famous examples of superintegrable systems are the Coulomb-Kepler [2,3] and the harmonic oscillator [4,5].

A systematic algebraic investigation is performed for superintegrable Hamiltonian systems on 2D and 3D Euclidean spaces in [6C8]. The algebraic computations were more or less completed for the constants of motion which are first- or second-order polynomials of the momenta. Over the year, much work has been done on the complete classifications of second-order classical and quantum superintegrable systems [9C14]. Nowadays, the search for arbitrary dimensional quantum superintegrable systems and their higher-order constants of motion is a paramount research area (see for examples [15-24]). In the context of the algebraic perspective, the higher-order polynomial algebras with structure constants of certain Casimir invariants are constructed by using the integrals of the d-dimensional superintegrable systems [25-30]. However, the classification of 3D superintegrable Hamiltonian systems is still an active field of research in particular for nondegenerate quantum superintegrable systems and their symmetry algebras [31-34]. The four parameters depending potentials are classified as the nondegenerate potentials, and less than four parameters depending potentials are classified as the degenerate potentials of the 3D superintegrable systems. Such classifications have been explicitly studied in [31,35]. The systems with degenerate and nondegenerate potentials were investigated in the seminal paper [8]. It is established that any 3d nondegenerate classical superintegrable system with five second-order constants of motion (including the Hamiltonian) allows an additional integral, which is linearly independent to others. All these integrals of the 3D nondegenerate superintegrable system [32,32] close to form a parafermionic-like Poisson algebras [36]. The energy spectra of the generalized Coulomb-Kepler system in Euclidean space have been studied using the methods of separation of variables in [37]. The energy eigenvalues of the generalized quantum Kepler-Coulomb system with nondegenerate potentials were calculated algebraically in [38]. However, the superintegrable systems with nondegenerate potential, their corresponding quadratic integrals and the symmetry algebras investigation are still interesting problems in quantum mechanics [32]. We introduce the 3D nondegenerate quantum superintegrable system depending on four parameters, which is known as the *KKM Potential* V_{IV} [31] with quadratic integrals to present full symmetry quadratic algebra structure. We calculate the energy eigenvalues of the system algebraically. We also show the multiseparability of the system in cylindrical polar and paraboloidal coordinates and solve the Schrödinger equation of the system. We compare the result with those obtained from algebraic calculations.

We present this paper in the following form. In section 2, we present a 3D nondegenerate quantum Hamiltonian system in a flat space, and its superintegrability for the set of algebraically independent quadratic integrals. In section 3, we construct the quadratic full symmetry algebra structure generated by the quadratic constants of motion of the system. Section 4 contains a brief discussion on the quadratic algebra $Q(3)$ related to the symmetry algebra. In section 5, we recall quadratic subalgebras which are generated by three generators involving structure constants from symmetry algebra and present their corresponding Casimir operators. In section 6, we present the algebraic realizations of the quadratic subalgebras in the context of deformed oscillators of Daskaloyannis's approach [39,40] and obtain the energy spectrum of the 3D system. Section 7 contains the solutions of the Schrödinger equation of the 3D Hamiltonian system in cylindrical polar and paraboloidal coordinates. Section 8 contains the concluding remarks.

§2. The 3D Nondegenerate Quantum Superintegrable System

The 3D superintegrable systems with the Hamiltonian

$$H = \frac{1}{2}\left(p_{x_1}^2 + p_{x_2}^2 + p_{x_3}^2\right) + V(x_1, x_2, x_3) \tag{2.1}$$

on the flat space have been initially studied in [8]. All these 3D systems with quadratic integrals have been classified in the complex Euclidean space and distinguished to the so-called nondegenerate potentials [31]. These nondegenerate potentials are linear combinations of four parameters, while degenerate potentials depend on less than four parameters. Kalnins, Kress and Miller [31] also established a general result for the nondegenerate potentials: if V is a 3D nondegenerate potential depending on four parameters considered on a conformally flat space with the metric

$$ds^2 = g(x_1, x_2, x_3)(dx_1^2 + dx_2^2 + dx_3^2), \tag{2.2}$$

then the classical analog of the Hamiltonian associated with the above metric is maximally superintegrable with five functionally independent quadratic integrals $\mathcal{L} = \{S_k : k = 1, \cdots, 5\}$ (including $S_1 \equiv H$) and there always exists an extra quadratic integral S_6 which is linearly independent to the others. These classical nondegenerate systems and their corresponding quadratic integrals have been performed a parafermionic-like quadratic Poisson algebra on conformally flat space [32]. However, the associative quadratic ternary symmetry algebras for the nondegenerate superintegrable systems in quantum mechanics and their degeneracy of the energy spectra remain largely unknown. Their analytic solutions to the Schrödinger equations via separation of variables would be of much interest. At the first attempt, we thus introduce a nondegenerate quantum system, which is known as the *KKM potential* V_{IV}, with the Hamiltonian operator in real Euclidean space $\mathbb{E}^3$ [31],

$$H = p_{x_1}^2 + p_{x_2}^2 + p_{x_3}^2 + c_1(4x_1^2 + x_2^2 + x_3^2) + c_2 x_1 + \frac{c_3}{x_2^2} + \frac{c_4}{x_3^2}, \tag{2.3}$$

where $p_{x_i} = -i\frac{\partial}{\partial x_i}$ and we set $m = \hbar = 1$. It is remarked that c_2 can be eliminated by a shift of coordinates under the condition that c_1 is not zero and it is possible to reduce the potential to 3 parameters, in this case, the algebra must transform correspondingly. In this paper, we present the full symmetry quadratic algebra of the 3D nondegenerate quantum superintegrable system (2.3) and obtain the energy spectrum applying the Daskaloyannis deformed oscillator algebra approach [39] on the symmetry algebras.

The quantum Hamiltonian system (2.3) has the following four algebraically independent quadratic integrals,

$$\begin{aligned} A_1 &= p_{x_1}^2 + 4c_1 x_1^2 + c_2 x_1, \quad A_2 = p_{x_2}^2 + c_1 x_2^2 + \frac{c_3}{x_2^2}, \\ B_1 &= J_2 p_{x_3} + p_{x_3} J_2 + 2c_1 x_1 x_3^2 + \frac{c_2 x_3^2}{2} - \frac{2c_4 x_1}{x_3^2}, \quad B_2 = J_1^2 + \frac{c_3 x_3^2}{x_2^2} + \frac{c_4 x_2^2}{x_3^2}, \end{aligned} \tag{2.4}$$

and one additional quadratic integral,

$$F = p_{x_2} J_3 + J_3 p_{x_2} - 2c_1 x_1 x_2^2 - \frac{c_2 x_2^2}{2} + \frac{2c_3 x_1}{x_2^2}, \tag{2.5}$$

where

$$J_1 = x_2 p_{x_3} - x_3 p_{x_2}, \quad J_2 = x_3 p_{x_1} - x_1 p_{x_3}, \quad J_3 = x_1 p_{x_2} - x_2 p_{x_1}. \tag{2.6}$$

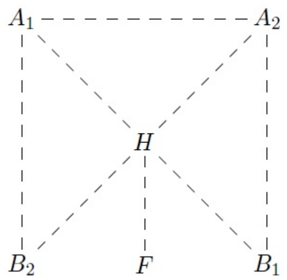
All these integrals are linearly independent including the Hamiltonian H. The Hamiltonian system is maximally superintegrable. It can be proved by the following commutation relations

$$[A_i, H] = 0, \quad [B_i, H] = 0, \quad i = 1, 2, \quad [H, F] = 0. \tag{2.7}$$

We also found that

$$[A_1, B_2] = 0, \quad [A_1, A_2] = 0, \quad [A_2, B_1] = 0. \tag{2.8}$$

The above commutativity relations can be expressed as the following diagram to easily understand,



(2.9)

where the dashed lines indicate that the commutator of the corresponding integrals is zero and the absence of dashed lines among the integrals means the commutator is nonzero. The presence of F ensures that the integrals generate a ternary type quadratic algebra involving six generators including the Hamiltonian operator. We can also define

$$C_1 = [A_1, B_1], \quad C_2 = [A_2, B_2], \quad D = [B_1, B_2]. \tag{2.10}$$

It is shown that the new integrals of motion C_1 and C_2 are cubic functions of momenta, which can not be expressed as a polynomial function in terms of other integrals of motion that are the second structure of momenta. We may also define

$$E_1 = [A_1, F], \quad E_2 = [A_2, F], \quad E_3 = [B_1, F], \quad E_4 = [B_2, F]. \tag{2.11}$$

§3. **Quadratic Symmetry Algebra**

We now derive the full quadratic symmetry algebra of the quantum superintegrable Hamiltonian system (2.3). After a long direct computation and using different commutation relations and Jacobi identities, the six integrals of motion including the Hamiltonian H close to form the following quadratic symmetry algebra,

$$[A_1, C_1] = 4c_2A_1 + 16c_1B_1 + 4c_2(A_2 - H), \tag{3.1}$$

$$[B_1, C_1] = 24A_1^2 + 32(A_2 - H)A_1 - 4c_2B_1 + 8H^2 - 16HA_2 -8c_1(4c_4 - 3) + 8A_2^2, \tag{3.2}$$

$$[A_2, C_2] = 8A_2^2 + 8(A_1 - H)A_2 + 8c_1(2B_2 + 1), \tag{3.3}$$

$$[B_2, C_2] = -16(c_3 + c_4 - 1)A_2 - 8\{A_2, B_2\} - 8(A_1 - H)B_2 -8(2c_3 - 1)A_1 + 8(2c_3 - 1)H, \tag{3.4}$$

$$[A_1, D] = 8A_2B_1 - 8FA_1 - 8A_2F + 8HF, \tag{3.5}$$

$$[C_1, F] = 8HA_2 - 8A_2A_1 - 8A_2^2 - 16c_1B_2 - 8c_1, \tag{3.6}$$

$$[C_1, B_2] = 8A_2B_1 - 8A_1F - 8A_2F + 8HF, \tag{3.7}$$

$$[E_1, A_2] = 16c_1F + 4c_2A_2, \tag{3.8}$$

$$[E_1, B_2] = 8HF - 8FA_1 - 8A_2F + 8B_1A_2, \quad (3.9)$$
$$[E_1, F] = 16A_2A_1 - 4c_2F - 8A_2^2 + 8c_1(4c_3 - 3), \quad (3.10)$$
$$[E_1, A_1] = -16c_1F - 4c_2A_2, \quad (3.11)$$
$$[C_2, B_1] = 4c_2B_2 - 8FA_2 - 8A_1F + 8HF + 2c_2, \quad (3.12)$$
$$[E_2, A_2] = -4c_2A_2 - 16c_1F, \quad (3.13)$$
$$[E_2, B_1] = 8A_2^2 + 8(A_1 - H)A_2 + 16c_1B_2 + 8c_1, \quad (3.14)$$
$$[E_2, B_2] = 8A_2F + 8A_1F - 8HF - 8B_1A_2, \quad (3.15)$$
$$[E_2, F] = 8A_2^2 - 16A_1A_2 + 4c_2F - 8c_1(4c_3 - 3), \quad (3.16)$$

and

$$[B_1, D] = 8FB_1 - 8(A_2 + 3A_1 - H)B_2 - 8(2c_4 - 1)A_2 - 12A_1 + 4H, \quad (3.17)$$
$$[D, B_2] = 8(B_1B_2 + B_2B_1) + 8FB_2 + 8(2c_3 - 1)B_1 + 8(2c_4 - 1)F - 8B_1B_2, \quad (3.18)$$
$$[E_3, B_1] = 8(A_1 + A_2 - H)F - 4c_2B_2 - 2c_2, \quad (3.19)$$
$$[E_3, B_2] = 8FB_1 - 8(2c_3 - 1)A_1 - 16(c_3 + c_4 - 1)A_2 + 8(2c_3 - 1)H$$
$$-8A_1B_2 - 16A_2B_2 + 8B_2 - 8B_1F, \quad (3.20)$$
$$[E_3, F] = 8A_2B_1 - 4c_2B_2 - 2c_2, \quad (3.21)$$
$$[E_4, A_1] = -8(A_1 + A_2 - H)F + 8B_1A_2, \quad (3.22)$$
$$[E_4, B_2] = -8(B_2F + FB_2) + 8FB_2 - 8(2c_4 - 1)F - 8B_1B_2 - 8(2c_3 - 1)B_1, \quad (3.23)$$
$$[E_4, F] = -8(2A_1 - A_2)B_2 + 8B_1F - 4(4c_3 - 3)H + 4(4c_3 - 5)A_1 + 8(2c_3 - 1)A_2. \quad (3.24)$$

We can also present a second algebra in terms of C_1, C_2 and D with coefficients in linear combinations of integrals A_1, A_2, B_1, B_2, F, H as

$$[C_1, C_2] = 8A_2C_1 - 4c_2C_2 - 16c_1D,$$
$$[C_1, D] = 8FC_1 - 8A_2C_2 - 24A_1C_2 + 8HC_2 + 4c_2D,$$
$$[C_2, D] = -8B_2C_1 - 8FC_2 + 8A_2D - 8(2c_3 - 1)C_1. \quad (3.25)$$

To the observation, the relations (3.1) - (3.2) and (3.3)- (3.4), respectively, involving the integrals set $\{A_1, B_1, C_1\}$ and $\{A_2, B_2, C_2\}$, defined by the subalgebras, $Q_1(3)$ and $Q_2(3)$, have a connection to the quadratic algebra $Q(3)$ of 2D superintegrable systems with quadratic integrals of motion [39]. It is stimulating to see the subalgebras $Q_i(3), i = 1, 2$ that are embedded in the symmetry algebra of the 3D nondegenerate superintegrable system (2.3). In the following section, we consider the subalgebras $Q_i(3), i = 1, 2$ to calculate the spectrum of the system (2.3) by using the Daskaloyannis approach and the deformed oscillator algebra realizations in [39, 40].

§4. **The Quadratic Algebra** $Q(3)$

In the above section, we successfully obtain the symmetry algebra structures of the 3D superintegrable system (2.3) for nondegenerate potential. We now have to calculate the energy spectrum of the system based on the symmetry algebra. In order to derive the spectrum algebraically, we demonstrate the existence of a set of subalgebra structures $Q_i(3), i=1,2$, involving three generators and compared them with the quadratic algebra $Q(3)$ presented by Daskaloyannis in the context of 2D superintegrable systems [39]. We recall briefly this algebraic method for two subalgebras $Q_i(3), i=1,2$ which involves three operators $\{\mathcal{A}_i,\mathcal{B}_i,\mathcal{C}_i\}$ for $i=1,2$ and $[\mathcal{A}_i,\mathcal{A}_j]=0$, for all i,j [30]. They are close to form the following quadratic algebras,

$$\begin{aligned}
[\mathcal{A}_i,\mathcal{B}_i] &= \mathcal{C}_i,\\
[\mathcal{A}_i,\mathcal{C}_i] &= \alpha_i\mathcal{A}_i^2+\gamma_i\{\mathcal{A}_i,\mathcal{B}_i\}+\delta_i\mathcal{A}_i+\epsilon_i\mathcal{B}_i+\zeta_i,\\
[\mathcal{B}_i,\mathcal{C}_i] &= a_i\mathcal{A}_i^2-\gamma_i\mathcal{B}_i^2-\alpha_i\{\mathcal{A}_i,\mathcal{B}_i\}+d_i\mathcal{A}_i-\delta_i\mathcal{B}_i+z_i,
\end{aligned}\tag{4.1}$$

where $i=1,2$. The coefficients α_i, γ_i, a_i are constants and d_i, δ_i, ϵ_i, ζ_i, z_i are polynomials of central elements: the Hamiltonian H and the generator $\mathcal{A}_j$ of the j-th subalgebra, which commutes with the generators of the i-th subalgebra. The generators $\{\mathcal{A}_i,\mathcal{B}_i,\mathcal{C}_i\}$ of the subalgebras $Q_i(3), i=1,2$ form a Casimir invariant

$$\begin{aligned}
\mathcal{K}_i &= \mathcal{C}_i^2-\alpha_i\{\mathcal{A}_i^2,\mathcal{B}_i\}-\gamma_i\{\mathcal{A}_i,\mathcal{B}_i^2\}+(\alpha_i\gamma_i-\delta_i)\{\mathcal{A}_i,\mathcal{B}_i\}+(\gamma_i^2-\epsilon_i)\mathcal{B}_i^2\\
&\quad+(\gamma_i\delta_i-2\zeta_i)\mathcal{B}_i+\frac{2a_i}{3}\mathcal{A}_i^3+\left(d_i+\frac{a_i\gamma_i}{3}+\alpha_i^2\right)\mathcal{A}_i^2+\left(\frac{a_i\epsilon_i}{3}+\alpha_i\delta_i+2z_i\right)\mathcal{A}_i.
\end{aligned}\tag{4.2}$$

It is pointed out that this Casimir invariant is possible to reform in terms of only central elements of the corresponding subalgebras. To determine the spectrum of the Hamiltonian operator H, the algebra $Q_i(3)$ (4.1) $i=1,2$ needs to realize in terms of the deformed oscillator algebras [39,40],

$$[\aleph_i,b_i^\dagger]=b_i^\dagger,\quad [\aleph_i,b_i]=-b_i,\quad b_ib_i^\dagger=\phi(\aleph_i+1),\quad b_i^\dagger b_i=\phi(\aleph_i),\tag{4.3}$$

and the structure function ϕ for $\gamma_i\neq 0$ is given by

$$\begin{aligned}
\phi_i(n_i) &= \gamma_i^8(3\alpha_i^2+4a_i\gamma_i)[2(n_i+u_i)-3]^2[2(n_i+u_i)-1]^4[2(n_i+u_i)+1]^2-3072\gamma_i^6\mathcal{K}_i[2(n_i+u_i)-1]^2\\
&\quad-48\gamma_i^6(\alpha_i^2\epsilon_i-\alpha_i\gamma_i\delta_i+a_i\gamma_i\epsilon_i-\gamma_i^2d_i)[2(n_i+u_i)-1]^4[2(n_i+u_i)+1]^2[2(n_i+u_i)-3]\\
&\quad+32\gamma_i^4\left(3\alpha_i^2\epsilon_i^2+4\alpha_i\gamma_i^2\zeta_i-6\alpha_i\gamma_i\delta_i\epsilon_i+2a_i\gamma_i\epsilon_i^2+2\gamma_i^2\delta_i^2-4\gamma_i^2d_i\epsilon_i+8\gamma_i^3z_i\right)\times\\
&\quad[2(n_i+u_i)-1]^2[12(n_i+u_i)^2-12(n_i+u_i)-1]+768(\alpha_i\epsilon_i^2+4\gamma_i^2\zeta_i-2\gamma_i\delta_i\epsilon_i)^2\\
&\quad-256\gamma_i^2[2(n_i+u_i)-1]^2(3\alpha_i^2\epsilon_i^3+4\alpha_i\gamma_i^4\zeta_i+12\alpha_i\gamma_i^2\zeta_i\epsilon_i-9\alpha_i\gamma_i\delta_i\epsilon_i^2+a_i\gamma_i\epsilon_i^3+2\gamma_i^4\delta_i^2\\
&\quad-12\gamma_i^3\delta_i\zeta_i+6\gamma_i^2\delta_i^2\epsilon_i+2\gamma_i^4d_i\epsilon_i-3\gamma_i^2d_i\epsilon_i^2-4\gamma_i^5z_i+12\gamma_i^3z_i\epsilon_i)
\end{aligned}\tag{4.4}$$

and the eigenvalues of the operator $\mathcal{A}_i$,

$$e(\mathcal{A}_i)=\mathcal{A}_i(q_i)=\sqrt{\epsilon_i}(q_i+u_i),\qquad \gamma_i=0,\quad \epsilon_i\neq 0;\tag{4.5}$$

and for the case $\gamma_i = 0$, $\epsilon_i \neq 0$, the structure function is given by

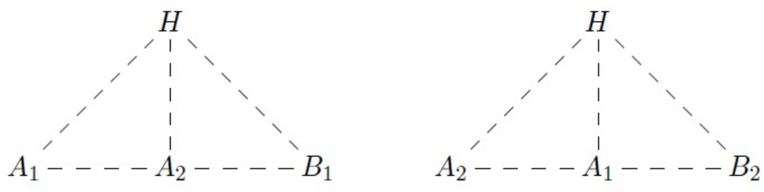
$$\begin{aligned}\Phi_i(n_i) &= \frac{1}{4}\left[-\frac{\mathcal{K}_i}{\epsilon_i}-\frac{z_i}{\sqrt{\epsilon_i}}-\frac{\delta_i}{\sqrt{\epsilon_i}}\frac{\zeta_i}{\epsilon_i}+\left(\frac{\zeta_i}{\epsilon_i}\right)^2\right]\\ &-\frac{1}{12}\left[3d_i-a_i\sqrt{\epsilon_i}-3\alpha_i\frac{\delta_i}{\sqrt{\epsilon_i}}+3\frac{\delta_i^2}{\epsilon_i}-6\frac{z_i}{\sqrt{\epsilon_i}}+6\alpha_i\frac{\zeta_i}{\epsilon_i}-6\frac{\delta_i}{\sqrt{\epsilon_i}}\frac{\zeta_i}{\epsilon_i}\right](n_i+u_i)\\ &+\frac{1}{4}\left[\alpha_i^2+d_i-a_i\sqrt{\epsilon_i}-3\alpha_i\frac{\delta_i}{\sqrt{\epsilon_i}}+\frac{\delta_i^2}{\epsilon_i}+2\alpha_i\frac{\zeta_i}{\epsilon_i}\right](n_i+u_i)^2\\ &-\frac{1}{6}\left[3\alpha_i^2-a_i\sqrt{\epsilon_i}-3\alpha_i\frac{\delta_i}{\sqrt{\epsilon_i}}\right](n_i+u_i)^3+\frac{1}{4}\alpha^2(n_i+u_i)^4 \qquad (4.6)\end{aligned}$$

and the eigenvalues of the operator $\mathcal{A}_i$,

$$e(\mathcal{A}_i) = \mathcal{A}_i(q_i) = \frac{\gamma_i}{2}\left((q_i+u_i)^2-\frac{\epsilon_i}{\gamma_i^2}-\frac{1}{4}\right), \quad \gamma_i \neq 0. \qquad (4.7)$$

§5. The Subalgebras $Q_i(3)$, $i = 1, 2$

The relations (3.1) - (3.2) and (3.3)- (3.4) of the quadratic symmetry algebras formed similar quadratic structure $Q(3)$ (4.1) involving three generators sets $\{A_1, B_1, C_1\}$ and $\{A_2, B_2, C_2\}$. The subalgebras $Q_i(3), i = 1, 2$ can be presented in the following diagrams,



(5.1)

The left figure shows that A_2 and H are central elements and the right figure shows that A_1 and H are central elements of the corresponding subalgebra structures. It is seen as a fact that one integral plays a role as a generator in a subalgebra structure while it plays a role as a central element in another subalgebra structure. In account to obtain the spectrum, we manipulate the subalgebras $Q_i(3)$, $i = 1, 2$ and it is clear that each of these subalgebras has a relationship with the quadratic algebra (4.1) and Casimir operator (4.2) presented in [39] for the 2D superintegrable system. We rewrite the relations (3.1) - (3.2) as the subalgebra structure $Q_1(3)$,

$$\begin{aligned}&[A_1, B_1] = C_1,\\ &[A_1, C_1] = 4c_2A_1 + 16c_1B_1 + 4c_2(A_2 - H), \qquad (5.2)\\ &[B_1, C_1] = 24A_1^2 + 32(A_2 - H)A_1 - 4c_2B_1 + 8H^2 - 16HA_2\\ &\qquad -8c_1(4c_4 - 3) + 8A_2^2,\end{aligned}$$

and the relations (3.3)- (3.4) as the subalgebra structure $Q_2(3)$,

$$\begin{aligned}
&[A_2, B_2] = C_2,\\
&[A_2, C_2] = 8A_2^2 + 8(A_1 - H)A_2 + 16c_1B_2 + 8c_1,\\
&[B_2, C_2] = -8\{A_2, B_2\} - 16(c_3 + c_4 - 1)A_2 - 8(A_1 - H)B_2\\
&\qquad\qquad +8(2c_3 - 1)(H - A_1).
\end{aligned} \tag{5.3}$$

There are Casimir operators K_1 of $Q_1(3)$ and K_2 of $Q_2(3)$ satisfying $[K_1, A_1] = 0 = [K_1, B_1]$ and $[K_2, A_2] = 0 = [K_2, B_2]$, respectively,

$$\begin{aligned}
K_1 = &C_1^2 - 4c_2\{A_1, B_1\} - 16c_1B_1^2 - 8c_2(A_2 - H)B_1 + 16A_1^3\\
&+32(A_2 - H)A_1^2 + [128c_1 + 16H^2 - 32HA_2 - 16c_1(4c_4 - 3) + 16A_2^2]A_1,
\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}
K_2 = &C_2^2 - 8\{A_2^2, B_2\} - 8(A_1 - H)\{A_2, B_2\} - 16c_1B_2^2 - 16c_1B_2\\
&-16(c_3 + c_4 - 5)A_2^2 - 16(2c_2 - 5)(A_1 - H)A_2.
\end{aligned} \tag{5.5}$$

Moreover, the quadratic subalgebras $Q_1(3)$ and $Q_2(3)$ possess corresponding Casimir invariants in terms of only central elements in the following forms, respectively,

$$K_1' = 128c_1H - 128c_1A_2 - 3c_2^2 + 4c_2^2c_4, \tag{5.6}$$

and

$$K_2' = 4(4c_3 - 3)(H - A_1)^2 - 16(2c_1 - 3c_1c_3 - 3c_1c_4 + 4c_1c_3c_4). \tag{5.7}$$

It is seen that the Casimir operator K_1' depends on only central elements H and A_2 for the subalgebra $Q_1(3)$, and the Casimir operator K_2' depends on only central elements H and A_1 of the subalgebra $Q_2(3)$. These two forms of the Casimir invariants will be used to realize these subalgebras in terms of the deformed oscillator algebras (4.3).

§6. Deformed Oscillators Realizations and Energy Spectrum

In order to obtain the energy spectrum of the superintegrable system (2.3), we realize the subalgebra structures $Q_1(3)$ and $Q_2(3)$ in terms of deformed oscillator algebra [39,40] $\{\aleph_i, b_i^\dagger, b_i\}$ of (4.3) with satisfying the real-valued function,

$$\phi(0) = 0, \quad \phi(n_i) > 0, \quad \forall n_i > 0. \tag{6.1}$$

The $\phi(n_i)$ is known as structure function. We first investigate the realization of the quadratic subalgebra structure $Q_1(3)$ (5.2). The realization of $Q_1(3)$ is of the form $A_1 = A_1(\aleph_1)$, $B_1 = b_1(\aleph_1) + b_1^\dagger\rho_1(\aleph_1) + \rho_1(\aleph_1)b_1$, where $A_1(x)$, $b_1(x)$ and $\rho_1(x)$ are functions to lead the following forms [30],

$$A_1(\aleph_1) = \sqrt{16c_1}(\aleph_1 + u_1), \tag{6.2}$$

$$b_1(\aleph_1) = -\frac{c_2}{\sqrt{c_1}}(\aleph_1 + u_1) - \frac{c_2(A_2 - H)}{4c_1}, \tag{6.3}$$

$$\rho_1(\aleph_1) = 1, \tag{6.4}$$

where u_1 is an arbitrary constant to be determined later. The following structure function $\phi(n_1, u_1, H)$ of the subalgebra (5.2) is constructed by using the deformed oscillators (4.3) and the Casimir invariants (5.4) and (5.6),

$$\begin{aligned}\phi_1(n_1, u_1, H) = \frac{1}{1024m_1^5}&\Big[(A_2 - H) - m_1\left(2 + m_4 - 4\left(n_1 + u_1\right)\right)\Big]\\ &\Big[(A_2 - H) + m_1\left(-2 + m_4 + 4\left(n_1 + u_1\right)\right)\Big]\\ &\Big[m_2^2 + 32m_1^3\left(-1 + 2\left(n_1 + u_1\right)\right)\Big],\end{aligned} \tag{6.5}$$

where $m_1^2 = c_1$, $m_2 = c_2$, $m_4^2 = 4c_4 + 1$. To obtain the eigenvalues of the central element A_2 of this subalgebra and the values of parameter u_1 by requiring that the unitary representations (unirreps) to be a finite, we should impose the following constraints on the structure function:

$$\phi(p_1 + 1; u_1, E) = 0; \quad \phi(0; u_1, E) = 0; \quad \phi(x) > 0, \quad \forall x > 0, \tag{6.6}$$

where p_1 is a positive integer. We also replace the eigenvalue E of H in $\phi_1(n_1, u_1, H)$. The constraints guarantee the structure functions are finite $(p_1 + 1)$-dimensional unirreps. We solve the constraints (6.6), which give information of the eigenvalues $e(A_2)$ of A_2 and the values of the constant u_1. Imposing the condition (6.6) to the structure functions (6.5) for unirreps of finite-dimensional $(p_1 + 1)$ and positive values of the structure function, we obtain the solutions with $\varepsilon_1 = +1$, $\varepsilon_2 = \pm 1$, $\varepsilon_3 = \pm 1$,

$$u_1 = \frac{1}{2} - \frac{m_2^2}{64m_1^3}, \quad \text{or} \quad u_1 = \frac{1}{4m_1}\left[E + 2m_1 + \varepsilon_1 m_1 m_4 - e(A_2)\right], \tag{6.7}$$

$$e(A_2) = 4\varepsilon_2 m_1(p_1 + 1) + E + \varepsilon_3 m_1 m_4 + \frac{m_2^2}{16m_1^2}. \tag{6.8}$$

It is remarked that the subalgebra $Q_1(3)$ is the algebra of the superintegrable two-dimensional subsystem, depending on the variables x_1, x_3. The above computation shows that the energy of the subsystem equal to $E - e(A_2)$ and the quadratic algebra depends on $H - A_2$, it is thus clearly understandable the existence of the Casimir invariant (5.6). A similar reason is applicable for the subalgebra $Q_2(3)$ and the Casimir invariant (5.7). We now obtain the eigenvalues of A_1 from the relations (6.2) and (6.7) as follows,

$$e(A_1) = 2m_1(2n_1 + 1) - \frac{m_2^2}{16m_1^2}, \quad \text{or} \quad e(A_1) = 2m_1(2n_1 + 1) + E + \varepsilon_1 m_1 m_4 - e(A_2). \tag{6.9}$$

We now turn to the quadratic subalgebra structure $Q_2(3)$ (5.3). Similar to $Q_1(3)$, the realizations of $Q_2(3)$ present the functions

$$A_2(\aleph_2) = \sqrt{16c_1}(\aleph_2 + u_2), \tag{6.10}$$

$$b_2(\aleph_2) = -8(\aleph_2 + u_2)^2 - \frac{2(A_1 - H)}{\sqrt{c_1}}(\aleph_2 + u_2) - \frac{1}{2}, \tag{6.11}$$

$$\rho_2(\aleph_2) = 1 \tag{6.12}$$

and the structure function

$$\begin{aligned}\phi(n_2, u_2, H) = &\frac{1}{256m_1^4}[-2 - m_3 + 4(n_2 + u_2)][-2 + m_3 + 4(n_2 + u_2)]\\ &[A_1 m_1 - m_1 H - m_1^2 m_4 + m_1^2(-2 + 4(n_2 + u_2)]\\ &[A_1 m_1 - m_1 H + m_1^2 m_4 + m_1^2(-2 + 4(n_2 + u_2)],\end{aligned} \tag{6.13}$$

where $m_3^2 = 4c_3 + 1$. We now impose the constraints (6.6) on the structure function (6.13) for unirreps of finite-dimensional $(p_2 + 1)$ and positive values of the structure function, giving the following solutions,

$$u_2 = \frac{1}{2} + \frac{\varepsilon_1 m_3}{4}, \quad \text{or} \quad u_2 = \frac{1}{4m_1}\left[E + 2m_1 + \varepsilon_1 m_1 m_4 - e(A_1)\right], \tag{6.14}$$

$$e(A_1) = 4\varepsilon_1 m_1(p_2 + 1) + E + \varepsilon_2 m_1 m_3 + \varepsilon_3 m_1 m_4, \tag{6.15}$$

where $\varepsilon_1 = +1$, $\varepsilon_2 = \pm 1$ and $\varepsilon_3 = \pm 1$. Similar, from the relations (6.10) and (6.14), we obtain the eigenvalues $e(A_2)$ of A_2 as

$$e(A_2) = 2m_1(2n_2 + 1) + \varepsilon_1 m_1 m_3, \quad e(A_2) = 2m_1(2n_2 + 1) + E + \varepsilon_1 m_1 m_4 - e(A_1). \tag{6.16}$$

The energy eigenvalues of the superintegrable system (2.3) are calculated using the relations (6.9), (6.15) and (6.8), (6.16), and choosing the suitable sign of $\varepsilon_i, i = 1, 2, 3$ for positive energy levels,

$$E = 4(p_2 + 1)m_1 + 2(2n_1 + 1)m_1 + m_1 m_3 + m_1 m_4 - \frac{m_2^2}{16m_1^2}, \tag{6.17}$$

$$E = 4(p_1 + 1)m_1 + 2(2n_2 + 1)m_1 + m_1 m_3 + m_1 m_4 - \frac{m_2^2}{16m_1^2}. \tag{6.18}$$

It is a fact that the elimination of the energy E from the above relations (6.17) and (6.18) leads to a relation,

$$p_1 - p_2 = n_1 - n_2, \tag{6.19}$$

which is valid, because the two deformed oscillators are treated as independent ones,

$$n_1 = 0, 1, 2, \cdots, p_1, \quad \text{and} \quad n_2 = 0, 1, 2, \cdots, p_2. \tag{6.20}$$

The mean value of the relations (6.17) and (6.18) reduce to the energy eigenvalues of the superintegrable Hamiltonian system (2.3),

$$E = 2(p_1 + p_2 + 2)m_1 + 2(n_1 + n_2 + 1)m_1 + m_1 m_3 + m_1 m_4 - \frac{m_2^2}{16 m_1^2}. \tag{6.21}$$

It is a fact that the quadratic subalgebra structures of the symmetry algebra provide us the energy spectrum for the 3D nondegenerate potential of the maximally quantum superintegrable system (2.3) purely algebraic computations. It is shown that the energy spectrum of the system in the algebraic investigation depends only on differential operators and their corresponding operator algebra in symmetry forms without knowledge of wave functions and calculus.

§7. Separation of Variables

We now demonstrate analytic calculations of the 3D superintegrable system (2.3) via separation of variables in the cylindrical polar and paraboloidal coordinates. The results will be compared with those obtained from algebraic derivations.

7.1 Cylindrical Polar Coordinates

The cylindrical polar coordinates are given by

$$x_1 = z, \quad x_2 = \rho \sin^2\theta, \quad x_3 = \rho\cos^2\theta, \tag{7.1}$$

where $\rho > 0$, $-\infty < z < \infty$ and $\theta \in [0, 2\pi]$ [37]. The Schrodinger equation $H\psi = E\psi$ of the system (2.3) in the coordinates can be expressed as

$$\begin{aligned}\Bigg[-\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\right) + c_1(4z^2+\rho^2) + c_2 z + \frac{c_3}{\rho^2\sin^2\theta} \\ + \frac{c_4}{\rho^2\cos^2\theta} - E\Bigg]\psi(\rho, z, \theta) = 0. \end{aligned} \tag{7.2}$$

The separation of variables of (7.2)

$$\psi(\rho, z, \theta) = R(\rho, z) Y(\theta) \tag{7.3}$$

gives rise to the angular and radial parts with separation constant A,

$$\left[-\frac{\partial^2}{\partial\theta^2} + \frac{c_3}{\sin^2\theta} + \frac{c_4}{\cos^2\theta} - A\right] Y(\theta) = 0, \tag{7.4}$$

$$\left[-\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right) + c_1(4z^2+\rho^2) + c_2 z - E + \frac{A}{\rho^2}\right] R(\rho, z) = 0. \tag{7.5}$$

We now take the equation (7.5) to separate the variables

$$R(\rho, z) = G(\rho) F(z) \tag{7.6}$$

and obtain,

$$\left[-\frac{\partial^2}{\partial z^2}+c_1 4z^2+c_2 z-E-A_1\right]F(z)=0, \tag{7.7}$$

$$\left[-\frac{\partial^2}{\partial \rho^2}-\frac{1}{\rho}\frac{\partial}{\partial\rho}+c_1\rho^2+\frac{A}{\rho^2}+A_1\right]G(\rho)=0, \tag{7.8}$$

where A_1 is a separation constant.

We now turn to (7.4), which converts to, by setting $\upsilon=\sin^2\theta$ and $g(\upsilon)=\upsilon^{\frac{1}{4}(1+2\gamma_3)}(1-\upsilon)^{\frac{1}{4}(1+2\gamma_4)}g_1(\upsilon)$,

$$\begin{aligned}&\upsilon(1-\upsilon){g_1}''(\upsilon)+\left[(1\pm\gamma_3)-(1+1\pm\gamma_3\pm\gamma_4)\,\upsilon\right]{g_1}'(\upsilon)-\\&\left[\left(\frac{1}{4}\pm\frac{\gamma_3}{2}+\frac{1}{4}\pm\frac{\gamma_4}{2}\right)^2-\frac{A}{4}\right]g_1(\upsilon)=0,\end{aligned} \tag{7.9}$$

where $\gamma_3=\pm\frac{1}{2}\sqrt{1+4c_3}$ and $\gamma_4=\pm\frac{1}{2}\sqrt{1+4c_4}$. By comparing with the Jacobi differential equation [41]

$$z\,(1-z)\,y''+\left[\gamma-(\alpha+1)\,z\right]y'+n\,(\alpha+n)\,y=0, \tag{7.10}$$

we find the separation constant

$$A=\left(2n\pm\gamma_3\pm\gamma_4+1\right)^2, \tag{7.11}$$

where n is positive integers. Hence we have the solutions of (7.9) as follows

$$\begin{aligned}Y\,(\theta)\quad&=(\sin^2\theta)^{\frac{1}{4}\pm\frac{\gamma_3}{2}}(1-\sin^2\theta)^{\frac{1}{4}\pm\frac{\gamma_4}{2}}\\&\times\Big[C_{12}F_1\left(-n,n+1\pm\gamma_3\pm\gamma_4,1\pm\gamma_3,\sin^2\theta\right)-(-1)^{-(1\pm\gamma_3)}\left(\sin^2\theta\right)^{\pm\gamma_3}\\&\times C_{22}F_1\left[(\pm\gamma_3-n),1+n\pm 2\gamma_3\pm\gamma_4,(1\pm\gamma_3),\sin^2\theta\right]\Big].\end{aligned} \tag{7.12}$$

Let us now turn to the equation (7.8). Putting the separation constant $A=(2n\pm\gamma_3\pm\gamma_4+1)^2$ into (7.8), we have

$$\left[-\frac{\partial^2}{\partial \rho^2}-\frac{1}{\rho}\frac{\partial}{\partial\rho}+c_1\rho^2+\frac{1}{\rho^2}\left(2n\pm\gamma_3\pm\gamma_4+1\right)^2+A_1\right]G(\rho)=0. \tag{7.13}$$

By setting $\xi=\varepsilon\rho^2$, $G(\xi)=\xi^\alpha G_1(\xi)$ and $G_1(\xi)=e^{-\frac{1}{2}\xi}G_2(\xi)$, (7.13) can be converted to the form

$$\xi{G_2}''(\xi)+\left[(2\alpha+1)-\xi\right]G_2'(\xi)-\left[\frac{1}{2}\left(2\alpha+1\right)+\frac{A_1}{4\varepsilon}\right]G_2\left(\xi\right)=0, \tag{7.14}$$

where $\alpha=\frac{1}{2}\left(1\pm\gamma_3\pm\gamma_4+2n\right)$ and $\varepsilon^2=c_1$. Comparing (7.14) with the confluent hypergeometric differential equation [41],

$$zf''(z)+(c-z)\,f'(z)-af(z)=0, \tag{7.15}$$

we obtain the separation constant,

$$A_1 = 4\varepsilon a - 2\varepsilon(2n \pm \gamma_3 \pm \gamma_4 + 1) \tag{7.16}$$

and the solution of (7.13) can be written as

$$\begin{aligned} G(\rho) = & \left[\left(\varepsilon\rho^2\right)^{\frac{1}{2}(1\pm\gamma_3\pm\gamma_4+2n)} e^{-\frac{1}{2}\varepsilon\rho^2} C_1 \phi\left(a + \frac{1}{2}; (2\pm\gamma_3\pm\gamma_4+2n); \varepsilon\rho^2\right)\right] \\ & + \left[C_2 \left(\varepsilon\rho^2\right)^{1-(2\pm\gamma_3\pm\gamma_4+2n)} \phi\left(a+1; (2\pm\gamma_3\pm\gamma_4+2n); \varepsilon\rho^2\right)\right]. \end{aligned} \tag{7.17}$$

Putting (7.16) into (7.7), we can be reformed as follows

$$\left[\frac{\partial^2}{\partial z^2} - 4c_1 z^2 - c_2 z + E + 4\varepsilon a - 2\varepsilon\left(2n \pm \gamma_3 \pm \gamma_4 + 1\right)\right] F(z) = 0, \tag{7.18}$$

which is one linear differential equation. Such linear differential equation

$$f^{''}(z) + \left(-q^2 z^2 - 2qsz + t\right) f(z) = 0, \tag{7.19}$$

solved in [42] with the condition $q^{-1}\left(s^2 + t\right)$ is an odd integer. Let us consider

$$q^{-1}\left(s^2 + t\right) = 2\tau + 1, \quad \tau \text{ is an integer.} \tag{7.20}$$

The elementary solution of (7.19) is given [42] as

$$F_1(z) = h(z).e^{-\frac{q}{2}\left(\left(z+\left(\frac{s}{q}\right)^2\right)\right)}, \tag{7.21}$$

$$F_2(z) = F_1(z). \int h(\xi)^{-2} e^{q\left(\xi+\frac{s}{q}\right)^2} d\xi, \tag{7.22}$$

where

$$h(z) = \left(z + \frac{s}{q}\right)^j + \sum_{1\leqslant l \leqslant \frac{j}{2}} \frac{j!}{l!(j-2l)! z^{2l} (-q)^l} \left(z + \frac{s}{q}\right)^{j-2l}, \quad j = 0, 1, 2, \cdots. \tag{7.23}$$

Hence we obtain the solution of (7.18) as follows

$$F_1(z) = h(z).e^{-\sqrt{c_1}\left(\left(z+\frac{c_2}{4c_1}\right)^2\right)}, \tag{7.24}$$

$$F_2(z) = F_1(z). \int h(\xi)^{-2} e^{2\sqrt{c_1}\left(\xi+\frac{c_2}{4c_1}\right)^2} d\xi, \tag{7.25}$$

$$h(z) = \left(z + \frac{c_2}{4c_1}\right)^j + \sum_{1\leqslant i \leqslant \frac{j}{2}} \frac{j!}{i!(j-2i)! z^{2i} \left(-2\sqrt{c_1}\right)^i} \left(z + \frac{c_2}{4c_1}\right)^{j-2i}. \tag{7.26}$$

Comparing (7.18) with (7.19), (7.20) and substituting $c_1 = \gamma_1^2$, $c_2 = \gamma_2$, we can obtain the eigenvalues of the nondegenerate system (2.3) in terms of quantum numbers involving the four parameters as follows,

$$E = 2(2\tau + 1)\gamma_1 + 2(2n - 2a \pm \gamma_3 \pm \gamma_4 + 1)\gamma_1 - \frac{\gamma_2^2}{16\gamma_1^2}. \tag{7.27}$$

Making identification $2\tau = p_1 + p_2 + 1$, $2(n - a) = n_1 + n_1$, $2\gamma_3 = m_3$ and $2\gamma_4 = m_4$, the energy spectrum (7.27) becomes (6.21).

7.2 Paraboloidal Coordinates

The paraboloidal coordinates are considered by

$$x_1 = \frac{1}{2}(u^2 - v^2), \qquad x_2 = uv\sin\phi, \qquad x_3 = uv\cos\phi, \tag{7.28}$$

where $0 \leqslant \phi < 2\pi$, $u \geqslant 0$ and $v \geqslant 0$. Now the Schrödinger eigenvalue equation $H\psi = E\psi$ of the system (**??**) in these coordinates leads to the following structure,

$$\Bigg[- \left[\frac{\partial^2}{\partial\phi^2} + \frac{u^2v^2}{u^2+v^2}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + \frac{u^2v^2}{u^2+v^2}\left(\frac{1}{u}\frac{\partial}{\partial u} + \frac{1}{v}\frac{\partial}{\partial v}\right)\right] + \frac{c_2}{2}u^2v^2\left(u^2 - v^2\right)$$
$$+ c_1u^2v^2\left(u^4 + v^4 - u^2v^2\right) + \frac{c_3}{\sin^2\phi} + \frac{c_4}{\cos^2\phi}\Bigg]\psi(u,v,\phi) = 0. \tag{7.29}$$

To separate the Schrödinger equation (7.29), the ansatz

$$\psi(u,v,\phi) = R(u,v)\,Y(\phi) \tag{7.30}$$

gives rise to the following differential equations,

$$\left[-\frac{\partial^2}{\partial\phi^2} + \frac{c_3}{\sin^2\phi} + \frac{c_4}{\cos^2\phi} - A\right] Y(\phi) = 0, \tag{7.31}$$

$$\Bigg[-\frac{u^2v^2}{u^2+v^2}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) - \frac{u^2v^2}{u^2+v^2}\left(\frac{1}{u}\frac{\partial}{\partial u} + \frac{1}{v}\frac{\partial}{\partial v}\right)$$
$$+ c_1u^2v^2\left(u^4 + v^4 - u^2v^2\right) + \frac{c_2}{2}u^2v^2\left(u^2 - v^2\right) - Eu^2v^2 + A\Bigg] R(u,v) = 0, \tag{7.32}$$

where A is a separation constant. Again taking the ansatz

$$R(u,v) = R_1(u)R_2(v) \tag{7.33}$$

for the separation of (7.32), it leads to

$$\left[-\frac{\partial^2}{\partial u^2} - \frac{1}{u}\frac{\partial}{\partial u} + c_1u^6 + \frac{c_2}{2}u^4 - Eu^2 + \frac{A}{u^2} - A_1\right] R_1(u) = 0, \tag{7.34}$$

$$\left[-\frac{\partial^2}{\partial v^2} - \frac{1}{v}\frac{\partial}{\partial v} + c_1v^6 + \frac{c_2}{2}v^4 - Ev^2 + \frac{A}{v^2} + A_1\right] R_2(v) = 0. \tag{7.35}$$

We now first change to (7.31) by setting $w = \sin^2\phi$ and $g(w) = w^{\alpha_1}(1-w)^{\alpha_2}g_1(w)$, which can be reduced to the following form

$$w(1-w)g_1^{''}+\left[(1\pm\gamma_3)-(2\pm\gamma_3\pm\gamma_4)\,w\right]g_1^{'}-\left[\left(\frac{1}{2}\pm\frac{\gamma_3}{2}\pm\frac{\gamma_4}{2}\right)^2-\frac{A}{4}\right]g_1(w)=0, \quad (7.36)$$

where $\gamma_3=\pm\frac{1}{2}\sqrt{1+4c_3}$ and $\gamma_4=\pm\frac{1}{2}\sqrt{1+4c_4}$, $\alpha_1=\frac{1}{4}\pm\frac{\gamma_3}{2}$ and $\alpha_2=\frac{1}{4}\pm\frac{\gamma_4}{2}$. Comparing (7.36) in terms of the Jacobi differential equation [41],

$$\mathcal{X}\left(1-\mathcal{X}\right)\mathcal{Y}^{''}+\left[\gamma-\left(\alpha+1\right)\mathcal{X}\right]\mathcal{Y}^{'}+\eta\left(\alpha+\eta\right)\mathcal{Y}=0 \quad (7.37)$$

and its solution,

$$\begin{aligned} y &= C_{1\,2}F_1\left(-\eta,\eta+\alpha,\gamma,\mathcal{X}\right) \\ &\quad -\left(-1\right)\mathcal{X}^{1-\gamma}C_{2\,2}F_1\left(1-\eta-\gamma,1+\eta+\alpha-\gamma,2-\gamma,\mathcal{X}\right), \end{aligned} \quad (7.38)$$

we obtain the separation constant

$$A=\left(2\eta\pm\gamma_3\pm\gamma_4+1\right)^2, \quad (7.39)$$

and the solution of (7.31),

$$\begin{aligned} &Y\left(\phi\right)=\left(\sin^2\phi\right)^{\frac{1}{4}\pm\frac{\gamma_3}{2}}\left(1-\sin^2\phi\right)^{\frac{1}{4}\pm\frac{\gamma_4}{2}}\Big[C_{1\,2}F_1\left(-\eta,\eta+1\pm\gamma_3\pm\gamma_4,1\pm\gamma_3,\sin^2\phi\right) \\ &-\left(-1\right)^{-(1\pm\gamma_3)}\left(\sin^2\phi\right)^{\pm\gamma_3}C_{2\,2}F_1\left[(\pm\gamma_3-\eta),1+\eta\pm2\gamma_3\pm\gamma_4,(1\pm\gamma_3),\sin^2\phi\right]\Big]. \end{aligned} \quad (7.40)$$

To solve the differential equations (7.34) and (7.35), let us set $z_1=u^2$ in (7.34) and $z_2=v^2$ in (7.35), the the couple equations become

$$\left[z_i^2\frac{\partial^2}{\partial z_i^2}+z_i\frac{\partial}{\partial z_i}+\left(-\frac{c_1}{4}z_i^4-\frac{c_2}{8}z_i^3+\frac{E}{4}z_i^2-\frac{A}{4}+\frac{A_i}{4}z_i\right)\right]R_i\left(z_i\right)=0, \quad (7.41)$$

where $A_1=-A_2$ and $i=1,2$. The equation (7.41) can be transformed into a Bi-Confluent Heun differential equation [43,44] of type

$$\mathcal{X}F^{''}+\left(1+p-q\mathcal{X}-2\mathcal{X}^2\right)F^{'}+\left[\left(r-p-2\right)\mathcal{X}-\frac{1}{2}\left(s+q\left(1+p\right)\right)\right]F\left(\mathcal{X}\right)=0, \quad (7.42)$$

which has a solution in terms of Hermite functions,

$$F=\sum_{n=0}^{\infty}c_nH_{n+1+p+\frac{1}{2}(r-p-2)}\left(\mathcal{X}+\frac{q}{2}\right), \quad (7.43)$$

where c_n satisfies the three terms recurrence formulas,

$$c_nL_n+c_{n-1}Q_{n-1}+c_{n-2}P_{n-2}=0 \quad (7.44)$$

with the relations

$$L_n = 2n\left(p + n + \frac{(r-p-2)}{2} + 1\right), \qquad P_n = p + n + 1,$$
$$Q_n = -\frac{1}{2}\left(s + q\left(p+1\right)\right) + q\left(p+n+1\right). \tag{7.45}$$

By setting $R_i\left(z_i\right) = z_i^{\rho}\, e^{az_i + \frac{b}{2}z_i^2}\, f_i\left(y\right)$ and $z_i = ky_i$ into (7.41), it leads to

$$y_i^2 f_i^{''}\left(y_i\right) + y_i\left(1 + 2\rho + 2aky_i + 2bk^2y_i^2\right) f_i^{'}\left(y_i\right) + \left[\left(\rho^2 - \frac{A}{4}\right) + \left(2a\rho + a + \frac{A_1}{4}\right)ky_i\right.$$
$$\left. + \left(2b\rho + 2b + a^2 + \frac{E}{4}\right)k^2y_i^2 + \left(2ab - \frac{c_2}{8}\right)k^3y_i^3 + \left(b^2 - \frac{c_1}{4}\right)k^4y_i^4\right] f_i\left(y_i\right) = 0. \tag{7.46}$$

To compare (7.46) and (7.42) with the suitable choice of signs, we have the following conditions,

$$\rho = \sqrt{\frac{A}{4}}, \qquad b = -\frac{\sqrt{c_1}}{2}, \qquad a = -\frac{c_2}{8\sqrt{c_1}}, \qquad k^2 = \frac{2}{\sqrt{c_1}}, \tag{7.47}$$

Using the above conditions, we can rewrite (7.46) as follows

$$y_i f_i^{''}\left(y_i\right) + \left(1 + 2\sqrt{\frac{A}{4}} - \frac{c_2}{2c_1}y_i - 2y_i^2\right) f_i^{'}\left(y_i\right) + \left[\left(-\frac{c_2\,A}{16\sqrt{c_1}} - \frac{c_2}{8\sqrt{c_1}} + \frac{A_1}{4}\right)\left(\sqrt{\frac{2}{\sqrt{c_1}}}\right)\right.$$
$$\left. + \left(-\sqrt{\frac{c_1\,A}{4}} - \sqrt{c_1} + \frac{c_2^2}{64c_1} + \frac{E}{4}\right)\left(\frac{2}{\sqrt{c_1}}\right)y_i\right] f_i\left(y_i\right) = 0. \tag{7.48}$$

Again by comparing (7.48) and (7.42), we obtain

$$r - p - 2 = \left(-\sqrt{\frac{c_1\,A}{4}} - \sqrt{c_1} + \frac{c_2^2}{64c_1} + \frac{E}{4}\right)\left(\frac{2}{\sqrt{c_1}}\right). \tag{7.49}$$

One can shown that the solution of a Bi-Confluent Heun equation [43,44] is the n degree polynomial, then it allows the condition $r - p - 2 = 2\mu$, μ is an integer. Using this condition, the solution of (7.34) is given as

$$f_i\left(y_i\right) = y_i^{\sqrt{A}}\, e^{\left(\frac{c_2}{8\sqrt{c_1}}y_i^2 + \frac{\sqrt{c_1}}{4}y_i^4\right)} \sum_{n=0}^{\infty} c_n H_{n+1+p+\frac{1}{2}(r-p-2)}\left(y_i^2 + \frac{q}{2}\right), \tag{7.50}$$

where

$$p = \sqrt{A}, \quad r = \frac{c_2^2}{32c_1^{\frac{3}{2}}} + \frac{E}{2\sqrt{c_1}}, \quad q = \frac{c_2}{2c_1}, \quad A = \left(2\eta \pm \gamma_3 \pm \gamma_4 + 1\right)^2. \tag{7.51}$$

Then we can present the spectrum explicit relation,

$$E = \sqrt{c_1}\left(4 + 4\mu\right) + 2\sqrt{c_1\,A} - \frac{c_2^2}{16c_1}. \tag{7.52}$$

Substituting $c_1 = \gamma_1^2$, $c_2 = \gamma_2$ and $A = \left(2\eta \pm \gamma_3 \pm \gamma_4 + 1\right)^2$ into (7.52), we have the required eigenvalue of the system (2.3),

$$E = 4(\mu+1)\gamma_1 + 2(2\eta \pm \gamma_3 \pm \gamma_4 + 1)\gamma_1 - \frac{\gamma_2^2}{16\gamma_1^2}. \tag{7.53}$$

Making identification $2\mu = p_1 + p_2 + 1$, and $2\eta = n_1 + n_1$, the energy spectrum (7.53) becomes (6.21).

§8. Conclusions

We constructed the quadratic full symmetry algebra for the 3D nondegenerate quantum superintegrable system generated by six linearly independent integrals of motion including the Hamiltonian. The symmetry algebra contains quadratic subalgebra structures generated by three generators with structure constants connected to the quadratic algebra of the two-dimensional quantum superintegrable system [39]. The algebraic calculations of the symmetry algebra to the quantum superintegrable system enable us to obtain the energy spectrum. We have presented corresponding Casimir invariants and derived the structure functions of the quadratic subalgebras of the symmetry algebra in the realizations of deformed oscillators. The finite-dimensional unirreps of these structure functions yield the energy spectrum of the model algebraically. We also showed that the system is multiseparable in cylindrical polar and paraboloidal coordinates. We solved the Schrödinger equation of the system and expressed the wave functions in terms of special functions, and obtained the physical spectrum. The results are compared with those spectrum obtained from the algebraic computation.

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Common Fixed Point Theorems on S-Metric Spaces Via C-Class Functions

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Abstract: In this work, we prove some common fixed point theorems on S-metric spaces via C-class functions and give some consequences of the main result. We also give some examples in support of the results. The results obtained in this article generalize, extend and improve several results from the existing literature regarding S-metric spaces.

Key Words: Common fixed point, S-metric space, C-class functions.

AMS(2010): 47H10, 54H25.

§1. Introduction

The fixed point theory one of the most important research fields in nonlinear analysis. In the last decades, many number of authors have published papers and battened continuously. The application potential is the main cause for this involvement. Fixed point theory has an application in many areas such as chemistry, physics, biology, computer science and many branches of mathematics. The Banach contraction mapping principle ([3]) or the Banach fixed point theorem is the most celebrated and pioneer result in a complete metric space. The famous Banach contraction mapping principle states that every self mapping $\mathcal{Q}$ defined on a complete metric space (X, d) satisfying the condition:

$$d(\mathcal{Q}(x), \mathcal{Q}(y)) \leq r\, d(x, y) \tag{1.1}$$

for all $x, y \in X$, where $r \in (0, 1)$ is a constant, has a unique fixed point and for every $x_0 \in X$ a sequence $\{\mathcal{Q}^n x_0\}_{n\geq 1}$ is convergent to the fixed point.

Most of the works after this were basically generalizations of the work of Banach. These generalizations include more general metric spaces, or more general contractions etc. One of the generalizations of the metric space is the S-metric space.

In 2012, Sedghi et al. [28] introduced the concept of a S-metric space which is different from other spaces and proved fixed point theorems in such spaces. They also give some examples of a S-metric space which shows that the S-metric space is different from other spaces. They built up some topological properties in such spaces and proved some fixed point theorems in

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the framework of S-metric spaces. After this grateful beginning work of Sedghi et al. [28] many authors attracted to study the problems of the fixed point, common fixed point, coupled fixed point and common coupled fixed point by using various contractive conditions for mappings (see, for examples, [5, 6, 8, 13, 18, 29, 30, 31]).

Recently, a large number of authors have published many papers on S-metric spaces in different directions (see, e.g.,g.,[9, 10, 11, 12, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27, 32, 33] and many others).

In 2014, Ansari [1] introduced the notion of C-class function that is pivotal result in fixed point theory.

In this work, we prove some common fixed point theorems on S-metric spaces via C-class functions and give some consequences of the main result. We also give some examples to demonstrate the validity of the result. Our results generalize, extend and improve several results from the existing literature.

§2. Preliminaries

In this section, we recall some basic definitions, lemmas and auxiliary results to prove our main results.

Definition 2.1([28]) *Let X be a nonempty set and let $S\colon X^3 \to [0,\infty)$ be a function satisfying the following conditions for all u, v, w, $t \in X$ hold with*

($S1$) $S(u,v,w) = 0$ *if and only if* $u = v = w$;

($S2$) $S(u,v,w) \le S(u,u,t) + S(v,v,t) + S(w,w,t)$.

Then, the function S is called an S-metric on X and the pair (X,S) is called an S-metric space or simply SMS.

Example 2.2([28]) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(u,v,w) = \|v+w-2u\| + \|v-w\|$ is an S-metric on X.

Example 2.3([28]) Let X be a nonempty set and d be an ordinary metric on X. Then $S(u,v,w) = d(u,w) + d(v,w)$ for all $u,v,w \in X$ is an S-metric on X.

Example 2.4([28]) Let $X = \mathbb{R}$ be the real line. Then $S(u,v,w) = |u-w| + |v-w|$ for all $u,v,w \in \mathbb{R}$ is an S-metric on X. This S-metric on X is called the usual S-metric on X.

Definition 2.5 *Let (X,S) be an S-metric space. For $\varepsilon > 0$ and $u \in X$ we define respectively the open ball $\mathcal{B}_S(u,\varepsilon)$ and closed ball $\mathcal{B}_S[u,\varepsilon]$ with center u and radius ε as follows:*

$$\mathcal{B}_S(u,\varepsilon) = \{v \in X : S(v,v,u) < \varepsilon\},$$

$$\mathcal{B}_S[u,\varepsilon] = \{v \in X : S(v,v,u) \le \varepsilon\}.$$

Example 2.6([29]) Let $X = \mathbb{R}$. Denote $S(u,v,w) = |v+w-2u| + |v-w|$ for all $u,v,w \in \mathbb{R}$.

Then

$$\begin{aligned} \mathcal{B}_S(1,2) &= \{v \in \mathbb{R} : S(v,v,1) < 2\} = \{v \in \mathbb{R} : |v-1| < 1\} \\ &= \{v \in \mathbb{R} : 0 < v < 2\} = (0,2), \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}_S[2,4] &= \{v \in \mathbb{R} : S(v,v,2) \leq 4\} = \{v \in \mathbb{R} : |v-2| \leq 2\} \\ &= \{v \in \mathbb{R} : 0 \leq v \leq 4\} = [0,4]. \end{aligned}$$

Definition 2.7([28],[29]) *Let (X,S) be an S-metric space and $A \subset X$.*

(Υ_1) *The subset A is said to be an open subset of X, if for every $x \in A$ there exists $c > 0$ such that $\mathcal{B}_S(x,c) \subset A$.*

(Υ_2) *A sequence $\{r_n\}$ in X converges to $r \in X$ if $S(r_n, r_n, r) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(r_n, r_n, r) < \varepsilon$. We denote this by $\lim_{n\to\infty} r_n = r$ or $r_n \to r$ as $n \to \infty$.*

(Υ_3) *A sequence $\{r_n\}$ in X is called a Cauchy sequence if $S(r_n, r_n, r_m) \to 0$ as $n, m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $S(r_n, r_n, r_m) < \varepsilon$.*

(Υ_4) *The S-metric space (X,S) is called complete if every Cauchy sequence in X is convergent.*

(Υ_5) *Let τ be the set of all $A \subset X$ having the property that for every $x \in A$, A contains an open ball centered in x. Then τ is a topology on X (induced by the S-metric space).*

(Υ_6) *A nonempty subset A of X is S-closed if closure of A is equal to A.*

Definition 2.8 *Let X be a non-empty set and let $A, B \colon X \to X$ be two self mappings of X. Then a point $u \in X$ is called a (Ω_1) fixed point of operator A if $A(u) = u$ and a (Ω_2) common fixed point of A and B if $A(u) = B(u) = u$.*

Definition 2.9([28]) *Let (X,S) be an S-metric space. A mapping $\mathcal{A} \colon X \to X$ is said to be a contraction if there exists a constant $0 \leq k < 1$ such that*

$$S(\mathcal{A}u, \mathcal{A}v, \mathcal{A}w) \leq k\, S(u,v,w) \tag{2.1}$$

for all $u, v, w \in X$.

Remark 2.10([28]) *If the S-metric space (X,S) is complete and $\mathcal{A} \colon X \to X$ is a contraction mapping, then $\mathcal{A}$ has a unique fixed point in X.*

Definition 2.11([28]) *Let (X,S) and (X', S') be two S-metric spaces. A function $R \colon X \to X'$ is said to be continuous at a point $x_0 \in X$ if for every sequence $\{r_n\}$ in X with $S(r_n, r_n, x_0) \to 0$, $S'(R(r_n), R(r_n), R(x_0)) \to 0$ as $n \to \infty$. We say that R is continuous on X if R is continuous at every point $x_0 \in X$.*

Definition 2.12([1]) *A mapping $F \colon [0,\infty) \times [0,\infty) \to R$ is called a C-class function if it is continuous and satisfies the following axioms:*

(*i*) $F(s,t) \leq s$;
(*ii*) $F(s,t) = s$ *implies that either* $s = 0$ *or* $t = 0$, *for all* $s, t \in [0, \infty)$.

Note that for some F, we have that $F(0,0) = 0$. The letter $\mathcal{C}$ denotes the set of all C-class functions. The following example shows that $\mathcal{C}$ is nonempty.

Example 2.13([1]) Each of the functions $F\colon [0,\infty) \times [0,\infty) \to R$ defined below are elements of $\mathcal{C}$.

(*i*) $F(s,t) = s - t$;
(*ii*) $F(s,t) = m\,s$, $0 < m < 1$;
(*iii*) $F(s,t) = \frac{s}{(1+t)^r}$, $r \in (0,\infty)$;
(*iv*) $F(s,t) = \frac{log(t+a^s)}{1+t}$, $a > 1$;
(*v*) $F(s,t) = \frac{ln(1+a^s)}{2}$, $a > e$;
(*vi*) $F(s,t) = (s+l)^{(1/(1+t)^r)} - l$, $l > 1$, $r \in (0,\infty)$;
(*vii*) $F(s,t) = s\,log_{t+a}\,a$, $a > 1$;
(*viii*) $F(s,t) = s - \big(\frac{1+s}{2+s}\big)\big(\frac{t}{1+t}\big)$;
(*ix*) $F(s,t) = s\beta(s)$, where $\beta\colon [0,\infty) \to [0,\infty)$ and is continuous;
(*x*) $F(s,t) = s - \big(\frac{t}{k+t}\big)$;
(*xi*) $F(s,t) = \frac{s}{(1+s)^r}$, $r \in (0,\infty)$.

Remark 2.14 *The items* (*i*), (*ii*) *and* (*ix*) *in Example* 2.13 *are pivotal results in fixed point theory ([1]). Also see [2] and [7].*

Definition 2.15([1]) *A function* $\psi\colon [0,\infty) \to [0,\infty)$ *is called an altering distance function if the following properties are satisfied:*

(ψ_1) ψ *is non-decreasing and continuous function;*
(ψ_2) $\psi(t) = 0$ *if and only if* $t = 0$.

Remark 2.16 *We denote* Ψ *the class of all altering distance functions.*

Definition 2.17([1]) *A function* $\varphi\colon [0,\infty) \to [0,\infty)$ *is said to be an ultra altering distance function, if it is continuous, non-decreasing such that* $\varphi(t) > 0$ *for* $t > 0$.

We denote by Φ_u the class of all ultra altering distance functions.

Lemma 2.18([28], Lemma 2.5) *Let* (X,S) *be an* S*-metric space. Then,* $S(u,u,v) = S(v,v,u)$ *for all* $u, v \in X$.

Lemma 2.19([28], Lemma 2.12) *Let* (X,S) *be an* S*-metric space. If* $r_n \to r$ *and* $p_n \to p$ *as* $n \to \infty$ *then* $S(r_n, r_n, p_n) \to S(r,r,p)$ *as* $n \to \infty$.

Lemma 2.20([6], Lemma 8) *Let* (X,S) *be an* S*-metric space and* A *be a nonempty subset of* X*. Then* A *is* S*-closed if and only if for any sequence* $\{r_n\}$ *in* A *such that* $r_n \to r$ *as* $n \to \infty$, *then* $r \in A$.

Lemma 2.21([28]) *Let* (X,S) *be an* S*-metric space. If* $c > 0$ *and* $x \in X$, *then the ball* $\mathcal{B}_S(x,c)$ *is a subset of* X.

Lemma 2.22([29]) *The limit of a convergent sequence in a S-metric space (X, S) is unique.*

Lemma 2.23([28]) *In a S-metric space (X, S), any convergent sequence is Cauchy.*

§3. Main Results

In this section, we shall prove some common fixed point theorems on S-metric spaces via C-class functions.

Theorem 3.1 *Let (X, S) be a complete S-metric space and $f, g\colon X \to X$ be two self-mappings satisfying the inequality:*

$$\psi(S(fx, fy, gz)) \leq F\Big(\psi(\Theta(x, y, z)), \varphi(\Theta(x, y, z))\Big), \tag{3.1}$$

where

$$\begin{aligned}\Theta(x, y, z) &= a_1\, S(x, y, z) + a_2\, S(x, x, fx) + a_3\, S(z, z, gz) \\ &\quad + a_4[S(z, z, fx) + S(x, x, gz)] + a_5\left(\frac{S(z, z, gz)}{[1 + S(x, y, z)]}\right)\end{aligned}$$

for all $x, y, z \in X$, where $a_1, a_2, a_3, a_4, a_5 > 0$ are nonnegative reals with $a_1 + a_2 + a_3 + 3a_4 + a_5 < 1$, $\psi \in \Psi$, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$. Then f and g have a unique common fixed point in X.

Proof For each $x_0 \in X$. Let $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for $n = 0, 1, 2, \ldots$. We prove that $\{x_n\}$ is a Cauchy sequence in (X, S). It follows from (3.1) for $x = y = x_{2n}$, $z = x_{2n-1}$ and using $(S1)$, $(S2)$ and Lemma 2.18, we have

$$\begin{aligned}\psi(S(x_{2n+1}, x_{2n+1}, x_{2n})) &= \psi(S(fx_{2n}, fx_{2n}, gx_{2n-1})) \\ &\leq F\Big(\psi(\Theta(x_{2n}, x_{2n}, x_{2n-1})), \varphi(\Theta(x_{2n}, x_{2n}, x_{2n-1}))\Big),\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}\Theta(x_{2n}, x_{2n}, x_{2n-1}) &= a_1\, S(x_{2n}, x_{2n}, x_{2n-1}) + a_2\, S(x_{2n}, x_{2n}, fx_{2n}) \\ &\quad + a_3\, S(x_{2n-1}, x_{2n-1}, gx_{2n-1}) \\ &\quad + a_4\, [S(x_{2n-1}, x_{2n-1}, fx_{2n}) + S(x_{2n}, x_{2n}, gx_{2n-1})] \\ &\quad + a_5\left(\frac{S(x_{2n-1}, x_{2n-1}, gx_{2n-1})}{[1 + S(x_{2n}, x_{2n}, x_{2n-1})]}\right) \\ &= a_1\, S(x_{2n}, x_{2n}, x_{2n-1}) + a_2\, S(x_{2n}, x_{2n}, x_{2n+1}) \\ &\quad + a_3\, S(x_{2n-1}, x_{2n-1}, x_{2n})\end{aligned}$$

$$
\begin{aligned}
& +a_4\,[S(x_{2n-1},x_{2n-1},x_{2n+1})+S(x_{2n},x_{2n},x_{2n})] \\
& +a_5\Big(\frac{S(x_{2n-1},x_{2n-1},x_{2n})}{[1+S(x_{2n},x_{2n},x_{2n-1})]}\Big) \\
\leq\;& a_1\,S(x_{2n},x_{2n},x_{2n-1})+a_2\,S(x_{2n+1},x_{2n+1},x_{2n}) \\
& +a_3\,S(x_{2n},x_{2n},x_{2n-1}) \\
& +a_4\,[2S(x_{2n-1},x_{2n-1},x_{2n})+S(x_{2n+1},x_{2n+1},x_{2n})] \\
& +a_5\Big(\frac{S(x_{2n},x_{2n},x_{2n-1})}{[1+S(x_{2n},x_{2n},x_{2n-1})]}\Big) \\
\leq\;& a_1\,S(x_{2n},x_{2n},x_{2n-1})+a_2\,S(x_{2n+1},x_{2n+1},x_{2n}) \\
& +a_3\,S(x_{2n},x_{2n},x_{2n-1}) \\
& +a_4\,[2S(x_{2n-1},x_{2n-1},x_{2n})+S(x_{2n+1},x_{2n+1},x_{2n})] \\
& +a_5\,S(x_{2n},x_{2n},x_{2n-1})\Big) \\
=\;& (a_1+a_3+2a_4+a_5)S(x_{2n},x_{2n},x_{2n-1}) \\
& +(a_2+a_4)S(x_{2n+1},x_{2n+1},x_{2n}). \qquad (3.3)
\end{aligned}
$$

Using equation (3.3) in equation (3.2) and using the property of F, we get

$$
\begin{aligned}
\psi(S(x_{2n+1},x_{2n+1},x_{2n})) \;\leq\;& F\Big(\psi\Big((a_1+a_3+2a_4+a_5)S(x_{2n},x_{2n},x_{2n-1}) \\
& +(a_2+a_4)S(x_{2n+1},x_{2n+1},x_{2n})\Big), \\
& \varphi\Big((a_1+a_3+2a_4+a_5)S(x_{2n},x_{2n},x_{2n-1}) \\
& +(a_2+a_4)S(x_{2n+1},x_{2n+1},x_{2n})\Big)\Big) \\
\leq\;& \psi\Big((a_1+a_3+2a_4+a_5)S(x_{2n},x_{2n},x_{2n-1}) \\
& +(a_2+a_4)S(x_{2n+1},x_{2n+1},x_{2n})\Big). \qquad (3.4)
\end{aligned}
$$

Since $\psi\in\Psi$, so using the property of ψ, we deduce that

$$
\begin{aligned}
S(x_{2n+1},x_{2n+1},x_{2n}) \;\leq\;& (a_1+a_3+a_4+a_5)S(x_{2n},x_{2n},x_{2n-1}) \\
& +(a_2+a_4)S(x_{2n+1},x_{2n+1},x_{2n}),
\end{aligned}
$$

or

$$
\begin{aligned}
S(x_{2n+1},x_{2n+1},x_{2n}) \;\leq\;& \Big(\frac{a_1+a_3+2a_4+a_5}{1-a_2-a_4}\Big)S(x_{2n},x_{2n},x_{2n-1}) \\
=\;& t\,S(x_{2n},x_{2n},x_{2n-1}), \qquad (3.5)
\end{aligned}
$$

where

$$t=\Big(\frac{a_1+a_3+2a_4+a_5}{1-a_2-a_4}\Big)<1,$$

since $a_1+a_2+a_3+3a_4+a_5<1$. This implies that

$$S(x_{n+1},x_{n+1},x_n)\leq t\,S(x_n,x_n,x_{n-1}) \qquad (3.6)$$

for $n = 0, 1, 2, \cdots$.

Let $\mathcal{D}_n = S(x_{n+1}, x_{n+1}, x_n)$ and $\mathcal{D}_{n-1} = S(x_n, x_n, x_{n-1})$. Then from equation (3.6), we conclude that

$$\mathcal{D}_n \leq t\,\mathcal{D}_{n-1} \leq t^2\,\mathcal{D}_{n-2} \leq \cdots \leq t^n\,\mathcal{D}_0. \tag{3.7}$$

Therefore, since $0 \leq t < 1$, taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, x_n) = 0. \tag{3.8}$$

Now, we shall show that $\{x_n\}$ is a Cauchy sequence in (X, S).

Thus for any $n, m \in \mathbb{N}$ with $m > n$ and using Lemma 2.18, then we have

$$\begin{aligned}
S(x_n, x_n, x_m) &\leq 2S(x_n, x_n, x_{n+1}) + S(x_m, x_m, x_{n+1})\\
&= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m)\\
&\leq 2S(x_n, x_n, x_{n+1}) + 2S(x_{n+1}, x_{n+1}, x_2)\\
&\quad +S(x_{n+2}, x_{n+2}, x_m)\\
&\leq 2S(x_n, x_n, x_{n+1}) + 2S(x_{n+1}, x_{n+1}, x_2)\\
&\quad +2S(x_{n+2}, x_{n+2}, x_{n+3}) + \cdots + S(x_{m-1}, x_{m-1}, x_m)\\
&\leq 2\big(t^n + t^{n+1} + t^{n+2} + \cdots + t^{m-1}\big)S(x_0, x_0, x_1)\\
&= 2\big(t^n + t^{n+1} + t^{n+2} + \cdots + t^{m-1}\big)\mathcal{D}_0\\
&\leq \Big(\frac{2t^n}{1-t}\Big)\mathcal{D}_0 \to 0 \text{ as } n, m \to \infty
\end{aligned}$$

since $0 \leq t < 1$. Thus, the sequence $\{x_n\}$ is a Cauchy sequence in the space (X, S). By the completeness of the space, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Now, we shall show that u is a fixed point of g. For this, using the given inequality (3.1) for $x = y = x_{2n}$ and $z = u$, we have

$$\begin{aligned}
\psi(S(x_{2n+1}, x_{2n+1}, gu)) &= \psi(S(fx_{2n}, fx_{2n}, gu))\\
&\leq F\Big(\psi(\Theta(x_{2n}, x_{2n}, u)), \varphi(\Theta(x_{2n}, x_{2n}, u))\Big),
\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}
\Theta(x_{2n}, x_{2n}, u) &= a_1\,S(x_{2n}, x_{2n}, u) + a_2\,S(x_{2n}, x_{2n}, fx_{2n}) + a_3\,S(u, u, gu)\\
&\quad +a_4[S(u, u, fx_{2n}) + S(x_{2n}, x_{2n}, gu)]\\
&\quad +a_5\Big(\frac{S(u, u, gu)}{[1 + S(x_{2n}, x_{2n}, u)]}\Big)\\
&= a_1\,S(x_{2n}, x_{2n}, u) + a_2\,S(x_{2n}, x_{2n}, x_{2n+1}) + a_3\,S(u, u, gu)\\
&\quad +a_4[S(u, u, x_{2n+1}) + S(x_{2n}, x_{2n}, gu)]\\
&\quad +a_5\Big(\frac{S(u, u, gu)}{[1 + S(x_{2n}, x_{2n}, u)]}\Big).
\end{aligned}$$

Letting $n \to \infty$ in the above inequality and using $(S1)$, we get

$$\Theta(x_{2n}, x_{2n}, u) = (a_3 + a_4 + a_5)S(u, u, gu). \tag{3.10}$$

Using equation (3.10) in equation (3.9) and using the property of F, we have

$$\begin{aligned} \psi(S(x_{2n+1}, x_{2n+1}, gu)) &\leq F\Big(\psi((a_3 + a_4 + a_5)S(u, u, gu))), \varphi((a_3 + a_4 + a_5)S(u, u, gu))\Big) \\ &\leq \psi((a_3 + a_4 + a_5)S(u, u, gu)). \end{aligned} \tag{3.11}$$

Letting $n \to \infty$ in equation (3.11), we obtain

$$\psi(S(u, u, gu)) \leq \psi((a_3 + a_4 + a_5)S(u, u, gu)). \tag{3.12}$$

Since $\psi \in \Psi$, so using the property of ψ in equation (3.12), we deduce that

$$\begin{aligned} S(u, u, gu) &\leq (a_3 + a_4 + a_5)S(u, u, gu) \\ &\leq (a_1 + a_2 + a_3 + 3a_4 + a_5)S(u, u, gu) \\ &< S(u, u, gu), \text{ since } a_1 + a_2 + a_3 + 2a_4 + a_5 < 1, \end{aligned}$$

which is a contradiction. Hence $S(u, u, gu) = 0$, that is, $gu = u$. This shows that u is a fixed point of g. By similar fashion, we can show that $fu = u$. Consequently, u is a common fixed point of f and g.

Now, we shall show the uniqueness. Let u_1 be another common fixed point of f and g such that $fu_1 = u_1 = gu_1$ with $u_1 \neq u$. Using given contractive condition (3.1) for $x = y = u$, $z = u_1$ and using $(S1)$ and Lemma 2.18, we obtain

$$\begin{aligned} \psi(S(u, u, u_1)) &= \psi(S(fu, fu, gu_1)) \\ &\leq F\Big(\psi(\Theta(u, u, u_1)), \varphi(\Theta(u, u, u_1))\Big), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} \Theta(u, u, u_1) &= a_1 S(u, u, u_1) + a_2 S(u, u, fu) + a_3 S(u_1, u_1, gu_1) \\ &\quad + a_4[S(u_1, u_1, fu) + S(u, u, gu_1)] + a_5 \left(\frac{S(u_1, u_1, gu_1)}{[1 + S(u, u, u_1)]}\right) \\ &= a_1 S(u, u, u_1) + a_2 S(u, u, u) + a_3 S(u_1, u_1, u_1) \\ &\quad + a_4[S(u_1, u_1, u) + S(u, u, u_1)] + a_5 \left(\frac{S(u_1, u_1, u_1)}{[1 + S(u, u, u_1)]}\right) \\ &= (a_1 + 2a_4)S(u, u, u_1). \end{aligned}$$

Substituting in equation (3.13) and using the property of F, we have

$$\begin{aligned} \psi(S(u, u, u_1)) &\leq F\Big(\psi((a_1 + 2a_4)S(u, u, u_1)), \varphi((a_1 + 2a_4)S(u, u, u_1))\Big) \\ &\leq \psi((a_1 + 2a_4)S(u, u, u_1)). \end{aligned} \tag{3.14}$$

Since $\psi \in \Psi$, so using the property of ψ in equation (3.14), we deduce that

$$\begin{aligned} S(u,u,u_1) &\leq (a_1+2a_4)S(u,u,u_1) \\ &\leq (a_1+a_2+a_3+3a_4+a_5)S(u,u,u_1) \\ &< S(u,u,u_1), \text{ since } a_1+a_2+a_3+2a_4+a_5<1, \end{aligned} \tag{3.15}$$

which is a contradiction. Hence $S(u,u,u_1)=0$, that is, $u=u_1$. This shows the uniqueness of the common fixed point of f and g. This completes the proof. □

If we take $F(s,t)=ms$ for some $m\in[0,1)$ and $\psi(t)=t$ for all $t\geq 0$ in Theorem 3.1, then we have the following result (with $ma_1\to a_1$, $ma_2\to a_2$, $ma_3\to a_3$, $ma_4\to a_4$, $ma_5\to a_5$).

Corollary 3.2 *Let (X,S) be a complete S-metric space and $f,g\colon X\to X$ be two self-mappings satisfying the inequality:*

$$\begin{aligned} S(fx,fy,gz) \leq\; & a_1\,S(x,y,z)+a_2\,S(x,x,fx)+a_3\,S(z,z,gz) \\ &+a_4[S(z,z,fx)+S(x,x,gz)]+a_5\left(\frac{S(z,z,gz)}{[1+S(x,y,z)]}\right) \end{aligned} \tag{3.16}$$

for all $x,y,z\in X$, where $a_1,a_2,a_3,a_4,a_5>0$ are nonnegative reals with $a_1+a_2+a_3+3a_4+a_5<1$. Then f and g have a unique common fixed point in X.

Proof Follows from Theorem 3.1 by taking $F(s,t)=ms$ for some $m\in[0,1)$ and $\psi(t)=t$ for all $t\geq 0$ with $ma_1\to a_1$, $ma_2\to a_2$, $ma_3\to a_3$, $ma_4\to a_4$, $ma_5\to a_5$. □

Putting $g=f$ in Theorem 3.1, then we obtain the following result.

Corollary 3.3 *Let (X,S) be a complete S-metric space and $f\colon X\to X$ be a self-mapping satisfying the following inequality:*

$$\psi(S(fx,fy,fz))\leq F\Big(\psi(\Lambda(x,y,z)),\varphi(\Lambda(x,y,z))\Big), \tag{3.17}$$

where

$$\begin{aligned} \Lambda(x,y,z) =\; & a_1\,S(x,y,z)+a_2\,S(x,x,fx)+a_3\,S(z,z,fz) \\ &+a_4[S(z,z,fx)+S(x,x,fz)]+a_5\left(\frac{S(z,z,fz)}{[1+S(x,y,z)]}\right) \end{aligned}$$

for all $x,y,z\in X$, where $a_1,a_2,a_3,a_4,a_5>0$ are nonnegative reals with $a_1+a_2+a_3+3a_4+a_5<1$, $\psi\in\Psi$, $\varphi\in\Phi_u$ and $F\in\mathcal{C}$. Then f has a unique fixed point in X.

Proof This result immediately follows from Theorem 3.1 by taking $g=f$. □

Corollary 3.4 *Let (X,S) be a complete S-metric space such that for some positive integer n, f^n satisfies the contraction condition (*3.17*) for all $x,y,z\in X$, where $\Lambda(x,y,z)$, $\psi\in\Psi$, $\varphi\in\Phi_u$ and $F\in\mathcal{C}$ are as in Corollary* 3.3*. Then f has a unique fixed point in X.*

Proof From Corollary 3.3, let z_0 be the unique fixed point of f^n, that is, $f^n(z_0) = z_0$. Then

$$f(f^n z_0) = fz_0 \quad \text{or} \quad f^n(fz_0) = fz_0.$$

This gives $fz_0 = z_0$. This shows that z_0 is a unique fixed point of f and completes the proof.□

If we take $F(s,t) = ms$ for some $m \in [0,1)$, $\psi(t) = t$ for all $t \geq 0$ and putting $a_1 = k$, where $k \in [0,1)$ and $a_2 = a_3 = a_4 = a_5 = 0$ in Corollary 3.3, then we have the following result (with $mk \to k$).

Corollary 3.5 *Let (X,S) be a complete S-metric space and $f\colon X \to X$ be a self-mapping satisfying the inequality:*

$$S(fx, fy, fz)) \leq k\, S(x,y,z) \tag{3.18}$$

for all $x,y,z \in X$, where $k \in [0,1)$ is a constant. Then f has a unique fixed point in X.

Remark 3.6 *Corollary* 3.5 *extends the well-known Banach fixed point theorem [3] from complete metric space to the setting of complete S-metric space.*

If we take $F(s,t) = s - t$ in Theorem 3.1, then we obtain the following result.

Corollary 3.7 *Let (X,S) be a complete S-metric space and $f,g\colon X \to X$ be two self-mappings of X satisfying the inequality:*

$$\psi(S(fx, fy, gz)) \leq \psi(\Theta(x,y,z)) - \varphi(\Theta(x,y,z)) \tag{3.19}$$

for all $x,y,z \in X$, where $\Theta(x,y,z)$, ψ, φ and $a_1, a_2, a_3, a_4, a_5 > 0$ are as in Theorem 3.1*. Then f and g have a unique common fixed point in X.*

Proof This result follows from Theorem 3.1. □

If we take $F(s,t) = s$ in Theorem 3.1, then we obtain the following result.

Corollary 3.8 *Let (X,S) be a complete S-metric space and $f,g\colon X \to X$ be two self-mappings of X satisfying the inequality:*

$$\psi(S(fx, fy, gz)) \leq \psi(\Theta(x,y,z)) \tag{3.20}$$

for all $x,y,z \in X$, where $\Theta(x,y,z)$, ψ and $a_1, a_2, a_3, a_4, a_5 > 0$ are as in Theorem **??***. Then f and g have a unique common fixed point in X.*

Proof This result follows from Theorem 3.1. □

If we take $\psi(t) = t$ for all $t \geq 0$ in Corollary 3.8, then we obtain the following result.

Corollary 3.9 *Let (X,S) be a complete S-metric space and $f,g\colon X \to X$ be two self-mappings of X satisfying the inequality:*

$$S(fx, fy, gz) \leq \Theta(x,y,z) \tag{3.21}$$

for all $x, y, z \in X$, where $\Theta(x, y, z)$ and $a_1, a_2, a_3, a_4, a_5 > 0$ are as in Theorem 3.1. Then f and g have a unique common fixed point in X.

Proof It follows from Theorem 3.1. □

If we take $g = f$ in Corollary 3.2, then we have the following result.

Corollary 3.10 *Let (X, S) be a complete S-metric space and $f\colon X \to X$ be a self-mapping satisfying the inequality:*

$$\begin{aligned} S(fx, fy, fz)) &\leq a_1\, S(x, y, z) + a_2\, S(x, x, fx) + a_3\, S(z, z, fz) \\ &\quad + a_4[S(z, z, fx) + S(x, x, fz)] + a_5\Big(\frac{S(z, z, fz)}{[1 + S(x, y, z)]}\Big) \end{aligned} \tag{3.22}$$

for all $x, y, z \in X$, where $a_1, a_2, a_3, a_4, a_5 > 0$ are nonnegative reals with $a_1 + a_2 + a_3 + 3a_4 + a_5 < 1$. Then f has a unique fixed point in X.

Other consequences of our results are the following.

Denote $\mathbf{\Gamma}$ the set of functions $\varphi\colon [0, \infty) \to [0, \infty)$ satisfying the following hypothesis:

$(h1)$ φ is a Lebesgue-integrable mapping on each compact subset of $[0, \infty)$;
$(h2)$ for any $\varepsilon > 0$ we have $\int_0^\varepsilon \varphi(t)dt > 0$.

Then, we get the result following.

Theorem 3.11 *Let (X, S) be a complete S-metric space. Suppose that the mappings f, $g\colon X \to X$ satisfy the following inequality:*

$$\int_0^{\psi(S(fx,fy,gz))} \phi(t)dt \leq F\Big(\psi\big(\int_0^{\Theta(x,y,z)} \phi(t)dt\big), \varphi\big(\int_0^{\Theta(x,y,z)} \phi(t)dt\big)\Big)$$

for all $x, y, z \in X$, where φ, ψ, F, $\Theta(x, y, z)$, a_1, a_2, a_3, a_4, $a_5 > 0$ are as in Theorem 3.1 and $\phi \in \mathbf{\Gamma}$. Then, f and g have a unique common fixed point in X.

If we take $F(s, t) = ms$ for some $m \in [0, 1)$, $g = f$ and $\psi(t) = t$ for all $t \geq 0$ in Theorem 3.11, then we have the following result (with $ma_1 \to a_1$, $ma_2 \to a_2$, $ma_3 \to a_3$, $ma_4 \to a_4$, $ma_5 \to a_5$).

Corollary 3.12 *Let (X, S) be a complete S-metric space. Suppose that the mapping $f\colon X \to X$ satisfying the following inequality:*

$$\begin{aligned} \int_0^{S(fx,fy,fz)} \phi(t)dt &\leq a_1 \int_0^{S(x,y,z)} \phi(t)dt + a_2 \int_0^{S(x,x,fx)} \phi(t)dt \\ &\quad + a_3 \int_0^{S(z,z,fz)} \phi(t)dt + a_4 \int_0^{[S(z,z,fx)+S(x,x,fz)]} \phi(t)dt \\ &\quad + a_5 \int_0^{\left(\frac{S(z,z,fz)}{[1+S(x,y,z)]}\right)} \phi(t)dt \end{aligned}$$

for all $x, y, z \in X$, where a_1, a_2, a_3, a_4, $a_5 > 0$ are as in Theorem 3.1 and $\phi \in \mathbf{\Gamma}$. Then f has a unique fixed point in X.

If we take $a_1 = k$ and $a_2 = a_3 = a_4 = a_5 = 0$ in Corollary 3.12, then we have the following result.

Corollary 3.13 *Let (X, S) be a complete S-metric space. Suppose that the mapping $f\colon X \to X$ satisfying the following inequality:*

$$\int_0^{S(fx,fy,fz)} \phi(t)dt \le k \int_0^{S(x,y,z)} \phi(t)dt$$

for all $x, y, z \in X$, where $k \in [0, 1)$ is a constant and $\phi \in \mathbf{\Gamma}$. Then f has a unique fixed point in X.

Remark 3.14 *Corollary* 3.13 *extends Theorem 2.1 of Branciari [4] from complete metric space to that setting of complete S-metric space.*

Remark 3.15 *Corollary* 3.13 *also extends Banach contraction mapping principle [3] from complete metric space to that setting of complete S-metric space for integral type contraction.*

Now, we give some examples in support of our results.

Example 3.16 Let $X = [0, 1]$ and f, $g\colon X \to X$ be given by $f(x) = \frac{x}{2}$ and $g(x) = \frac{x}{4}$for all $x \in X$. Define the function $S\colon X^3 \to [0, \infty)$ by $S(x, y, z) = \max\{x, y, z\}$ for all $x, y, z \in X$, then S is an S-metric on X. Let $x, y, z \in X$ such that $x \ge y \ge z$.

(1) We have

$$\begin{aligned}
S(fx, fy, gz) &= \max\left\{\frac{x}{2}, \frac{y}{2}, \frac{z}{4}\right\} = \frac{x}{2},\\
S(x, y, z) &= \max\{x, y, z\} = x,\\
S(x, x, fx) &= \max\{x, x, \frac{x}{2}\} = x,\\
S(z, z, gz) &= \max\{z, z, \frac{z}{4}\} = z,\\
S(z, z, fx) &= \max\{z, z, \frac{x}{2}\} = \frac{x}{2},\\
S(x, x, gz) &= \max\{x, x, \frac{z}{4}\} = x,\\
S(x, x, fz) &= \max\{x, x, \frac{z}{2}\} = x,\\
S(z, z, fz) &= \max\{z, z, \frac{z}{2}\} = z.
\end{aligned}$$

Consider the inequality (3.16) of Corollary 3.2,we have

$$\frac{x}{2} \le a_1.x + a_2.x + a_3.z + \frac{3a_4}{2}.x + a_5.\frac{z}{1+x},$$

Putting $x = 1$, $y = \frac{1}{2}$ and $z = \frac{1}{3}$, then we have

$$\frac{1}{2} \leq a_1 + a_2 + a_3.\frac{1}{3} + a_4.\frac{1}{2} + a_5.\frac{1}{6} \quad \text{or} \quad 3 \leq 6a_1 + 6a_2 + 2a_3 + 9a_4 + a_5.$$

The above inequality is satisfied for: (1) $a_1 = \frac{1}{3}$, $a_2 = \frac{1}{4}$ and $a_3 = a_4 = a_5 = 0$; (2) $a_1 = \frac{1}{3}$, $a_3 = \frac{1}{4}$, $a_4 = \frac{1}{8}$ and $a_2 = a_5 = 0$; (3) $a_2 = \frac{1}{3}$, $a_3 = \frac{1}{2}$ and $a_1 = a_4 = a_5 = 0$ with $a_1 + a_2 + a_3 + 3a_4 + a_5 < 1$. Thus all the conditions of Corollary 3.2 are satisfied. Hence by applying Corollary 3.2, f and g have a unique common fixed point in X. Indeed, $0 \in X$ is the unique common fixed point of f and g in this case.

(2) Now, consider the inequality (3.18) of Corollary 3.5, we have

$$\frac{x}{2} \leq k\,x, \quad \text{or} \quad k \geq \frac{1}{2}.$$

If we take $0 < k < 1$, then all the conditions of Corollary 3.5 are satisfied and $0 \in X$ is the unique fixed point of f.

Example 3.17 Let $X = [0,1]$ and $S\colon X^3 \to \mathbb{R}_+$ be given by

$$S(x,y,z) = \begin{cases} |x-z| + |y-z|, & \text{if } x,y,z \in [0,1), \\ 1, & \text{if } x = 1 \text{ or } y = 1 \text{ or } z = 1, \end{cases}$$

for all $x, y, z \in X$. Then (X,S) is a complete S-metric space.

Let the mapping $f\colon X \to X$ be given by

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } x,y,z \in [0,1), \\ \\ \frac{1}{6}, & \text{if } x = y = z = 1. \end{cases}$$

Now, we consider the following cases for verification of inequality (3.22) of Corollary 3.10.

Case 1. If $x, y \in [0, \frac{1}{2}]$, $z \in [\frac{1}{2}, 1)$ or $z \in [0, \frac{1}{2}]$, $x, y \in [\frac{1}{2}, 1)$. Then

$$\begin{aligned} S(fx,fy,fz) &= S\Big(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\Big) = 0 \\ &\leq a_1\, S(x,y,z) + a_2\, S(x,x,fx) + a_3\, S(z,z,fz) \\ &\quad + a_4[S(z,z,fx) + S(x,x,fz)] + a_5\left(\frac{S(z,z,fz)}{[1 + S(x,y,z)]}\right). \end{aligned}$$

Thus, the inequality (3.22) of Corollary 3.10 is trivially satisfied.

Case 2. If $x, y \in [0, \frac{1}{2}]$ and $z = 1$. Then,

$$S(fx,fy,fz) = S\Big(\frac{1}{2}, \frac{1}{2}, \frac{1}{6}\Big) = \frac{2}{3}.$$

Taking $x = y = \frac{1}{2}$,

$$S(x,y,z) = 1,\ S(x,x,fx) = 0,\ S(z,z,fz) = \frac{5}{3},$$

$$S(z,z,fx) = 1,\ S(x,x,fz) = \frac{2}{3}.$$

Now

$$\begin{aligned} \frac{2}{3} &\leq a_1\, S(x,y,z) + a_2\, S(x,x,fx) + a_3\, S(z,z,fz) \\ &\quad + a_4[S(z,z,fx) + S(x,x,fz)] \\ &\quad + a_5\Big(\frac{S(z,z,fz)}{[1+S(x,y,z)]}\Big) \\ &= a_1.1 + a_2.0 + a_3.\frac{5}{3} + a_4.\frac{5}{3} + a_5.\frac{5}{6}, \end{aligned}$$

or

$$4 \leq 6a_1 + 10a_3 + 10a_4 + 5a_5.$$

The above inequality is satisfied for: (1) $a_1 = \frac{1}{3}$, $a_3 = \frac{1}{5}$ and $a_2 = a_4 = a_5 = 0$, (2) $a_1 = \frac{1}{3}$, $a_4 = \frac{1}{5}$ and $a_2 = a_3 = a_5 = 0$ and (3) $a_3 = \frac{1}{10}$, $a_4 = \frac{1}{5}$; $a_5 = \frac{1}{5}$ and $a_1 = a_2 = 0$ with $a_1 + a_2 + a_3 + 3a_4 + a_5 < 1$. Thus all the conditions of Corollary 3.10 are satisfied. Hence by applying Corollary 3.10, f has a unique fixed point in X. Indeed, $\frac{1}{2} \in X$ is the unique fixed point of f in this case.

Case 3. If $x, z \in [0, \frac{1}{2}]$ and $y = 1$. Then

$$S(fx, fy, fz) = S\Big(\frac{1}{2}, \frac{1}{6}, \frac{1}{2}\Big) = \frac{1}{3}.$$

Taking $x = z = \frac{1}{2}$,

$$S(x,y,z) = \frac{1}{2},\ S(x,x,fx) = 0,\ S(z,z,fz) = 0,\ S(z,z,fx) = 0,$$

$$S(x,x,fz) = 0.$$

Now

$$\begin{aligned} \frac{1}{3} &\leq a_1\, S(x,y,z) + a_2\, S(x,x,fx) + a_3\, S(z,z,fz) \\ &\quad + a_4[S(z,z,fx) + S(x,x,fz)] + a_5\Big(\frac{S(z,z,fz)}{[1+S(x,y,z)]}\Big) \\ &= \frac{a_1}{2} + a_2.0 + a_3.0 + a_4.0 + a_5.0 \\ &= \frac{a_1}{2} \quad \text{or} \quad \frac{2}{3} \leq a_1. \end{aligned}$$

The above inequality is satisfied for $a_1 = \frac{2}{3}$ and $a_2 = a_3 = a_4 = a_5 = 0$ with $a_1 + a_2 + a_3 + 3a_4 + a_5 < 1$. Thus all the conditions of Corollary 3.10 are satisfied. Hence by applying Corollary 3.10, f has a unique fixed point in X. Indeed, $\frac{1}{2} \in X$ is the unique fixed point of f in this case.

Case 4. If $y, z \in [0, \frac{1}{2}]$ and $x = 1$. Then

$$S(fx, fy, fz) = S\Big(\frac{1}{6}, \frac{1}{2}, \frac{1}{2}\Big) = \frac{1}{3}.$$

Taking $y = z = \frac{1}{2}$,

$$S(x, y, z) = \frac{1}{2},\ S(x, x, fx) = \frac{5}{3},\ S(z, z, fz) = 0,$$

$$S(z, z, fx) = \frac{2}{3},\ S(x, x, fz) = 1.$$

Now

$$\begin{aligned}
\frac{1}{3} &\leq a_1\, S(x, y, z) + a_2\, S(x, x, fx) + a_3\, S(z, z, fz) \\
&\quad + a_4[S(z, z, fx) + S(x, x, fz)] + a_5 \Big(\frac{S(z, z, fz)}{[1 + S(x, y, z)]}\Big) \\
&= \frac{a_1}{2} + a_2 . \frac{5}{3} + a_3 . 0 + a_4 . \frac{5}{3} + a_5 . 0 \\
&= \frac{a_1}{2} + \frac{5a_2}{3} + \frac{5a_4}{3}
\end{aligned}$$

or

$$2 \leq 3a_1 + 10a_2 + 10a_4.$$

The above inequality is satisfied for: (1) $a_1 = \frac{1}{3}$, $a_2 = \frac{1}{10}$, $a_3 = a_4 = a_5 = 0$; (2) $a_1 = \frac{1}{3}$, $a_4 = \frac{1}{10}$, $a_2 = a_3 = a_5 = 0$ and (3) $a_2 = a_4 = \frac{1}{10}$ and $a_1 = a_3 = a_5 = 0$ with $a_1 + a_2 + a_3 + 3a_4 + a_5 < 1$. Thus all the conditions of Corollary 3.10 are satisfied. Hence by applying Corollary 3.10, f has a unique fixed point in X. Indeed, $\frac{1}{2} \in X$ is the unique fixed point of f in this case.

Considering all the above cases, we conclude that the inequality used in Corollary 3.10 remains valid for mapping f constructed in the above example and consequently by applying Corollary 3.10, f has a unique fixed point. One can easily see that $u = \frac{1}{2} \in X$ is the unique fixed point of f.

§4. **Conclusion**

In this paper, we prove some common fixed point theorems in the setting of complete S-metric spaces via C-class functions and we give some examples in support of our results. Also, we give some consequences as corollaries of the established results. The results obtained in this paper extend, generalize and enrich several results from the existing literature regarding complete S-metric spaces.

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$(LCS)_n$-Manifold Endowed with Torseforming Vector Field and Concircular Curvature Tensor

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Abstract: This paper reviews few curvature properties of $(LCS)_n$-manifold endowed with torseforming vector field and concicrular curvature tensor. Moreover, we have proved several interesting results in this study.

Key Words: $(LCS)_n$-manifold, torseforming vector field, concircular vector field, concircular curvature tensor, scalar curvature.

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§1. Introduction

In 2003, Shaikh [14] has first developed and studied the structure of Lorentzian concircular structure manifolds (briefly $(LCS)_n$-manifolds) with several examples, which generalizes the concept of LP-Sasakian manifolds given by Matsumoto [9] and by Mihai and Rosca [10]. Later on Shaikh et al., [16] proved the existence of ϕ-recurrent $(LCS)_n$-manifolds. Recently the same author studied invariant submanifolds of $(LCS)_n$-manifolds. The notion of $(LCS)_n$-manifolds have been intensively studied by several geometers such as Hui and Atceken [7], Prakasha [13], Venkatesha et. al., [28] and many others.

A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle is called a concircular transformation [31]. A concircular transformation is always a conformal transformation [8]. An invariant of a concircular transformation is the concircular curvature tensor C given by [31]

$$C(X,Y)U = R(X,Y)U - \frac{r}{n(n-1)}[g(Y,U)X - g(X,U)Y]. \tag{1.1}$$

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Now we can easily obtained From (1.1) that

$$(\nabla_W C)(X,Y)U = (\nabla_W R)(X,Y)U - \frac{dr(W)}{n(n-1)}[g(Y,U)X - g(X,U)Y]. \tag{1.2}$$

The study of torseforming vector field has a long history starting in 1925 by the work of Brinkmann [6], Shirokov [17] and Yano [30, 31]. Torseforming vector field in a Riemannian manifold has been introduced by Yano in 1944 [30] and the complex analogue of a torseforming vector field was introduced by Yamaguchi [29] in 1979. The geometry of torseforming vector field in a Riemannian manifold with different structures have been studied extensively by many geometers such as Bagewadi et. al., [5].

The paper is organized in the following way: In Section 2, we recall the basic definitions and formulas of $(LCS)_n$-manifold needed throughout the paper. The next Section is devoted to the study of $(LCS)_n$-manifold admitting unit torseforming vector field. Here we have shown that an $(LCS)_n$-manifold admits a concircular vector field. In Section 4, we consider globally ϕ-Concircularly symmetric $(LCS)_n$-manifold. Thus, we have obtain that the manifold is of constant scalar curvature provided $2\alpha\rho = \beta$.

For readers who are unfamiliar with terminology, notations, recent overviews and introductions, we suggest the auhthors to refer the papers [1, 2, 3, 4, 11, 12, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27].

§2. **Preliminaries**

An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to R$ is a non-degenerate inner product of signature $(-,+,\cdots,+)$, where T_pM denotes the tangent vector space of M at p and R is the real number space.

A Lorentzian manifold endowed with a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold, gives

$$g(\xi,\xi) = -1. \tag{2.1}$$

Since ξ is a unit concircular vector field, there exists a non-zero 1-form η such that

$$g(V,\xi) = \eta(X), \tag{2.2}$$

from which the following equation holds

$$(\nabla_U \eta)(V) = \alpha[g(U,V) + \eta(U)\eta(V)], \quad (\alpha \neq 0) \tag{2.3}$$

for all vector fields U, V, where ∇ denotes the operator of covariant differentiation with respect to Lorentzian metric g and α is a non-zero scalar function satisfying

$$\nabla_V \alpha = (V\alpha) = d\alpha(V) = \rho\eta(V), \tag{2.4}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\phi V = \frac{1}{\alpha}\nabla_V \xi, \tag{2.5}$$

then from (2.3) and (2.4), we have

$$\phi V = V + \eta(V)\xi, \tag{2.6}$$

from which it follows that ϕ is a symmetric $(1,1)$ tensor field. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ, its associated 1-form η and $(1,1)$ tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly $(LCS)_n$-manifold) [14]. In a $(LCS)_n$-manifold, the following relations hold ([14, 15]):

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi V) = 0, \tag{2.7}$$

$$g(\phi U, \phi V) = g(U,V) + \eta(U)\eta(V), \tag{2.8}$$

$$R(U,V)W = (\alpha^2 - \rho)[g(V,W)U - g(U,W)V], \tag{2.9}$$

$$S(U,\xi) = (n-1)(\alpha^2 - \rho)\eta(U), \tag{2.10}$$

$$S(\phi U, \phi V) = S(U,V) + (n-1)(\alpha^2 - \rho)\eta(U)\eta(V), \tag{2.11}$$

$$Q\xi = (n-1)(\alpha^2 - \rho)\xi. \tag{2.12}$$

for any vector fields U, V, W, where R, S denote respectively the curvature tensor and the Ricci tensor of the manifold.

§3. Torseforming Vector Field in a $(LCS)_n$-Manifold

Definition 3.1 *A vector field γ on a Riemannian manifold is said to be torseforming vector field if the 1-form $\omega(V) = g(V,\gamma)$ satisfies the equation of the form*

$$(\nabla_U \omega)(V) = \beta g(U,V) + \pi(U)\omega(V), \tag{3.1}$$

where β is a non-vanishing scalar and π is a non zero 1-form given by $\pi(V) = g(V,P)$.

Let us consider an $(LCS)_n$ manifold admitting a unit torseforming vector field $\tilde{\gamma}$ corresponding to the torseforming vector field γ. Suppose $g(V,\tilde{\gamma}) = T(V)$, then we have

$$T(V) = \frac{\omega(V)}{\sqrt{\omega(\gamma)}}. \tag{3.2}$$

Now by considering the relation (3.1), we have

$$\frac{(\nabla_U \omega)(V)}{\sqrt{\omega(\gamma)}} = \frac{\beta}{\sqrt{\omega(\gamma)}} g(U,V) + \frac{\pi(U)}{\sqrt{\omega(\gamma)}} \omega(V).$$

Using (3.2) in the above equation, we obtain

$$(\nabla_U T)(V) = a g(U,V) + \pi(U) T(V), \tag{3.3}$$

where $a = \frac{\beta}{\sqrt{\omega(\gamma)}}$.

Plugging $Y = \tilde{\gamma}$ in (3.3) and using $T(\tilde{\gamma}) = g(\tilde{\gamma}, \tilde{\gamma}) = 1$, we get

$$\pi(U) = -aT(U), \tag{3.4}$$

and hence relation (3.3) can be written in the form

$$(\nabla_U T)(V) = a[g(U,V) - T(U)T(V)], \tag{3.5}$$

which implies that T is closed.

Taking covariant differential of (3.5) with respect to W and using Ricci identity, we get

$$\begin{aligned} -T(R(U,V)W) &= (Ua)[g(V,W) - T(V)T(W)] \\ &\quad -(Va)[g(U,W) - T(U)T(W)] \\ &\quad +a^2[g(V,W)T(U) - g(U,W)T(V)]. \end{aligned} \tag{3.6}$$

Replacing $W = \xi$ in (3.6) and then by considering (2.5), we obtain

$$\begin{aligned} -(\alpha^2 - \rho)T(\eta(V)U - \eta(U)V) &= (Ua)[\eta(V) - T(V)T(\xi)] \\ &\quad -(Va)[\eta(U) - T(U)T(\xi)] \\ &\quad +a^2[\eta(V)T(U) - \eta(U)T(V)]. \end{aligned} \tag{3.7}$$

Again, replacing $U = \tilde{\gamma}$ in (3.7) and since $T(\tilde{\gamma}) = g(\tilde{\gamma}, \tilde{\gamma}) = 1$, we have

$$(\alpha^2 - \rho + a^2 + \tilde{\gamma}a)[\eta(V) - \eta(\tilde{\gamma})T(V)] = 0. \tag{3.8}$$

Thus, we can state the following result.

Theorem 3.1 *If a $(LCS)_n$-manifold endowed with a unit torseforming vector field $\tilde{\gamma}$, then the following conditions are occur:*

$$\begin{aligned} \eta(V) - \eta(\tilde{\gamma})T(V) &= 0, \qquad \textit{(I)} \\ (\alpha^2 - \rho + a^2 + \tilde{\gamma}a) &= 0. \qquad \textit{(II)} \end{aligned}$$

We first begin with the case where the condition (I) holds true, from which it follows that

$$\eta(V) = \eta(\tilde{\gamma})T(V).$$

Plugging $V = \xi$ in above equation, gives

$$\eta(\xi) = \eta(\tilde{\gamma})^2,$$

and thus $\eta(\tilde{\gamma}) = \pm\sqrt{-1}$, since $\eta(\xi) = -1$, we get

$$\eta(V) = \pm\sqrt{-1}T(V). \tag{3.9}$$

Using (3.9) in (2.3) and by virtue of (3.5), we have

$$\alpha[g(U,V) - T(U)T(V)] = \pm\sqrt{-1}a(g(U,V) - T(U)T(V)).$$

This implies that $a = \pm\sqrt{-1}\alpha$ and hence the expression (3.4) reduces to

$$\pi(V) = \pm\sqrt{-1}\alpha T(V). \tag{3.10}$$

If we consider $\alpha = 1$, above equation yields

$$\pi(V) = \pm\sqrt{-1}T(V). \tag{3.11}$$

Since T is closed, π is also closed. Hence we can state the following result.

Lemma 3.1 *In an $(LCS)_n$-manifold satisfying condition (I), the unit torseforming vector field $\tilde{\gamma}$ reduces to cocircular vector field provided the manifold becomes LP Sasakian Structure.*

Next, we claim that the case where the condition (II) holds true, then the case (I) does not occur. That is, it follows that

$$\eta(V) - \eta(\tilde{\gamma})T(V) \neq 0. \tag{3.12}$$

Now it can be easily obtained from (3.6) that

$$-(\alpha^2 - \rho)T(QU) = (n-1)aU - (aU) + (\tilde{\gamma}a)T(U) + a^2(n-1)T(U). \tag{3.13}$$

By considering $U = \xi$ in (3.13) and making use of (2.10), we obtain

$$a\xi = -(\alpha^2 - \rho + a^2)\eta(\tilde{\gamma}). \tag{3.14}$$

Plugging $V = \xi$ in (3.7) and in the view of (3.14) and $T(\xi) = \eta(\tilde{\gamma})$, we get

$$aU = -(\alpha^2 - \rho + a^2)T(U). \tag{3.15}$$

Now it can be seen from (3.4) that

$$V\pi(U) = -[(Va)T(U) + a(VT(U))].$$

By considering the equation (3.15), we follows that

$$V\pi(U) = -[-(\alpha^2 - \rho + a^2)T(V)T(U) + a(VT(U))], \tag{3.16}$$

$$U\pi(V) = -[-(\alpha^2 - \rho + a^2)T(U)T(V) + a(UT(V))] \tag{3.17}$$

from which we can easily obtained that

$$\pi([U,V]) = -aT([U,V]). \tag{3.18}$$

Now by using (3.16), (3.17) and (3.18), we have

$$(d\pi)(U,V) = -a[(dT)(U,V)].$$

Since T is closed, π is also closed. Hence we can state that

Lemma 3.2 *In an $(LCS)_n$-manifold satisfying condition (II), the unit torseforming vector field $\tilde{\gamma}$ reduces to cocircular vector field.*

§4. Globally ϕ-Concircularly Symmetric $(LCS)_n$-Manifold

Definition 4.1 *An $(LCS)_n$-manifold M is said to be globally ϕ-concirularly symmetric if the concircular curvature tensor C satisfies*

$$\phi^2((\nabla_X C)(U,V)W) = 0,$$

for all vector fields $X, U, V, W \in \chi(M)$.

If X, U, V and W are horizontal vector fields then the manifold is called locally ϕ-concircularly symmetric.

Let us consider an globally ϕ-concircularly symmetric $(LCS)_n$-manifold, then we have

$$\phi^2((\nabla_X C)(U,V)W) = 0. \tag{4.1}$$

Using (2.6) in equation (4.1), gives

$$(\nabla_X C)(U,V)W + \eta((\nabla_X C)(U,V)W)\xi = 0,$$

from which it follows that

$$g((\nabla_X R)(U,V)W,Y) - \frac{drX}{n(n-1)}[g(V,W)g(U,Y) - g(U,W)g(V,Y)]$$
$$+\eta((\nabla_X R)(U,V)W)\eta(Y) - \frac{drX}{n(n-1)}[g(V,W)\eta(U) - g(U,W)\eta(V)]\eta(Y) = 0. \tag{4.2}$$

Plugging $U = Y = e_i$, where e_i is an orthonormal basis and taking summation over i, we get

$$\begin{aligned}(\nabla_X S)(V,W) &- \frac{drX}{n} g(V,W) + \eta((\nabla_X R)(e_i,V)W)\eta(e_i)\\ &+ \frac{drX}{n(n-1)}[g(V,W) + \eta(V)\eta(W)] = 0.\end{aligned}$$

Considering $W = \xi$ in the above equation, gives

$$(\nabla_X S)(V,\xi) - \frac{drX}{n}\eta(V) + \eta((\nabla_X R)(e_i,V)\xi)\eta(e_i) = 0. \tag{4.3}$$

By considering the expression

$$\eta((\nabla_X R)(e_i,V)\xi)\eta(e_i) = g((\nabla_X R)(e_i,V)\xi,\xi)g(e_i,\xi). \tag{4.4}$$

Now above equation takes the form

$$\begin{aligned}g((\nabla_X R)(e_i,V)\xi,\xi) &= g(\nabla_X R(e_i,V)\xi,\xi) - g(R(\nabla_X e_i,V)\xi,\xi)\\ &\quad -g(R(e_i,\nabla_X V)\xi,\xi) - g(R(e_i,V)\nabla_X\xi,\xi).\end{aligned} \tag{4.5}$$

Since e_i is an orthonormal basis and by virtue of (2.9), we find that

$$g((\nabla_X R)(e_i,V)\xi,\xi) = g(\nabla_X R(e_i,V)\xi,\xi) - g(R(e_i,V)\nabla_X\xi,\xi), \tag{4.6}$$

$$g(\nabla_X R(e_i,V)\xi,\xi) + g(R(e_i,V)\xi,\nabla_X\xi) = 0. \tag{4.7}$$

By employing (4.7) in (4.6), gives

$$g((\nabla_X R)(e_i,V)\xi,\xi) = 0. \tag{4.8}$$

Taking an account of (4.4) and (4.8) in (4.3), turns into

$$(\nabla_X S)(V,\xi) = \frac{drX}{n}\eta(V). \tag{4.9}$$

If we take $V = \xi$ in (4.9), we found that $dr(X) = 0$. Then, from (4.9), we have $\nabla_X S(V,\xi) = 0$. This implies that

$$dr(X) = n(n-1)(2\alpha\rho - \beta). \tag{4.10}$$

Next claim that if $2\alpha\rho = \beta$, we get $dr(X) = 0$ and hence the scalar curvature r is constant.

This leads to the result following.

Theorem 4.2 *A globally ϕ-concircularly symmetric $(LCS)_n$-manifold is of constant scalar curvature provided $2\alpha\rho = \beta$.*

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Metric on L-Fuzzy Real Line

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Abstract: In this study, the concept of L-fuzzy real numbers which is given in [14] is extended by presenting the definition from both-sided. For each side, different functions are defined and it is proved that these functions are metrics. For that, it is shown that for a complete lattice L, given conditions in [14] for an equivalence relation $\sim$ on $md_{\mathbb{R}}(L)$ are equivalent. So condition is weakened in our work. A metric which is consistent with the Euclidean metric is defined by using two-sided metrics. Also, an example is given for L-Fuzzy metric.

Key Words: L-fuzzy real line, metric, chain.

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§1. Introduction

Firstly, we give the concepts of L-fuzzy real number and equivalence relation.

Definition 1.1([11]) *Let $(L, \leq)$ be a complete lattice. Then, $\lambda \in L^{\mathbb{R}}$ is called L-fuzzy real number $\Leftrightarrow$*

(*i*) *$\exists x_0 \in \mathbb{R}$ such that $\lambda(x_0) = 1$;*

(*ii*) *For all $a \in L$, $\lambda_{[a]}$ level subset is closed interval.*

Definition 1.2([9]) *Let $(L, \leq)$ be a complete lattice and*

$$md_{\mathbb{R}}(L) = \left\{ \lambda \in L^{\mathbb{R}} : \bigvee_{t \in \mathbb{R}} \lambda(t) = 1, \quad \bigwedge_{t \in \mathbb{R}} \lambda(t) = 0, \ \lambda \text{ monotonous decreasing} \right\}.$$

For $\forall \lambda \in md_{\mathbb{R}}(L)$ and all $t \in \mathbb{R}$, let

$$\lambda(t-) := \bigwedge_{s<t} \lambda(s) \quad \text{and} \quad \lambda(t+) := \bigvee_{s>t} \lambda(s).$$

Then, an equivalence relation "$\sim$" on $md_{\mathbb{R}}(L)$ is defined as follow:

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for $\lambda, \mu \in md_{\mathbb{R}}(L)$,

$$\lambda \sim \mu :\Leftrightarrow \forall t \in \mathbb{R},\ \lambda(t-) = \mu(t-)\ \ , \lambda(t+) = \mu(t+)\,.$$

The set of equivalence classes containing λ is denoted by

$$[\lambda] := \{\mu \in md_{\mathbb{R}}(L) : \mu \sim \lambda\}\,.$$

Let $\mathbb{R}[L]_{right} := \left\{[\lambda] : \lambda \in md_{\mathbb{R}}(L)_{right}\right\}$ be set of all equivalence classes with respect to " $\sim$ " equivalence relation on $md_{\mathbb{R}}(L)$. $\mathbb{R}[L]_{right}$ is called L-fuzzy real line.

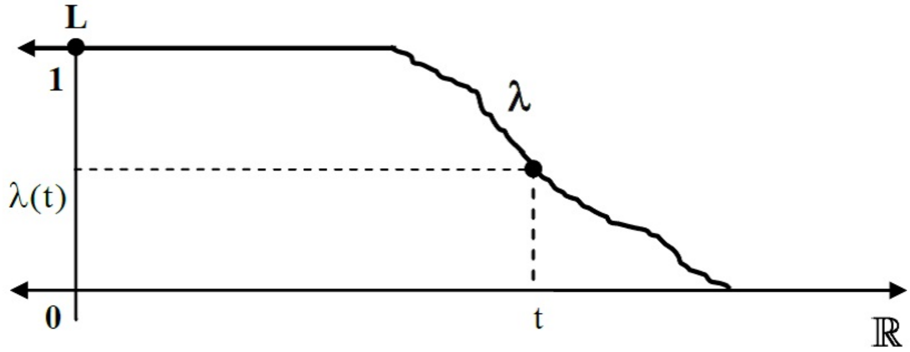

Figure 1. $[\lambda] \in \mathbb{R}[L]_{right}$

The concepts of L-fuzzy unit interval and L-fuzzy real line have an important place in L-fuzzy topoligical spaces. The metrics are very essential tools in various fields of sciences that measure the distance or difference between two points. Firstly, the concept of the L-fuzzy unit interval was given in 1975 by Hutton [7]. In 1982, Rodabaugh defined the fuzzy addition process on the fuzzy real line taking L-complete lattice instead of $[0, 1]$ [11].

In 1983, Lowen examined the algebraic structure of the L-fuzzy real line [10]. Wang gave the necessary and sufficient condition on the convergence of infinite sums by giving infinite additive concepts on the fuzzy real line [12]. S. Göhler ve Werner Göhler examined the topological properties of the fuzzy real line by defining two special fuzzy metrics on the fuzzy real line [4]. Diamond [2] defined a metric for the triangular fuzzy numbers. Kaufmann et al. considered a distance of two fuzzy numbers combined by the interval of a-cuts of fuzzy numbers [8]. Heilpern proposed three definitions of the distance between two fuzzy numbers [5].

In 2007, Han-Liang Huang and Fu-Gui Shi gave the concepts of L-fuzzy numbers and L-fuzzy convex sets on L completely distributive lattice [6]. Recently, Jian-zhong Xiao and Xing-hua Zhu have studied the metric structure of the fuzzy real line by giving the semi-metric concept on the fuzzy real towards the L completely distributive lattice [13]. Allahviranloo et al. gave a metric based on modified Euclidean metric on interval numbers, for $L - R$ fuzzy numbers with fixed $L(.)$ and $R(.)$ is introduced [1]. García, J.G. and Kubiak showed how the Hutton's concept evolved [3].

§2. Main Results for Metric on the L-Fuzzy Real Line

In this section, some new concepts and some theorems and results related to these concepts are given as parallel to the concepts given in Definition 1.2. In addition, using theorems and the results in this section, a metric on the L-fuzzy real line was created for complete lattice $(L,\leq)$.

Theorem 2.1 *Let $(L,\leq)$ be a complete lattice, $\lambda,\mu\in md_{\mathbb{R}}(L)$ and for $t_0\in\mathbb{R}$, $\lambda(t_0-)\neq\mu(t_0-)$. Then,*

(i) For $\forall\varepsilon>0,\ \exists s_0\in(t_0-\varepsilon,t_0):\lambda(t_0-)\nleq\mu(s_0)$ or $\mu(t_0-)\nleq\lambda(s_0)$;

(ii) For $\forall\varepsilon>0,\ \exists s_0\in(t_0-\varepsilon,t_0):\lambda(s_0+)\neq\mu(s_0+)$.

Proof (*i*) Let $\lambda(t_0-)\neq\mu(t_0-)$ for $t_0\in\mathbb{R}$. Let's assume that

$$\text{For } \exists\varepsilon>0:\forall s\in(t_0-\varepsilon,t_0),\quad \lambda(t_0-)\leq\mu(s)\ \text{ and }\ \mu(t_0-)\leq\lambda(s).$$

Then,

$$\lambda(t_0-)\leq\bigwedge_{t_0-\varepsilon<s<t_0}\mu(s)\ \text{ and }\ \mu(t_0-)\leq\bigwedge_{t_0-\varepsilon<s<t_0}\lambda(s).$$

Since λ and μ are decreasing,

$$\lambda(t_0-)\leq\bigwedge_{t_0-\varepsilon<s<t_0}\mu(s)=\bigwedge_{s<t_0}\mu(s)=\mu(t_0-)\Rightarrow\lambda(t_0-)\leq\mu(t_0-)\tag{2.1}$$

and

$$\mu(t_0-)\leq\bigwedge_{t_0-\varepsilon<s<t_0}\lambda(s)=\bigwedge_{s<t_0}\lambda(s)=\lambda(t_0-)\Rightarrow\mu(t_0-)\leq\lambda(t_0-).\tag{2.2}$$

From (2.1) and (2.2), $\lambda(t_0-)=\mu(t_0-)$. This contradicts the hypothesis of the theorem. So,

$$\forall\varepsilon>0,\ \exists s_0\in(t_0-\varepsilon,t_0):\lambda(t_0-)\nleq\mu(s_0)\ \text{ or }\ \mu(t_0-)\nleq\lambda(s_0).$$

(*ii*) Let $\lambda(t_0-)\neq\mu(t_0-)$ for $t_0\in\mathbb{R}$ and $\varepsilon>0$. From (*i*),

$$\exists s_0\in(t_0-\varepsilon,t_0):\lambda(t_0-)\nleq\mu(s_0)\ \text{ or }\ \mu(t_0-)\nleq\lambda(s_0).$$

Without loss of generality, let $\lambda(t_0-)\nleq\mu(s_0)$. Since μ is decreasing, $\mu(s_0+)\leq\mu(s_0)$. Hence,

$$\lambda(t_0-)\nleq\mu(s_0+).\tag{2.3}$$

On the other hand,

$$\{\lambda(s):s_0<s<t_0\}\subset\{\lambda(s):s<t_0\}.$$

Therefore,

$$\bigvee_{s_0<s<t_0} \lambda(s) \geq \bigwedge_{s<t_0} \lambda(s) = \lambda(t_0-)$$

is obtained. So,

$$\lambda(t_0-) \leq \bigvee_{s_0<s<t_0} \lambda(s).$$

Since λ is decreasing,

$$\lambda(t_0-) \leq \bigvee_{s_0<s<t_0} \lambda(s) = \bigvee_{s_0<s} \lambda(s) = \lambda(s_0+).$$

From (2.3), since $\lambda(t_0-) \nleq \mu(s_0+)$, $\lambda(s_0+) \nleq \mu(s_0+)$. So, $\lambda(s_0+) \neq \mu(s_0+)$. □

Theorem 2.2 *Let $(L, \leq)$ be a complete lattice, $\lambda, \mu \in md_{\mathbb{R}}(L)$ and $\lambda(t_0+) \neq \mu(t_0+)$ for $t_0 \in \mathbb{R}$. Then,*

(*i*) *For $\forall\varepsilon > 0,\ \exists s_0 \in (t_0, t_0+\varepsilon) : \mu(s_0) \nleq \lambda(t_0+)$ or $\lambda(s_0) \nleq \mu(t_0+)$;*

(*ii*) *For $\forall\varepsilon > 0,\ \exists s_0 \in (t_0, t_0+\varepsilon) : \lambda(s_0-) \neq \mu(s_0-)$.*

Proof (*i*) Let $\lambda(t_0+) \neq \mu(t_0+)$ for $t_0 \in \mathbb{R}$. Let's assume that

$$\text{For } \exists\varepsilon > 0 : \forall s \in (t_0, t_0+\varepsilon), \quad \mu(s) \leq \lambda(t_0+) \text{ and } \lambda(s) \leq \mu(t_0+).$$

Then,

$$\bigvee_{t_0<s<t_0+\varepsilon} \mu(s) \leq \lambda(t_0+) \quad \text{and} \quad \bigvee_{t_0<s<t_0+\varepsilon} \lambda(s) \leq \mu(t_0+).$$

Since λ and μ are decreasing,

$$\lambda(t_0+) = \bigvee_{t_0<s} \lambda(s) = \bigvee_{t_0<s<t_0+\varepsilon} \lambda(s) \leq \mu(t_0+) \Rightarrow \lambda(t_0+) \leq \mu(t_0+) \tag{2.4}$$

and

$$\mu(t_0+) = \bigvee_{t_0<s} \mu(s) = \bigvee_{t_0<s<t_0+\varepsilon} \mu(s) \leq \lambda(t_0+) \Rightarrow \mu(t_0+) \leq \lambda(t_0+). \tag{2.5}$$

From (2.4) and (2.5), $\lambda(t_0+) = \mu(t_0+)$ is obtained. This contradicts the hypothesis of the theorem. So,

$$\forall\varepsilon > 0,\ \exists s_0 \in (t_0, t_0+\varepsilon) : \mu(s_0) \nleq \lambda(t_0+) \text{ or } \lambda(s_0) \nleq \mu(t_0+).$$

(*ii*) Let $\lambda(t_0+) \neq \mu(t_0+)$ for $t_0 \in \mathbb{R}$ and $\varepsilon > 0$. From (*i*),

$$\exists s_0 \in (t_0, t_0+\varepsilon) : \mu(s_0) \nleq \lambda(t_0+) \text{ or } \lambda(s_0) \nleq \mu(t_0+).$$

Without loss of generality, let $\lambda(s_0) \nleq \mu(t_0+)$. Since λ is decreasing, $\lambda(s_0) \leq \lambda(s_0-)$. Hence,

$$\lambda(s_0-) \nleq \mu(t_0+). \tag{2.6}$$

On the other hand,

$$\{\mu(s) : t_0 < s < s_0\} \subset \{\mu(s) : t_0 < s\}.$$

Therefore,

$$\bigwedge_{t_0<s<s_0} \mu(s) \leq \bigvee_{t_0<s} \mu(s) = \mu(t_0+)$$

is obtained. So, $\bigwedge_{t_0<s<s_0} \mu(s) \leq \mu(t_0+)$. Since μ is decreasing,

$$\mu(s_0-) = \bigwedge_{s<s_0} \mu(s) = \bigwedge_{t_0<s<s_0} \mu(s) \leq \mu(t_0+).$$

Hence $\mu(s_0-) \leq \mu(t_0+)$.

From (2.6), since $\lambda(s_0-) \nleq \mu(t_0+)$, $\lambda(s_0-) \nleq \mu(s_0-)$. So, $\lambda(s_0-) \neq \mu(s_0-)$. □

The following result is obtained from Theorem 2.1(ii) and Theorem 2.2(ii).

Corollary 2.1 *Let* $(L, \leq)$ *be a complete lattice,* $\lambda, \mu \in md_{\mathbb{R}}(L)$. *Then*

$$\lambda(t_0-) \neq \mu(t_0-) \quad \textit{for a } t_0 \in \mathbb{R} \ \Leftrightarrow\ \lambda(s_0+) \neq \mu(s_0+) \quad \textit{for a } s_0 \in \mathbb{R}.$$

Theorem 2.3 *Defined as maping* $d_{right} : \mathbb{R}[L]_{right} \times \mathbb{R}[L]_{right} \to [0, +\infty)$ *is a metric on set* $\mathbb{R}[L]_{right}$.

$$d_{right}([\lambda], [\mu]) := \sup\left\{\left|\bigvee\{t : \lambda(t) \geq k\} - \bigvee\{t : \mu(t) \geq k\}\right| : k \in L\setminus\{0\}\right\}.$$

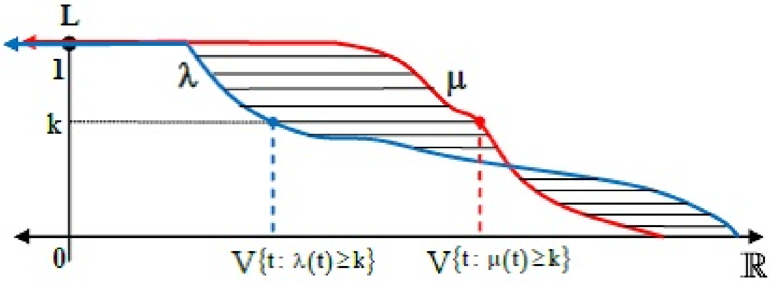

Figure 2. $\mathbb{R}[L]_{right}; d_{right}$

Proof (i) Let's show that $[\lambda] = [\mu] \Rightarrow d_{right}([\lambda], [\mu]) = 0$. Let $k \in L\setminus\{0\}$ be an arbitrary and constant. Let's define as

$$t_1 := \bigvee_{\lambda(t)\geq k} t, \quad t_2 := \bigvee_{\mu(t)\geq k} t$$

and

$$A := \{t : \lambda(t) \geq k\}, \qquad B := \{t : \mu(t) \geq k\}.$$

Then, $\exists t' \in B : t_2 - \varepsilon < t'$ for arbitrary and constant $\varepsilon > 0$. So, $\mu(t') \geq k$ and $t_2 - \varepsilon < t'$. Since $[\lambda] = [\mu]$, $\lambda(t'-) = \mu(t'-)$.

On the other hand, since μ is decreasing, $\mu(t') \leq \mu(t'-)$. Hence,

$$k \leq \mu(t') \leq \mu(t'-) = \lambda(t'-).$$

That is, $k \leq \lambda(t'-)$. On the other hand,

$$\exists t^* \in \mathbb{R} : t_2 - \varepsilon < t^* < t' \text{ for } \varepsilon > 0.$$

Since $k \leq \lambda(t'-)$ and λ is decreasing, $k \leq \lambda(t'-) = \bigwedge_{s<t'} \lambda(s) \leq \lambda(t^*)$. So, $t_2 - \varepsilon < t^*$ and $t^* \in A$ are obtained. Since $\varepsilon > 0$ is arbitrary, $t_2 \leq \bigvee A = t_1$. Hence,

$$t_2 \leq t_1. \tag{2.7}$$

Now, let's show that $t_1 \leq t_2$ and $\exists t' \in A : t_1 - \varepsilon < t'$ for arbitrary and constant $\varepsilon > 0$. So $\lambda(t') \geq k$ and $t_1 - \varepsilon < t'$. Since $[\lambda] = [\mu]$, $\lambda(t'-) = \mu(t'-)$. On the other hand, since λ is decreasing, $\lambda(t') \leq \lambda(t'-)$. So $k \leq \lambda(t') \leq \lambda(t'-) = \mu(t'-)$. That is, $k \leq \mu(t'-)$.

On the other hand,

$$\exists t^* \in \mathbb{R} : t_1 - \varepsilon < t^* < t' \text{ for } \varepsilon > 0.$$

Since, $k \leq \mu(t'-)$ and μ is decreasing $k \leq \mu(t'-) = \bigwedge_{s<t'} \mu(s) \leq \mu(t^*)$ is obtained. Hence, $t_1 - \varepsilon < t^*$ and $t^* \in B$ is obtained. Since $\varepsilon > 0$ is arbitrary, $t_1 \leq \bigvee B = t_2$. So

$$t_1 \leq t_2. \tag{2.8}$$

From (2.7) and (2.8), $t_1 = t_2$ is obtained. Since $k \in L\backslash\{0\}$ is arbitrary,

$$d_{right}([\lambda], [\mu]) = 0.$$

Conversely, let's show that $d_{right}([\lambda], [\mu]) = 0 \quad \Rightarrow \quad [\lambda] = [\mu]$. In fact, let's assume that $[\lambda] \neq [\mu]$. In this case,

$$\exists t_0 \in \mathbb{R} : \lambda(t_0-) \neq \mu(t_0-) \text{ or } \lambda(t_0+) \neq \mu(t_0+).$$

From Corollary 2.1, $\lambda(t_0-) \neq \mu(t_0-)$. Then, for $a := \mu(t_0-) = \bigwedge_{s<t_0} \mu(s)$,

$$a \nleq \bigwedge_{s<t_0} \lambda(s) \text{ or } a \ngeq \bigwedge_{s<t_0} \lambda(s)$$

is written. Without loss of generality, let's assume that $a \nleq \bigwedge_{s<t_0} \lambda(s)$. Then,

$$\exists s_0 \in \mathbb{R} : s_0 < t_0 \text{ and } a \nleq \lambda(s_0). \tag{2.9}$$

Let $A := \{t : \lambda(t) \geq a\}$ and $B := \{t : \mu(t) \geq a\}$. Then, there is the assertion $\bigvee B \geq t_0$. Notice that $\exists s' \in \mathbb{R} : t_0 - \varepsilon < s' < t_0$ for all $\varepsilon > 0$, $\mu(s') \geq \bigwedge_{s<t_0} \mu(s) = a$. Hence $\mu(s') \geq a$. So $s' \in B$. Since $\varepsilon > 0$ is arbitrary, $t_0 \leq \bigvee B$.

According to hypothesis, since

$$d_{right}([\lambda], [\mu]) = \sup\left\{\left|\bigvee\{t : \lambda(t) \geq k\} - \bigvee\{t : \mu(t) \geq k\}\right| : k \in L\setminus\{0\}\right\} = 0,$$

and $\bigvee_{\mu(t)\geq a} t = \bigvee_{\lambda(t)\geq a} t$. Hence,

$$t_0 \leq \bigvee B = \bigvee_{\mu(t)\geq a} t = \bigvee A \Rightarrow t_0 \leq \bigvee A.$$

So, $\exists t' \in A : t_0 - \varepsilon < t'$ for $\varepsilon := t_0 - s_0 > 0$. From here $a \leq \lambda(t')$ and $s_0 < t'$. Since λ is decreasing, $a \leq \lambda(t')$ and $\lambda(t') \leq \lambda(s_0)$. So $a \leq \lambda(s_0)$. This contradicts (2.9). So $[\lambda] = [\mu]$ is obtained.

(ii) It is clearly that $d_{right}([\lambda], [\mu]) = d_{right}([\mu], [\lambda])$.

(iii) Let's show that $d_{right}([\lambda], [\eta]) \leq d_{right}([\lambda], [\mu]) + d_{right}([\mu], [\eta])$ as follows:

Let's define

$$t_\lambda(k_0) := \bigvee_{\lambda(t)\geq k_0} t, \quad t_\mu(k_0) := \bigvee_{\mu(t)\geq k_0} t, \quad t_\eta(k_0) := \bigvee_{\eta(t)\geq k_0} t$$

for arbitrary $k_0 \in L\,(k_0 \neq 0)$. Then,

$$|t_\lambda(k_0) - t_\eta(k_0)| \leq |t_\lambda(k_0) - t_\mu(k_0)| + |t_\mu(k_0) - t_\eta(k_0)|.$$

Here,

$$\sup_{k\in L, k>0} |t_\lambda(k) - t_\eta(k)| \leq \sup_{k\in L, k>0} |t_\lambda(k) - t_\mu(k)| + \sup_{k\in L, k>0} |t_\mu(k) - t_\eta(k)|.$$

As a result,

$$d_{right}([\lambda], [\eta]) \leq d_{right}([\lambda], [\mu]) + d_{right}([\mu], [\eta])$$

is obtained. $\square$

Definition 2.3 *Let* $(L,\leq)$ *be a complete lattice,* $\lambda\in L^{\mathbb{R}}$ *and*

$$mi_{\mathbb{R}}(L):=\left\{\lambda\in L^{\mathbb{R}}:\bigvee_{t\in\mathbb{R}}\lambda(t)=1,\bigwedge_{t\in\mathbb{R}}\lambda(t)=0,\ \lambda \text{ is monotonous increasing}\right\}.$$

For all $\lambda\in mi_{\mathbb{R}}(L)$ *and all* $\forall t\in\mathbb{R}$ *let,*

$$\lambda(t-):=\bigvee_{s<t}\lambda(s)\quad\text{and}\quad\lambda(t+):=\bigwedge_{s>t}\lambda(s),$$

Then, an equivalence relation " $\sim$ *" on* $mi_{\mathbb{R}}(L)$ *is defined as following:*

for $\lambda,\mu\in mi_{\mathbb{R}}(L)$,

$$\lambda\sim\mu:\Leftrightarrow\forall t\in\mathbb{R},\lambda(t-)=\mu(t-),\lambda(t+)=\mu(t+)$$

and the set of equivalence classes containing λ *is defined as*

$$[\lambda]:=\{\mu\in mi_{\mathbb{R}}(L):\mu\sim\ \lambda\},$$

the set of all equivalence classes with respect to " $\sim$ *" equivalence relation on* $mi_{\mathbb{R}}(L)$ *is defined as*

$$\mathbb{R}[L]_{left}:=\{[\lambda]:\ \lambda\in mi_{\mathbb{R}}(L)\}.$$

Figure 3. $[\lambda]\in\mathbb{R}[L]_{left}$

Theorem 2.4 *Let* $(L,\leq)$ *be a complete lattice,* $\lambda,\mu\in mi_{\mathbb{R}}(L)$ *and for* $t_0\in\mathbb{R}$, $\lambda(t_0+)\neq\mu(t_0+)$. *Then,*

(*i*) *For* $\forall\varepsilon>0$, $\exists s_0\in(t_0,t_0+\varepsilon):\lambda(t_0+)\nleq\mu(s_0)$ *or* $\mu(t_0+)\nleq\lambda(s_0)$;
(*ii*) *For* $\forall\varepsilon>0$, $\exists s_0\in(t_0,t_0+\varepsilon):\lambda(s_0-)\neq\mu(s_0-)$.

Proof The proof of this theorem is similar to Theorem 2.1. □

Theorem 2.5 *Let* $(L,\leq)$ *be a complete lattice,* $\lambda,\mu\in mi_{\mathbb{R}}(L)_{left}$ *and for* $t_0\in\mathbb{R}$, $\lambda(t_0-)\neq\mu(t_0-)$. *Then,*

(*i*) *For* $\forall\varepsilon>0$, $\exists s_0\in(t_0-\varepsilon,t_0):\mu(s_0)\nleq\lambda(t_0-)$ *or* $\lambda(s_0)\nleq\mu(t_0-)$;

(*ii*) *For* $\forall\varepsilon > 0,\ \exists s_0 \in (t_0 - \varepsilon, t_0) : \mu(s_0+) \neq \lambda(s_0+)$.

Proof The proof is similar to Theorem 2.2. □

The following result can be obtained from the Theorem 2.4(*ii*) and Theorem 2.5(*ii*).

Corollary 2.2 *Let* $(L, \leq)$ *be a complete lattice,* $\lambda, \mu \in mi_{\mathbb{R}}(L)$. *Then*

$$\lambda(t_0-) \neq \mu(t_0-) \quad \text{for a } t_0 \in \mathbb{R} \Leftrightarrow \lambda(s_0+) \neq \mu(s_0+) \quad \text{for a } s_0 \in \mathbb{R}.$$

Theorem 2.6 *Defined as the mapping* $d_{left} : \mathbb{R}[L]_{left} \times \mathbb{R}[L]_{left} \to [0, +\infty)$ *is a metric on the set* $\mathbb{R}[L]_{left}$.

$$d_{left}([\lambda], [\mu]) := \sup\left\{\left|\bigwedge\{t : \lambda(t) \geq k\} - \bigwedge\{t : \mu(t) \geq k\}\right| : k \in L\backslash\{0\}\right\}.$$

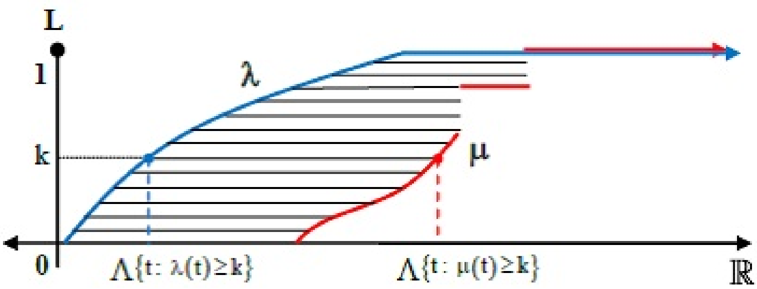

Figure 4. $\left(\mathbb{R}[L]_{left}, d_{left}\right)$

Definition 2.4 $\lambda \in L^{\mathbb{R}}$ *is called L-fuzzy number if the following conditions satisfy:*

(1) $\exists x_0 \in \mathbb{R} : \lambda(x_0) = 1$;

(2) $\forall s < s' \leq x_0,\ \lambda(s) \leq \lambda(s')$ *and* $\bigwedge_{t \leq x_0} \lambda(t) = 0$;

(3) $\forall x_0 \leq s < s',\ \lambda(s) \geq \lambda(s')$ *and* $\bigwedge_{x_0 \leq t} \lambda(t) = 0$.

The set of L-fuzzy numbers given this definition is denoted by $F\mathbb{R}[L]$.

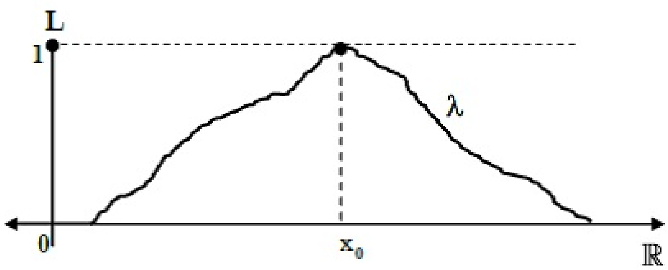

Figure 5. $[\lambda] \in F\mathbb{R}[L]$

Example 2.1 A real number $r_0 \in \mathbb{R}$ given in the classical sense is expressed as follows on the set $F\mathbb{R}[L]$:

$$\lambda(x) = \begin{cases} 1, & x = r_0 \\ 0, & x \neq r_0 \end{cases}.$$

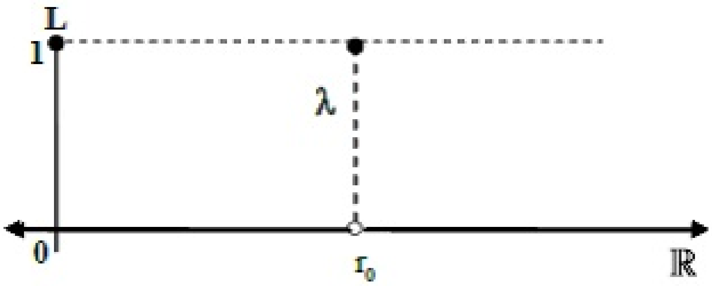

Figure 6. $r_0 \in F\mathbb{R}[L]$

After that, for ease of typing, we shall use λ instead of $[\lambda]$.

Definition 2.5 *Let $\lambda \in F\mathbb{R}[L]$. Then, mappings $\lambda_-, \lambda^- : \mathbb{R} \to L$ is defined as follows:*

$$\lambda_-(x) := \begin{cases} 1, & x < x_0 \\ \lambda(x), & x_0 \leq x \end{cases}, \qquad \lambda^-(x) := \begin{cases} \lambda(x), & x \leq x_0 \\ 1, & x_0 < x \end{cases}.$$

Theorem 2.7 *Let $\lambda, \mu \in F\mathbb{R}[L]$. Then*

$$\lambda = \mu \Leftrightarrow \lambda_- = \mu_- \;\; and \;\; \lambda^- = \mu^-.$$

Proof The "$\Rightarrow$" part is evident.

The "$\Leftarrow$" part should be $\exists x_1, x_2 \in \mathbb{R}$ such that $\lambda(x_1) = 1$ and $\mu(x_2) = 1$. Now, let

$$\lambda_-(x) = \begin{cases} 1, & x < x_1 \\ \lambda(x), & x_1 \leq x \end{cases}, \qquad \lambda^-(x) = \begin{cases} \lambda(x), & x \leq x_1 \\ 1, & x_1 < x \end{cases}$$

$$\mu_-(x) = \begin{cases} 1, & x < x_2 \\ \mu(x), & x_2 \leq x \end{cases}, \qquad \mu^-(x) = \begin{cases} \mu(x), & x \leq x_2 \\ 1, & x_2 < x \end{cases}.$$

Then

$$\lambda_-(x) = \mu_-(x) \Rightarrow \lambda(x) = \mu(x) \text{ for all } x \in \mathbb{R} \text{ satisfied } x_1 < x, \tag{2.10}$$

$$\lambda^-(x) = \mu^-(x) \Rightarrow \lambda(x) = \mu(x) \text{ for all } x \in \mathbb{R} \text{ satisfied } x < x_1. \tag{2.11}$$

This completes the proof. □

Theorem 2.8 *Let* $\lambda, \mu \in F\mathbb{R}[L]$ *and*

$$d_{right}(\lambda_-, \mu_-) = \sup\left\{\left|\bigvee\{t : \lambda_-(t) \geq k\} - \bigvee\{t : \mu_-(t) \geq k\}\right| : k \in L \backslash \{0\}\right\},$$

$$d_{left}\left(\lambda^-, \mu^-\right) = \sup\left\{\left|\bigwedge\left\{t : \lambda^-(t) \geq k\right\} - \bigwedge\left\{t : \mu^-(t) \geq k\right\}\right| : k \in L \backslash \{0\}\right\},$$

where d_{right} *and* d_{left} *are the metrics on the sets* $\mathbb{R}[L]_{right}$ *and* $\mathbb{R}[L]_{left}$ *respectively and* d *defined by*

$$d(\lambda, \mu) := \max\left\{d_{right}(\lambda_-, \mu_-), d_{left}\left(\lambda^-, \mu^-\right)\right\},$$

is a metric on $F\mathbb{R}[L]$.

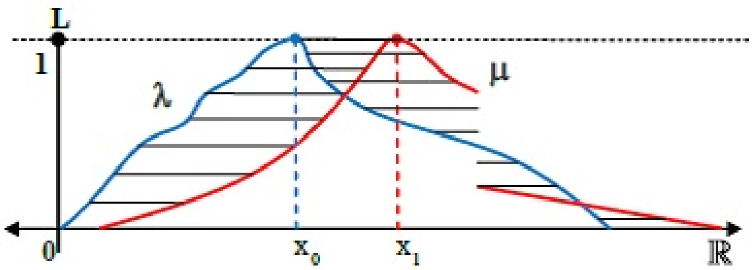

Figure 7. $(F\mathbb{R}[L], d)$

Proof (*i*) Let's show that $d(\lambda, \mu) = 0 \Leftrightarrow \lambda = \mu$.
First, the assertion "$\Rightarrow$". Let $d(\lambda, \mu) = 0$. From definition,

$$d_{right}(\lambda_-, \mu_-) = 0 \text{ and } d_{left}\left(\lambda^-, \mu^-\right) = 0.$$

Since d_{right} and d_{left} are metrics, $\lambda_- = \mu_-$ and $\lambda^- = \mu^-$. From Theorem 2.7, $\lambda = \mu$.

Second, the assertion "$\Leftarrow$". Let $\lambda = \mu$. From Theorem 2.7, $\lambda_- = \mu_-$ and $\lambda^- = \mu^-$. Since d_{right} and d_{left} are the metrics

$$d_{right}(\lambda_-, \mu_-) = 0 \quad \text{and} \quad d_{left}\left(\lambda^-, \mu^-\right) = 0.$$

Hence,

$$d(\lambda, \mu) = \max\left\{d_{right}(\lambda_-, \mu_-), d_{left}\left(\lambda^-, \mu^-\right)\right\}.$$

That is,

$$d(\lambda, \mu) = 0.$$

(*ii*) $d(\lambda, \mu) = d(\mu, \lambda)$.

(*iii*) Let's show that $d(\lambda, \eta) \leq d(\lambda, \mu) + d(\mu, \eta)$. Let

$$d(\lambda, \eta) = \max\left\{d_{right}(\lambda_-, \eta_-), d_{left}\left(\lambda^-, \eta^-\right)\right\}.$$

Without loss of generality, we can take

$$d_{right}(\lambda_-, \eta_-) \geq d_{left}\left(\lambda^-, \eta^-\right).$$

Since d_{right} is a metric and

$$d(\lambda,\eta) = d_{right}(\lambda_-,\eta_-) \leq d_{right}(\lambda_-,\mu_-) + d_{right}(\mu_-,\eta_-) \leq d(\lambda,\mu) + d(\mu,\eta),$$

following inequality is obtained:

$$d(\lambda,\eta) \leq d(\lambda,\mu) + d(\mu,\eta).$$

This completes the proof. □

Example 2.2 The real numbers $3, 7 \in \mathbb{R}$ given in the classical sense is expressed as follows on the set $F\mathbb{R}[L]$:

$$\lambda_3(x) := \begin{cases} 1, & x = 3 \\ 0, & x \neq 3 \end{cases} \quad \text{and} \quad \lambda_7(x) := \begin{cases} 1, & x = 7 \\ 0, & x \neq 7 \end{cases}$$

$\lambda_{3,}\lambda_7 \in F\mathbb{R}[L]$.

$$d(\lambda_3,\lambda_7) = \max\left\{d_{right}\left((\lambda_-)_3,(\lambda_-)_7\right), d_{left}\left(\left(\lambda^-\right)_3,\left(\lambda^-\right)_7\right)\right\} = \max\{4,4\} = 4,$$

where

$$(\lambda_-)_3(x) := \begin{cases} 1, & x \leq 3 \\ 0, & x > 3 \end{cases}, \qquad \left(\lambda^-\right)_3(x) := \begin{cases} 1, & x \geq 3 \\ 0, & x < 3 \end{cases}$$

and

$$(\lambda_-)_7(x) := \begin{cases} 1, & x \leq 7 \\ 0, & x > 7 \end{cases} \qquad \left(\lambda^-\right)_7(x) := \begin{cases} 1, & x \geq 7 \\ 0, & x < 7 \end{cases}$$

are defined. Specially, since $I = [0,1]$ is complete, $I = [0,1]$ can be taken instead of L-complete lattice. In this case, d_{right} defined in Theorem 2.3, d_{left} defined in Theorem 2.6 and d defined in Theorem 2.8 satisfy the metric conditions.

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Generating Functions of Group Codes

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Abstract: Let Σ^* is the free monoid over a finite alphabet Σ and H a subgroup of a given group G.A *group code* X is the minimal generator of X^* with $X^* = \Psi^{-1}(H)$, where Ψ is a morphism from the free monoid Σ^* to the group G. *A generating function* is just a different way of writing a sequence. Generating functions transform problems about sequence into problems about functions. In this paper, we will give a several formulas for the generating functions of $X$and X^*.

Key Words: Words and languages, free monoid, morphism of monoids, generating function.

AMS(2010): 94A60, 68Q42, 68Q70, 20M05.

§1. Introduction

Codes are an essential tool in information theory, and the theory of variable length codes is firmly related to combinatorics on words. The object of the theory is to study factorisation of words into sequences of words taken from a given set X. In a free monoid X^* generated by a code X there does not exist two distinct factorisations in X for any word. It is not always easy to verify a given set of words is a code. Some examples of the variable length codes are the Huffman coding, Limpel-Zev-Welch code and Arithmetic coding. The thoery of variable length codes takes its origin in the framework of the theory of information, since Shannon's early works in the 1950's. An algebraic theory of codes was subsequently initiated by M. P. Schutzenberger (see [17]). Variable-length codes occur frequently in the domain of data compression. Statistical data compression methods employ variable-length codes, with the short codes assigned to symbols or groups of symbols that appear more often in the data (have a higher probability of occurrence).

In this paper, we are interested in a particular type of variable lengths codes, called group codes, more precisely we describe in terms of different parameters the generating functions of the group codes and their stars.

The remainder of this paper is organized as follows. In Section 2, we introduce the notations for the rest of the paper and give basic definition of terms that will be helpful as we proceed. In Section 3, we show several formulas for the generating functions of the group codes and its

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stars. Finally, we draw our conclusions in Section 4.

§2. Preliminaries

There is an extremely powerful tool in discrete mathematics used to manipulate sequences called generating function. A generating function is just a different way of writing a sequence of numbers. Generating functions transform problems about sequence into problems about functions. This is great because we've got piles of mathematical machinery for manipulating functions. Let $(g_n)_{n\geq 0}$ be a sequence of numbers. The generating function associated to this sequence is the series $G(x) = \sum\limits_{n\geq 0} g_n x^n$. The correspondence between a sequence and its generating function with a double-sided arrow as follows:

$$\langle g_0, g_1, g_2, g_3, \cdots \rangle \longleftrightarrow G(x) = g_0 + g_1 x + g_2 x^2 + g_3 x^3 \cdots .$$

The magic of generating functions is that we can carry out all sorts of manipulations on sequences by performing mathematical operations on their associated generating functions. Let's experiment with various operations and characterize their effects in terms of sequences. Notice that,

1. If $\langle g_0, g_1, g_2, g_3, \cdots \rangle \longleftrightarrow G(x)$ and $a \in \mathbb{R}$ then,

$$\langle g_0, g_1, g_2, g_3, \cdots \rangle \longleftrightarrow G(x).$$

2. If $\langle g_0, g_1, g_2, g_3, \cdots \rangle \longleftrightarrow G(x)$ then

$$\left\langle \overbrace{0, 0, \cdots, 0}^{k\ zeroes}, g_0, g_1, g_2, g_3, \cdots, \right\rangle \longleftrightarrow x^k G(x).$$

3. If $\langle g_0, g_1, g_2, g_3, \cdots, \rangle \longleftrightarrow G(x)$ then $\langle g_1, 2g_2, 3g_3, \cdots, \rangle \longleftrightarrow G'(x)$.

4. If $\langle g_0, g_1, g_2, g_3, \cdots \rangle \longleftrightarrow G(x)$ and $\langle f_0, f_1, f_2, f_3, \cdots \rangle \longleftrightarrow F(x)$ then

$$\langle g_0 + f_0, g_1 + f_1, g_2 + f_2, g_3 + f_3, \cdots \rangle \longleftrightarrow G(x) + F(x).$$

5. If $\langle g_0, g_1, g_2, g_3, \cdots \rangle \longleftrightarrow G(x)$ and $\langle k_0, k_1, k_2, k_3, \cdots \rangle \longleftrightarrow K(x)$ then

$$\langle m_0, m_1, m_2, m_3, \cdots \rangle \longleftrightarrow G(x) K(x),$$

where

$$m_n = g_0 k_n + g_1 k_{n-1} + g_2 k_{n-2} + \cdots + g_n k_0.$$

Now, le us recall the power series expansion of $(1+x)^{\alpha}$, valid for $\alpha \in \mathbb{R}$,

$$(1+x)^{\alpha} = 1 + \alpha x + \cdots + \left(\begin{array}{c} \alpha \\ k \end{array} \right) + \cdots ,$$

where, by convention, $\left(\begin{array}{c} \alpha \\ k \end{array} \right) = \dfrac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$.

A semigroup is a pair $(S, \circ)$, where S is a set and $\circ$ an associative binary operation on S. If the set S contains an element 1_S such that $1_S \circ s = s \circ 1_S = s$ for all $s \in S$ we call $(S, \circ, 1_S)$ a monoid and refer to the element 1_S as the one ore the identity.

A semigroup morphism from a semigroup $(S, \circ)$ into a semigroup $(T, \triangle)$ is a mapping $h : S \longrightarrow T$ such that $h(u \circ v) = h(u) \triangle h(u)$.

Let X and Y be two subsets of a semigroup $(S, \circ)$. The product of X and Y is the set $X \circ Y = \{x \circ y : x \in X, y \in Y\}$. We denote by X^+ the subsemigroup generated by X, that is $X^+ = \{x_1 \circ \cdots \circ x_n : n \geq 1, x_1 \in X\}$. If S is a monoid, we also define $X^* = X^+ \cup \{1_S\}$ which is the submonoid of S generated by X.

Let A denote a finite set of symbols. The elements of A are called letters and the set A is called an alphabet. A finite word over A is a finite sequence of letters $u = (a_1, a_2, \cdots, a_n)$ of elements of A denoted by the concatenation $w = a_1 a_2 \cdots a_n$. The integer $n = |w|$ is the length of the word w. For example, the finite sequences 00110 and 110 are two words over the binary alphabet $\{0, 1\}$ with $|00110| = 5$ and $|110| = 3$. The empty sequence () of length 0 is called the empty word and is denoted by ϵ. The set A^* of all words over A equipped with the operation of concatenation has a structure of a monoid with the empty word ϵ as a neutral element, called the free monoid on A. We denote by $A^+ = A^* - \{\epsilon\}$ the free semigroup over A.

For example, $\{0, 1, 2\}^* = \{\epsilon, 0, 1, 2, 00, 01, 02, 11, 12, 20, 21, \cdots\}$. If a is a letter of the alphabet A, for any word $w = a_1 a_2 \cdots a_k$ of A^*, we denote by $|u|_a = Card\ \{i = 1, 2, \cdots, k : a_i = a\}$, the number of the occurrences of a in the word u. For example, we have $|00110|_0 = 3$ and $|00110|_1 = 2$ [8].

For $X \subset A^*$, we define $X^0 = \{\epsilon\}, X^{n+1} = X^n X\ (n \geq 0)$ and $X^* = \bigcup\limits_{n \geq 0} X^n$. Note that, any submonoid M of A^* has a unique minimal generating set $(M - \epsilon) - (M - \epsilon)^2$.

Given two words $u, w \in A^*$, we say that u is factor (prefix, suffix) of w if and only if we have $w \in A^* u A^*$ $(w \in uA^*, w \in A^* u)$. Given a subset L of A^*, we denote by $F(L)$ $(P(L), S(L))$, the set of the words are factor (prefix, suffix) of some word in L.

A homomorphism between the free monoids A^* and Γ^* is an application $h : A^* \longrightarrow B^*$ satisfying $h(uv) = h(u)h(v)$ for all $u, v \in A^*$. Note that, the homomorphism h is completely determined by the images of letters of A in B^*, i.e, $h(a)$ for any a belong to A.
For $x, y \in A^*$, we define

$$x^{-1}y = \{z \in A^* : xz = y\} \quad \text{and} \quad xy^{-1} = \{z \in A^* : x = zy\}.$$

For subsets X, Y of A^*, this notation is extended to

$$X^{-1}Y = \bigcup_{x\in X}\bigcup_{y\in Y} x^{-1}y \quad \text{and} \quad XY^{-1} = \bigcup_{x\in X}\bigcup_{y\in Y} xy^{-1}.$$

A set $X \subset A^*$ is a code if any word in X^+ can be written uniquely as a product of words in X, that is, has a unique factorisation in words in X, i.e., if for all $m, n \geq 1$ and $(x_i)_{i=1,\cdots,n}, (y_i)_{i=1,\cdots,m}$ the condition

$$x_1x_2\cdots x_n = y_1y_2\cdots y_m \text{ implies} n = m \text{ and } x_i = y_i \text{ for } i = 1, \cdots, n.$$

Any code X satisfy the Kraft inequality

$$\sum_{x\in X} (Card\,(A))^{-|x|} \leq 1$$

and for any sequence $l_1, \cdots, l_n$ of positive integers such that

$$\sum_{i=1}^{i=n} (Card\,(A))^{-l_i} \leq 1,$$

there exists a prefix code $X = \{x_1, \cdots, x_n\}$ over A such that $|x_i| = l_i$ for all $i \in \{1, \cdots, n\}$.The basic question to be asked is "When is a given subset X of A^* a variable length code?".

This was answered by Sardinas and Patterson [3]. Define recursively subsets U_n of A^* as follows:

$$\begin{cases} U_0 = X^{-1}X - \{\epsilon\} \\ U_{n+1} = U_n^{-1}X \cup X^{-1}U_n \quad \text{for} \quad n \geq 0, \end{cases}$$

where ϵ denotes the identity of A^* and $X^{-1}X = \bigcup_{x\in X} x^{-1}X$. We have

- If $\epsilon \in U_n$, then X is not a variable length code;
- If $U_{n+1} = U_n$, then X is a variable length code [1, 3].

We say that, a code X is maximal if and only if for any word $z \notin X$ the set $X \cup \{z\}$ cannot be a code.

A subset X of A^* is called prefix (suffix) if $X \cap XA^+ = \emptyset$ (resp. $X \cap XA^+ = \emptyset$). A subset X of A^* is bi-prefix if it is both suffix and prefix. A code X is complete if and only if any word of A^* is a factor of some word in X^*.

For any set $X \subset A^*$, the generating function of X is

$$G_X\,(z) = \sum_{n\geq 0} g_n z^n,$$

where

$$g_n = Card\,(X \cap A^n)\,.$$

Notice that if X is a code, then [4]

$$G_{X^*} = \frac{1}{1 - G_X}.$$

The sequence $(g_n)_{n \geq 0}$ is called the length distribution of X.

Let $X, Y \subset A^*$. If X and y are disjoint, then $G_{X \cup Y} = G_X + G_Y$. Similarly, if the product of X and Y is unambiguous, that is whenever $xy = x'y'$ with $x, x' \in X, y, y' \in Y$ imply $x = x', y = y'$, then $G_{XY} = G_X G_Y$.

§3. Group Codes and Their Generating Functions

The following propositions from [2] gives a methods to construct the group codes.

Proposition 3.1 *Let G be a group and H a subgroup of G. Let $\Psi : A^* \longrightarrow G$ be a morphism. Let $X^* = \Psi^{-1}(H)$ with X the minimal generator the set X^*. We have,*

(1) *The submoinoi X^* is unitary on the right and on the left, that is whenever $xy \in X^*, x \in X^*$ then $y \in X^*$ and that is whenever $xy \in X^*, y \in X^*$ then $x \in X^*$;*

(2) *The set X is bi-prefix code;*

(3) *If Ψ is surjective, then X is a maximal bi-prefix code;*

(4) *If X is a code, then $G_X = 1 - \frac{1}{G_{X^*}}$.*

Proof (1) Suppose that $xy \in X^*, y \in X^*$, i.e, $\Psi(xy) \in H$ and $\Psi(x) \in H$, from where $\Psi(y) = (\Psi(x))^{-1} \Psi(xy)$ is in H then $y \in X^*$, so X^* is unitary on the right. On the same way, we prove that X^* is unitary on the left.

(2) As X^* is unitary on the right and on the left, then the set X is bi-prefix code.

(3) Suppose now that Ψ is surjective. If $X^* = A^*$, then $X = A$, and hence (3) is proved. Otherwise let w be any word in $A^*, w \notin X^*$. Since Ψ is surjective ($\Psi(A^*) = G$), $\Psi(w)$ is an element of the group G, so there exists $v \in A^*$ such that $\Psi(v) = (\Psi(w))^{-1}$. The words vw and wv are in X^* since $\Psi(vw) = \Psi(wv) = 1_G$. Naturally $wvw \in (X \cup \{w\})^*$, but the word wvw admits two distinct factorizations in words of $X \cup \{w\}$, that is $wvw = w(vw) = (wv)w$. The set $X \cup \{w\}$ cannot be a code, for all $w \notin Z$. Finally X is a maximal code.

(4) As $G_{X^*} = \frac{1}{1-G_X}$, then $G_X = 1 - \frac{1}{G_{X^*}}$ ($G_{X^*} \neq 0$). □

Notation 3.2 In the last case the set X is called a group code denoted by $X(G, H)_\Psi$

Example 3.3 Consider the morphism of monoids $\Psi : \{a, b\}^* \longrightarrow (\mathbb{Z}, +)$ defined by

$$\Psi(a) = 1, \Psi(b) = -1, \Psi(\epsilon) = 0.$$

And then, $\forall w \in \{a, b\}^*$ we have $\Psi(w) = |w|_a - |w|_b$.

The mapping Ψ is surjective because $\forall m \in \mathbb{Z}, \exists w \in \{a, b\}^*$ such that $\Psi(w) = m$.

In fact, we have

(1) If $m = 0$ then $w = \epsilon$;

(2) If $m > 0$ then $w = a^m$;

(3) If $m < 0$ then $w = b^{-m}$.

Let $H = \{0\}$ the trivial subgroup of $(\mathbb{Z}, +)$. Then

$$X^* = \Psi^{-1}(\{0\}) = \left\{w \in \{a,b\}^* : |w|_a = |w|_b\right\}.$$

The set X^* are the words over $\{a,b\}$ having an equal number of occurrences of a and b is a submonoid of $\{a,b\}^*$ generated by a bi-prefix code. Since any word of X^* of length $2n$ is obtained by choosing n positions among $2n$, we have

$$G_{X^*}(z) = \sum_{n\geq 0} \binom{2n}{n} z^n.$$

Then, the sequence $\left\{\binom{2n}{n}\right\}_{n\geq 0}$ is the length distribution of X^*.

We show that

$$G_{X^*}(z) = \sum_{n\geq 0} \binom{2n}{n} z^n = \left(1-4z^2\right)^{-\frac{1}{2}}.$$

In fact, we have

$$\begin{aligned}
\left(1-4z^2\right)^{-\frac{1}{2}} &= \sum_{n\geq 0} \binom{-\frac{1}{2}}{n} \left(-4z^2\right)^n \\
&= \sum_{n\geq 0} \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!} \times (-1)^n \times (4)^n \times \left(z^2\right)^n \\
&= \sum_{n\geq 0} \frac{(-1)^n \times 1 \times 3 \times 5 \cdots (2n-1)}{2^n \times n! \times n!} \times (-1)^n \times (2)^n \times (2)^n \times n! \times \left(z^2\right)^n \\
&= \sum_{n\geq 0} \frac{(2n)!}{n! \times n!} \left(z^2\right)^n \quad \text{(note that } (2)^n \times n! = 2 \times 4 \times 6 \cdots \times 2n) \\
&= \sum_{n\geq 0} \binom{2n}{n} z^{2n}.
\end{aligned}$$

As

$$G_{X^*} = \left(1-4z^2\right)^{-\frac{1}{2}},$$

then

$$G_X(z) = \left(1 - \frac{1}{G_{X^*}}\right)(z) = 1 - \left(1-4z^2\right)^{\frac{1}{2}}.$$

We get that

$$\begin{aligned}
G_X(z) &= 1-\left(1-4z^2\right)^{\frac{1}{2}} = 1-\sum_{n\geq 0}\binom{\frac{1}{2}}{n}\left(-4z^2\right)^n \\
&= 1-\left(1+\sum_{n\geq 1}\binom{\frac{1}{2}}{n}\left(-4z^2\right)^n\right) = -\sum_{n\geq 1}\binom{\frac{1}{2}}{n}\left(-4z^2\right)^n \\
&= -\sum_{n\geq 1}\frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\cdots\left(\frac{-1}{2}-n+1\right)}{n!}\times(-1)^n\times(4)^n\times\left(z^2\right)^n \\
&= -\sum_{n\geq 1}\frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\cdots\left(\frac{-(2n-3)}{2}\right)}{n!}\times(-1)^n\times(2)^n\times(2)^n\times z^{2n} \\
&= -\sum_{n\geq 1}\frac{(-1)^n\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\cdots\left(\frac{(2n-3)}{2}\right)}{n!}\times(-1)^n\times(2)^n\times(2)^n\times z^{2n} \\
&= \sum_{n\geq 1}\frac{(1)(3)(5)\cdots(2n-3)}{n!\times n!}\times n!\times(2)^n\times z^{2n} \\
&= \sum_{n\geq 1}\frac{(2n)!}{(2n-1)\,n!\times n!}z^{2n} = \sum_{n\geq 1}\frac{2}{n}\binom{2n-2}{n-1}z^{2n}.
\end{aligned}$$

Finally the generating function of X is

$$G_X(z) = \sum_{n\geq 1}\frac{2}{n}\binom{2n-2}{n-1}z^{2n}.$$

Example 3.4 Let $\Psi : A^* \longrightarrow (\mathbb{Z}/n\mathbb{Z},\oplus)$ the morphism of monoids defined by

$\Psi(a) = \overline{1}$ *for all* $a \in A$, *and* $\Psi(\epsilon) = \overline{0}$. *And consequently,* $\forall w \in A^* : \Psi(w) = |w|\, mod(n)$.

We have the mapping Ψ is surjective because $\forall \overline{m} \in \mathbb{Z}/n\mathbb{Z}$, the word $w = \sigma^m \in A^*$, for all $\sigma \in A$, satisfies the condition $\Psi(\sigma^m) = \overline{m}$. And if $X^* = \Psi^{-1}(\{\overline{0}\}) = \{w \in A^* : |w| \equiv 0 \text{mod } (n)\}$ then, $X = A^n$. We have

$$G_X(z) = \sum_{n\geq 0} g_n z^n,$$

where $g_n = Card\,(A^n) = (Card\,(A))^n$, then

$$G_X(z) = \sum_{n\geq 0}(Card\,(A))^n z^n = \frac{1}{1-(Card\,(A))^n z}.$$

The generating series of X^* is

$$G_{X^*} = \frac{1}{1-G_X} = \frac{1}{1-\frac{1}{1-(Card(A))^n z}} = \frac{(Card\,(A))^n z - 1}{(Card\,(A))^n z}.$$

Proposition 3.5 *Let* $X(G,H)_\Psi$ *be an arbitrary group code. If the morphism* Ψ *is surjective, then* $X(G,H)_\Psi$ *is complete.*

Proof We show that any word of A^* is a factor of some word in X^*. Let $w \in A^*$, the word w is a factor of $uwv \in X^*$, where $\Psi(u) = (\Psi(w))^{-1}$ and $\Psi(v) = \Psi(\epsilon) = 1_G$. Consequently we obtain $A^* = F(X^*)$. □

Example 3.6 Let $\Psi : \{a,b\}^* \longrightarrow (\mathbb{Z},+)$ defined by: $\Psi(a) = 1, \Psi(b) = -1, \Psi(\epsilon) = 0$. And then, $\forall w \in \{a,b\}^*$ we have $\Psi(w) = |w|_a - |w|_b$.

The morphism Ψ is surjective. In fact, let $X^* = \Psi^{-1}(\{0\}) = \{w \in \{a,b\}^* : |w|_a = |w|_b\}$. Any word w of $\{a,b\}^*$ is a factor of $uwv \in X^*$, where $\Psi(u) = (\Psi(w))^{-1} = -(|w|_a - |w|_b) = |w|_b - |w|_a$ and $\Psi(v) = \Psi(\epsilon) = 1_G$. otherwise the word is the factor of $uwv \in X^*$, where $|v|_a = |w|_b, |v|_b = |w|_a$ and for example $v = ab$. In this case, we have $\Psi(uwv) = 0$.

Example 3.7 Let $\Psi : A^* \longrightarrow (\mathbb{Z}/n\mathbb{Z}, \oplus)$ the morphism of monoids defined by $\Psi(a) = \overline{1}$ for all $a \in A$ and $\Psi(\epsilon) = \overline{0}$. Consequently, $\forall w \in A^* : \Psi(w) = |w| \, mod(n)$. In fact, the morphism Ψ is surjective. Let $X^* = \Psi^{-1}(\{\overline{0}\}) = \{w \in A^* : |w| \equiv 0 \bmod (n)\}$. Any word w of A^* is a factor of $uwv \in X^*$, where $\Psi(u) = (\Psi(w))^{-1} = -(|w| \, mod(n))$ and $v = \epsilon$.

§4. Conclusion

In this work, we have calculated the generating functions of some group codes and its stars.

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Generalised Sasakian-Space-Form in Submanifolds

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Abstract: In this paper, we obtain necessary and sufficient condition for an invariant submanifold of generalised sasakian space form with semi-symmetric metric connections to be totally geodesic.

Key Words: Invariant submanifolds, generalised sasakian space form, totally geodesic, semi-symmetric metric connection.

AMS(2010): 53C15, 53C21, 53C25, 53C40.

§1. Introduction

In differential geometry, Invariant submanifolds (I.S.M.) of a contact manifold have been a major area of research for long time since the concept was borrowed from complex geometry. A submanifold of a contact manifold is said to be totally geodesic if every geodesic in that submanifold is also geodesic in the ambient manifold. The generalised Sasakian space forms (G.S.S.F.) have been investigated by numerous researchers like Alegre and Carriazo [1], [2], [3]. Thereafter, (G.S.S.F.) have been study by many authors [4], [9], [10], [14], [16], [19]. The conception of a semi-symmetric metric connection(S.S.M.C.) on a Riemannian manifold is introduced by H. A. Hayden [15] and studied by various authors [17], [18], [33] and [34]. Submanifolds of a Riemannian manifold with S.S.M.C. was studied by Z. Nakao [22] and I.S.M. which was established by B. Y. Chen [11], [12] and [13].

In this paper, we procure essential and competent condition for an I.S.M. of G.S.S.F with S.S.M.C.to be totally geodesic. We have considered many geometrical conditions by using

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different curvature tensors such as concircular, Weyl and Conformal curvature tensor on I.S.M. of G.S.S.F. with S.S.M.C.

An almost contact metric manifold $\overline{M}$ is called G.S.S.F if

$$\begin{aligned}\overline{R}(X,Y)Z =& f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y \\ &- g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \end{aligned} \quad (1.1)$$

for all vector fields X, Y, Z on $\overline{M}$, where $\overline{R}$ is the curvature tensor of $\overline{M}$ of dimension $(2n+1)$. It is indicated as

$$\overline{M}^{2n+1}(f_1, f_2, f_3), \quad f_1 = \frac{c+3}{4}, \quad f_2 = f_3 = \frac{c-1}{4}.$$

For readers who are unfamiliar with terminology, notations, recent overviews and introductions, we suggest the auhthors to refer the papers [5, 6, 7, 8, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32].

§2. **Preliminaries**

Let $(\overline{M})$ be a $(2n+1)$ dimensional manifold equipped with almost contact metric structure (ϕ, ξ, η, g) consisting of a (1,1) tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying

$$\eta(\xi) = 1, \ \eta(X) = g(X, \xi), \ \phi^2 X = -X + \eta(X)\xi, \ \phi\xi = 0, \quad (2.1)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2.2)$$

$$g(\phi X, Y) + g(X, \phi Y) = 0, \ \eta(\phi X) = 0, \quad (2.3)$$

for all vector fields X, Y.

In a G.S.S.F $\overline{M}^{2n+1}(f1_1, f_2, f_3)$, the following hold:

$$(\overline{\nabla}_X \phi)Y = (f_1 - f_3)[g(X, Y)\xi - \eta(Y)X], \quad (2.4)$$

$$\overline{\nabla}_X \xi = -(f_1 - f_3)\phi X, \quad (2.5)$$

$$\overline{S}(X, Y) = (2nf_1 + 3f_2 - f_3)g(X, Y) - \{3f_2 + (2n-1)f_3\}\eta(X)\eta(Y) \quad (2.6)$$

for all X, Y, Z on $\overline{M}^{2n+1}$ and $\overline{\nabla}$ is the Levi-Civita connection on $\overline{M}$ and $\overline{S}$ is the Ricci tensor and $\overline{r}$ is the scalar curvature of $\overline{M}$.

Let M be a submanifold immersed in a $(2n+1)$ dimensional contact metric manifold $\overline{M}$ induced with metric g. TM is the tangent bundle of the manifold M and $T^\perp M$ is the set of vector fields normal to M.

Gauss and Weingarten formula are given by,

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \overline{\nabla}_X N = \nabla^\perp_X N - A_N X, \quad (2.7)$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$, where $\nabla^{\perp}$ is the connection in the normal bundle. The second fundamental form h and A_N are related by

$$g(A_N X, Y) = g(h(X, Y), N) \tag{2.8}$$

for any $X, Y \in \Gamma(TM)$, $N \in T^{\perp}M$.

If $h = 0$, then the submanifold is said to be totally geodesic , which implies that the geodesics in M are also geodesics in $\overline{M}$. Also, we indicate $Q(E,T)$ a $(0, k+2)$-type tensor field interpret as follows

$$\begin{aligned} Q(E,T)(X_1, X_2, \ldots, X_k; X, Y) &= -T((X \wedge_E Y)X_1, X_2, \ldots, X_k) \\ - T(X_1, (X \wedge_E Y)X_2, \cdots, X_k) &- \cdots - T(X_1, X_2, \cdots, X_{k-1}, (X \wedge_E Y)X_k), \end{aligned} \tag{2.9}$$

where $(X \wedge_E Y)Z = E(Y, Z)X - E(X, Z)Y$.

A submanifold is said to be pseudo-parallel if

$$\overline{R}(X, Y) \cdot h = fQ(g, h). \tag{2.10}$$

In an (I.S.M.) of a (G.S.S.F.) N is identically zero. We have

$$h(X, \xi) = 0 \tag{2.11}$$

for any vector field X tangent to M. In a $(2n+1)$ dimensional Riemannian manifold, The concircular curvature tensor $\overline{C}$, Weyl curvature tensor $\overline{W}$ and Conformal curvature tensor $\overline{V}$ are given by,

$$\overline{C}(X, Y)Z = \overline{R}(X, Y)Z - \left(\frac{\overline{r}}{2n(2n+1)}\right)[g(Y, Z)X - g(X, Z)Y], \tag{2.12}$$

$$\overline{W}(X, Y)Z = \overline{R}(X, Y)Z - \frac{1}{2n}[\overline{S}(Y, Z)X - \overline{S}(X, Z)Y], \tag{2.13}$$

$$\begin{aligned} \overline{V}(X, Y)Z =& \overline{R}(X, Y)Z - \frac{1}{2n-1}[\overline{S}(Y, Z)X - \overline{S}(X, Z)Y + g(Y, Z)\overline{Q}X \\ &- g(X, Z)\overline{Q}Y] + \frac{\overline{r}}{2n(2n-1)}[g(Y, Z)X - g(X, Z)Y]. \end{aligned} \tag{2.14}$$

A semi-symmetric connection $\tilde{\overline{\nabla}}$ is called S.S.M.C. if it satisfies $\tilde{\overline{\nabla}}g = 0$.

The connection among the S.S.M.C. $\tilde{\overline{\nabla}}$ and the Riemannian connection $\overline{\nabla}$ of a G.S.S.F. $\overline{M}^{2n+1}(f_1, f_2, f_3)$ is given by

$$\tilde{\overline{\nabla}}_X Y = \overline{\nabla}_X Y + \eta(Y)X - g(X, Y)\xi. \tag{2.15}$$

If $\overline{R}$ and $\tilde{\overline{R}}$ are the Riemannian Curvature tensor of G.S.S.F. $\overline{M}^{2n+1}(f_1, f_2, f_3)$ with respect to Levi-civita connection and S.S.M.C. , then

$$\tilde{\overline{R}}(X,Y)Z = \overline{R}(X,Y)Z - \alpha(Y,Z)X + \alpha(X,Z)Y + g(Y,Z)JX + g(X,Z)JY, \tag{2.16}$$

where α is a $(0,2)$ tensor field given by,

$$\alpha(X,Y) = (\tilde{\overline{\nabla}}_X \eta)Y + \frac{1}{2}g(X,Y), \tag{2.17}$$

$$g(JX,Y) = g(\tilde{\overline{\nabla}}_X \xi, Y) + \frac{1}{2}g(X,Y) = \alpha(X,Y), \tag{2.18}$$

$$\tilde{\overline{S}}(X,Y) = \overline{S}(X,Y) - (2n-1)\alpha(X,Y) - cg(X,Y), \tag{2.19}$$

where $c = trace(\alpha)$, $\tilde{\overline{S}}$, $\tilde{\overline{r}}$ and $\overline{S}$,$\overline{r}$ are the Ricci tensor and scalar curvature with respect to S.S.M.C. $\tilde{\overline{\nabla}}$ and $\overline{M}^{2n+1}(f_1, f_2, f_3)$ with respect to Levi-civita connection respectively.

§3. Invariant Submanifolds of Generalised Sasakian Space Form Satisfying $\overline{C}(X,Y) \cdot h = fQ(g,h)$

Theorem 3.1 *Let M be an I.S.M. of a G.S.S.F. $\overline{M}$ with semi-symmetric metric connection. Then M satisfies $\overline{C}(X,Y) \cdot h = fQ(g,h)$ iff. M is totally geodesic provided*

$$f \neq \left[\left(f_1 - f_3 - \frac{1}{2}\right) + \phi(f_1 - f_3) - \frac{3}{2} - \left(\frac{r}{2n(2n+1)}\right)\right]. \tag{3.1}$$

Proof Let M be an I.S.M. of a G.S.S.F. with semi-symmetric metric connection satisfying

$$\overline{C}(X,Y) \cdot h = fQ(g,h), \tag{3.2}$$

Notice that (3.2) can be written as,

$$R^{\perp}(X,Y)h(U,V) - h(\overline{C}(X,Y)U,V) - h(U,\overline{C}(X,Y)V) = -f[h((X \wedge_g Y),V) + h(U,(X \wedge_g Y)V)]. \tag{3.3}$$

Using (3.3) and also putting $X = V = \xi$, we get,

$$R^{\perp}(\xi,Y)h(U,\xi) - h(\overline{C}(\xi,Y)U,\xi) - h(U,\overline{C}(\xi,Y)\xi) = -f[g(Y,U)h(\xi,\xi) - g(\xi,U)h(Y,\xi) + g(Y,\xi)h(U,\xi) - g(\xi,\xi)h(U,Y)]. \tag{3.4}$$

Applying (2.11) in (3.4), we acquire,

$$-h(\overline{C}(\xi,Y)U,\xi) - h(U,\overline{C}(\xi,Y)\xi) = f[h(U,Y)]. \tag{3.5}$$

By virtue of (2.12), (2.11), (2.15), (2.16), (2.17) (2.18) and (2.19), we obtain

$$h(\overline{C}(\xi, Y)U, \xi) = 0 \tag{3.6}$$

and

$$-h(U, \overline{C}(\xi, Y)\xi) = \Big[\Big(f_1 - f_3 - \frac{1}{2}\Big) + \phi(f_1 - f_3) - \frac{3}{2} - \Big(\frac{r}{2n(2n+1)}\Big)\Big] h(U, Y). \tag{3.7}$$

Substituting (3.6) and (3.7) in (3.5) we get

$$\Big[\Big(f_1 - f_3 - \frac{1}{2}\Big) + \phi(f_1 - f_3) - \frac{3}{2} - \Big(\frac{r}{2n(2n+1)}\Big)\Big] h(U, Y) = f[h(U, Y)]. \tag{3.8}$$

That is, $h(U, Y) = 0$ implies M is totally geodesic provided,

$$f \neq \Big[\Big(f_1 - f_3 - \frac{1}{2}\Big) + \phi(f_1 - f_3) - \frac{3}{2} - \Big(\frac{r}{2n(2n+1)}\Big)\Big]. \tag{3.9}$$

Conversely, If M is totally geodesic, then we obtain M fulfilling $\overline{C}(X, Y) \cdot h = fQ(g, h)$. This completes the proof. □

§4. Invariant Submanifolds of Generalised Sasakian Space Form Satisfying $\overline{W}(X, Y) \cdot h = fQ(g, h)$

Theorem 4.1 *Let M be an I.S.M. of a G.S.S.F. $\overline{M}$ with semi-symmetric connection. Then M satisfies $\overline{W}(X, Y) \cdot h = fQ(g, h)$ iff M is totally geodesic, provided,*

$$f \neq \Big[\Big(f_1 - f_3 - \frac{1}{2}\Big) + \phi(f_1 - f_3) - \frac{3}{2} - \frac{1}{2n}\Big(2n(f_1 - f_3) - c - n + \frac{1}{2}\Big)\Big]. \tag{4.1}$$

Proof Let M be an I.S.M. of a G.S.S.F. with semi-symmetric connection satisfying

$$\overline{W}(X, Y) \cdot h = fQ(g, h), \tag{4.2}$$

Notice that (3.11) which follows as,

$$R^{\perp}(X, Y)h(U, V) - h(\overline{W}(X, Y)U, V) - h(U, \overline{W}(X, Y)V = -f[h((X \wedge_g Y), V) + h(U, (X \wedge_g Y)V)]. \tag{4.3}$$

Taking $X = V = \xi$ and using (2.9) we obtain,

$$\begin{aligned} &R^{\perp}(\xi, Y)h(U, \xi) - h(\overline{W}(\xi, Y)U, \xi) - h(U, \overline{W}(\xi, Y)\xi) \\ &= -f[g(Y, U)h(\xi, \xi) - g(\xi, U)h(Y, \xi) + g(Y, \xi)h(U, \xi) - g(\xi, \xi)h(U, Y)]. \end{aligned} \tag{4.4}$$

Putting (2.11) in (4.4) we get,

$$-h(\overline{W}(\xi, Y)U, \xi) - h(U, \overline{W}(\xi, Y)\xi) = f[h(U, Y)]. \tag{4.5}$$

By virtue of (2.13),(2.11), (2.15), (2.16), (2.17), (2.18) and (2.19), we get

$$h(\overline{W}(\xi, Y)U, \xi) = 0 \tag{4.6}$$

and

$$\begin{aligned} -h(U, \overline{W}(\xi, Y)\xi) = \Bigg[&\left(f_1 - f_3 - \frac{1}{2}\right) + \phi(f_1 - f_3) - \frac{3}{2} \\ &- \frac{1}{2n}\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right)\Bigg] h(U, Y). \end{aligned} \tag{4.7}$$

Substituting (4.6) and (4.7) in (4.5) we get,

$$\begin{aligned} &\left(f_1 - f_3 - \frac{1}{2}\right)\phi(f_1 - f_3) - \frac{3}{2} - \frac{1}{2n}(2n(f_1 - f_3) - c - n \\ &\qquad + \frac{1}{2})h(U, Y) = f[h(U, Y). \end{aligned} \tag{4.8}$$

That is, $h(U, Y) = 0$ implies M is totally geodesic provided,

$$f \neq \left[\left(f_1 - f_3 - \frac{1}{2}\right) + \phi(f_1 - f_3) - \frac{3}{2} - \frac{1}{2n}\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right)\right]. \tag{4.9}$$

Conversely, If M is totally geodesic, then we get M satisfies $\overline{W}(X, Y) \cdot h = fQ(g, h)$. This completes the proof. □

§5. Invariant Submanifolds of Generalised Sasakian Space Form Satisfying $\overline{V}(X, Y) \cdot h = fQ(g, h)$

Theorem 5.1 *Let M be an I.S.M. of a G.S.S.F. $\overline{M}$ with semi-symmetric connection. Then, M satisfies $\overline{V}(X, Y) \cdot h = fQ(g, h)$ iff M is totally geodesic, provided,*

$$\begin{aligned} f \neq &\left(f_1 - f_3 - \frac{1}{2}\right) + \phi(f_1 - f_3) - \frac{3}{2} \\ &- \frac{2}{2n-1}\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right) + \left(\frac{r}{2n(2n-1)}\right). \end{aligned} \tag{5.1}$$

Proof Let M be an I.S.M. of a G.S.S.F. with semi-symmetric connection satisfying

$$\overline{V}(X,Y)\cdot h = fQ(g,h). \tag{5.2}$$

Notice that (5.2) can be written as,

$$\begin{aligned} &R^{\perp}(X,Y)h(U,V) - h(\overline{V}(X,Y)U,V) - h(U,\overline{V}(X,Y)V) \\ &\qquad = -f[h((X\wedge_g Y),V) + h(U,(X\wedge_g Y)V)]. \end{aligned} \tag{5.3}$$

Putting $X = V = \xi$ and using (2.9) we get,

$$\begin{aligned} &R^{\perp}(\xi,Y)h(U,\xi) - h(\overline{V}(\xi,Y)U,\xi) - h(U,\overline{V}(\xi,Y)\xi) \\ &\quad = -f[g(Y,U)h(\xi,\xi) - g(\xi,U)h(Y,\xi) + g(Y,\xi)h(U,\xi) - g(\xi,\xi)h(U,Y)]. \end{aligned} \tag{5.4}$$

Substituting (2.11) in (5.4) we obtain,

$$-h(\overline{V}(\xi,Y)U,\xi) - h(U,\overline{V}(\xi,Y)\xi) = f[h(U,Y)]. \tag{5.5}$$

By virtue of (2.14), (2.15), (2.16), (2.17), (2.18) and (2.19) we get

$$h(\overline{V}(\xi,Y)U,\xi) = 0 \tag{5.6}$$

and

$$\begin{aligned} -h(U,\overline{V}(\xi,Y)\xi) = &\Bigg[\left(f_1 - f_3 - \frac{1}{2}\right) + \phi(f_1 - f_3) - \frac{3}{2} \\ &- \frac{2}{2n-1}\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right) \\ &+ \left(\frac{r}{2n(2n-1)}\right)\Bigg] h(U,Y). \end{aligned} \tag{5.7}$$

Substituting (5.6) and (5.7) in (5.5) we get,

$$\begin{aligned} &\Bigg[\left(f_1 - f_3 - \frac{1}{2}\right) + \phi(f_1 - f_3) - \frac{3}{2} - \frac{2}{2n-1}\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right) \\ &\qquad + \left(\frac{r}{2n(2n-1)}\right)\Bigg] h(U,Y) = f[h(U,Y)]. \end{aligned} \tag{5.8}$$

That is, $h(U,Y) = 0$ implies M is totally geodesic provided,

$$\begin{aligned} f \neq &\left(f_1 - f_3 - \frac{1}{2}\right) + \phi(f_1 - f_3) - \frac{3}{2} - \frac{2}{2n-1}\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right) \\ &\qquad + \left(\frac{r}{2n(2n-1)}\right). \end{aligned} \tag{5.9}$$

Conversely, If M is totally geodesic, then we obtain M comply with

$$\overline{V}(X,Y)\cdot h = fQ(g,h).$$

□

§6. Invariant Submanifolds of Generalised Sasakian Space Form Satisfying $\overline{C}(X,Y)\cdot h = fQ(S,h)$

Theorem 6.1 *Let M be an I.S.M. of a G.S.S.F. $\overline{M}$ with semi-symmetric connection. Then M satisfies $\overline{C}(X,Y)\cdot h = fQ(S,h)$ iff. M is totally geodesic provided,*

$$f \neq \frac{1}{2n(f_1-f_3)-c-n+\frac{1}{2}}\left[\left(f_1-f_3-\frac{1}{2}\right)+\phi(f_1-f_3) - \frac{3}{2}-\left(\frac{r}{2n(2n+1)}\right)\right]. \tag{6.1}$$

Proof Let M be an I.S.M. of a G.S.S.F. with semi-symmetric connection satisfying

$$\overline{C}(X,Y)\cdot h = fQ(S,h). \tag{6.2}$$

Notice that (6.1) can be written as

$$R^{\perp}(X,Y)h(U,V) - h(\overline{C}(X,Y)U,V) - h(U,\overline{C}(X,Y)V) = -f[h((X\wedge_S Y),V) + h(U,(X\wedge_S Y)V)]. \tag{6.3}$$

Putting $X=V=\xi$ and using (2.9) we get,

$$R^{\perp}(\xi,Y)h(U,\xi) - h(\overline{C}(\xi,Y)U,\xi) - h(U,\overline{C}(\xi,Y)\xi) = -f[\tilde{\overline{S}}(Y,U)h(\xi,\xi) - \tilde{\overline{S}}(\xi,U)h(Y,\xi) + \tilde{\overline{S}}(Y,\xi)h(U,\xi) - \tilde{\overline{S}}(\xi,\xi)h(U,Y)]. \tag{6.4}$$

Substituting (2.11), (2.12) in (6.4) we obtain,

$$\left[\left(f_1-f_3-\frac{1}{2}\right)+\phi(f_1-f_3)-\frac{3}{2}-\left(\frac{r}{2n(2n+1)}\right)-f(\tilde{\overline{S}}(\xi,\xi))\right]h(U,Y)=0. \tag{6.5}$$

That is, $h(U,Y)=0$ implies M is totally geodesic provided,

$$f \neq \frac{1}{2n(f_1-f_3)-c-n+\frac{1}{2}}\left[\left(f_1-f_3-\frac{1}{2}\right) + \phi(f_1-f_3)-\frac{3}{2}-\left(\frac{r}{2n(2n+1)}\right)\right]. \tag{6.6}$$

This completes the proof. □

§7. Invariant Submanifolds of Generalised Sasakian Space Form Satisfying $\overline{W}(X,Y).h = fQ(S,h)$

Theorem 7.1 *Let M be an I.S.M. of a G.S.S.F. $\overline{M}$ with semi-symmetric connection. Then M satisfies $\overline{W}(X,Y)\cdot h = fQ(S,h)$ iff M is totally geodesic provided,*

$$f \neq \frac{1}{\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right)}\left[\left(f_1 - f_3 - \frac{1}{2}\right) + \phi(f_1 - f_3) - \frac{3}{2}\right] - \frac{1}{2n}. \tag{7.1}$$

Proof Let M be an I.S.M. of a G.S.S.F. with semi-symmetric connection satisfying

$$\overline{W}(X,Y).h = fQ(S,h). \tag{7.2}$$

We have

$$R^{\perp}(X,Y)h(U,V) - h(\overline{W}(X,Y)U,V) - h(U,\overline{W}(X,Y)V)$$
$$= -f[h((X \wedge_S Y),V) + h(U,(X \wedge_S Y)V)]. \tag{7.3}$$

Taking $X = V = \xi$ and using (2.9) we acquire

$$R^{\perp}(\xi,Y)h(U,\xi) - h(\overline{W}(\xi,Y)U,\xi) - h(U,\overline{W}(\xi,Y)\xi)$$
$$= -f[\tilde{\overline{S}}(Y,U)h(\xi,\xi) - \tilde{\overline{S}}(\xi,U)h(Y,\xi) + \tilde{\overline{S}}(Y,\xi)h(U,\xi) - \tilde{\overline{S}}(\xi,\xi)h(U,Y)]. \tag{7.4}$$

Substituting (2.11), (2.13) in (7.4) we obtain

$$\left[\left(f_1 - f_3 - \frac{1}{2}\right) + \phi(f_1 - f_3) - \frac{3}{2} - \frac{1}{2n}\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right)\right.$$
$$\left. - f(\tilde{\overline{S}}(\xi,\xi))\right]h(U,Y) = 0. \tag{7.5}$$

We now have $h(U,Y) = 0$ implies M^{2n+1} is totally geodesic provided,

$$f \neq \frac{1}{\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right)}\left[\left(f_1 - f_3 - \frac{1}{2}\right) + \phi(f_1 - f_3) - \frac{3}{2}\right] - \frac{1}{2n}. \tag{7.6}$$

This completes the proof. □

§8. Invariant Submanifolds of Generalised Sasakian Space Form Satisfying $\overline{V}(X,Y).h = fQ(S,h)$

Theorem 8.1 *Let M be an I.S.M. of a G.S.S.F. $\overline{M}$ with semi-symmetric connection. Then*

M satisfies $\overline{V}(X,Y).h = fQ(S,h)$ *iff. M is totally geodesic provided,*

$$f \neq \frac{1}{\left(2n(f_1-f_3)-c-n+\frac{1}{2}\right)}\Bigg[\left(f_1-f_3-\frac{1}{2}\right)+\phi(f_1-f_3)-\frac{3}{2} + \frac{r}{2n(2n-1)}\Bigg]-\frac{2}{2n-1}. \tag{8.1}$$

Proof Let M be an I.S.M. of a G.S.S.F. with semi-symmetric connection satisfying

$$\overline{V}(X,Y).h = fQ(S,h). \tag{8.2}$$

Notice that (8.2) can be written as

$$R^{\perp}(X,Y)h(U,V)-h(\overline{V}(X,Y)U,V)-h(U,\overline{V}(X,Y)V) = -f[h((X\wedge_S Y),V)+h(U,(X\wedge_S Y)V)]. \tag{8.3}$$

Putting $X = V = \xi$ and using (2.9), we have

$$R^{\perp}(\xi,Y)h(U,\xi)-h(\overline{V}(\xi,Y)U,\xi)-h(U,\overline{V}(\xi,Y)\xi) = -f[\tilde{\tilde{S}}(Y,U)h(\xi,\xi)-\tilde{\tilde{S}}(\xi,U)h(Y,\xi)+\tilde{\tilde{S}}(Y,\xi)h(U,\xi)-\tilde{\tilde{S}}(\xi,\xi)h(U,Y)]. \tag{8.4}$$

By putting (2.14), (2.11) and (2.19) in (8.4) we get

$$\Bigg[\left(f_1-f_3-\frac{1}{2}\right)+\phi(f_1-f_3)-\frac{3}{2}-\frac{2}{2n-1}\left(2n(f_1-f_3)-c-n+\frac{1}{2}\right) + \frac{r}{2n(2n-1)}-f(\tilde{\tilde{S}}(\xi,\xi))\Bigg]h(U,Y)=0. \tag{8.5}$$

That is, $h(U,Y) = 0$ implies M is totally geodesic provided

$$f \neq \frac{1}{\left(2n(f_1-f_3)-c-n+\frac{1}{2}\right)}\Bigg[\left(f_1-f_3-\frac{1}{2}\right)+\phi(f_1-f_3)-\frac{3}{2} + \frac{r}{2n(2n-1)}\Bigg]-\frac{2}{2n-1}. \tag{8.6}$$

This completes the proof. □

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On Block-Line Forest Signed Graphs

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Abstract: In this paper we introduced the new notion called block-line forest signed graph of a signed graph and its properties are studied. Also, we obtained the structural characterization of this new notion and presented some switching equivalent characterizations.

Key Words: Signed graphs, balance, switching, block-line forest signed graph.

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§1. Introduction

For standard terminology and notion in graph theory, we refer the reader to the text-book of Harary [1]. The non-standard will be given in this paper as and when required.

Given a graph $G = (V, E)$, the block-line forest graph of $G = (V, E)$, denoted $\mathcal{BLFG}(G)$, is defined to be that graph with $V(\mathcal{BLFG}(G)) = E(G) \cup B$, where B is set of blocks of G, and any two vertices in $V(\mathcal{BLFG}(G))$ are joined by an edge if, and only if, one corresponds to a block of G and other to a line incident with it (see [4]).

To model individuals' preferences towards each other in a group, Harary [2] introduced the concept of signed graphs in 1953. A signed graph $S = (G, \sigma)$ is a graph $G = (V, E)$ whose edges are labeled with positive and negative signs (i.e., $\sigma : E(G) \to \{+, -\}$). The vertices of a graph represent people and an edge connecting two nodes signifies a relationship between individuals. The signed graph captures the attitudes between people, where a positive (negative edge) represents liking (disliking). An unsigned graph is a signed graph with the signs removed. Similar to an unsigned graph, there are many active areas of research for signed graphs.

The sign of a cycle (this is the edge set of a simple cycle) is defined to be the product of the signs of its edges; in other words, a cycle is positive if it contains an even number of negative edges and negative if it contains an odd number of negative edges. A signed graph S is said to be balanced if every cycle in it is positive. A signed graph S is called totally unbalanced if every cycle in S is negative. A chord is an edge joining two non adjacent vertices in a cycle.

A *marking* of S is a function $\zeta : V(G) \to \{+, -\}$. Given a signed graph S one can easily

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define a marking ζ of S as follows: For any vertex $v \in V(S)$,

$$\zeta(v) = \prod_{uv \in E(S)} \sigma(uv),$$

the marking ζ of S is called *canonical marking* of S. For more new notions on signed graphs refer the papers (see [5, 9-22]).

The following are the fundamental results about balance, the second being a more advanced form of the first. Note that in a bipartition of a set, $V = V_1 \cup V_2$, the disjoint subsets may be empty.

Theorem 1.1 *A signed graph S is balanced if and only if either of the following equivalent conditions is satisfied:*

(i)(Harary [2]) *Its vertex set has a bipartition $V = V_1 \cup V_2$ such that every positive edge joins vertices in V_1 or in V_2, and every negative edge joins a vertex in V_1 and a vertex in V_2;*

(ii)(Sampathkumar [6]) *There exists a marking μ of its vertices such that each edge uv in Γ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$..*

Switching S with respect to a marking ζ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs.

Two signed graphs $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ are said to be *weakly isomorphic* (see [24]) or *cycle isomorphic* (see [25]) if there exists an isomorphism $\phi : G_1 \to G_2$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known (see [25]).

Theorem 1.2(T. Zaslavsky, [25]) *Given a graph G, any two signed graphs in $\psi(G)$, where $\psi(G)$ denotes the set of all the signed graphs possible for a graph G, are switching equivalent if and only if they are cycle isomorphic.*

§2. Block-Line Forest Signed Graph of a Signed Graph

Motivated by the existing definition of complement of a signed graph, we now extend the notion of block-line forest graphs to signed graphs as follows: The *block-line forest signed graph* $\mathcal{BLFS}(S) = (\mathcal{BLFG}(G), \sigma')$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $\mathcal{BLFG}(G)$ and sign of any edge uv is $\mathcal{BLFS}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called a block-line forest signed graph, if $S \cong \mathcal{BLFS}(S')$ for some signed graph S'. The following result restricts the class of block-line forest signed graphs.

Theorem 2.1 *For any signed graph $S = (G, \sigma)$, its block-line forest signed graph $\mathcal{BLFS}(S)$ is balanced.*

Proof Since sign of any edge $e = uv$ in $\mathcal{BLFS}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S, by Theorem 1.1, $\mathcal{BLFS}(S)$ is balanced. □

For any positive integer k, the k^{th} iterated line-block signed graph, $\mathcal{BLFS}^k(S)$ of S is defined as follows:

$$\mathcal{BLFS}^0(S) = S, \quad \mathcal{BLFS}^k(S) = \mathcal{BLFS}(\mathcal{BLFS}^{k-1}(S)).$$

Corollary 2.2 *For any signed graph $S = (G, \sigma)$ and for any positive integer k, $\mathcal{BLFS}^k(S)$ is balanced.*

Corollary 2.3 *For any two signed graphs S_1 and S_2 with the same underlying graph, $\mathcal{BLFS}(S_1) \sim \mathcal{BLFS}(S_2)$.*

In [23], Swamy et al. defined the line-block signed graph of a signed graph as follows:

The line-block signed graph $\mathcal{LBS}(S) = (\mathcal{LBG}(G), \sigma')$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $\mathcal{LBG}(G)$ and sign of any edge uv is $\mathcal{LBS}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S.

Further, a signed graph $S = (G, \sigma)$ is called a line-block signed graph, if $S \cong \mathcal{LBS}(S')$ for some signed graph S'. The following result restricts the class of line-block signed graphs.

Theorem 2.4(Swamy et al., [24]) *For any signed graph $S = (G, \sigma)$, its line-block signed graph $\mathcal{LBS}(S)$ is balanced.*

In [4], the authors remarked that $\mathcal{BLFG}(G)$ and $\mathcal{LBG}(G)$ are isomorphic if and only if G is a block. We now characterize the signed graphs such that the block-line forest signed graphs and its line-block signed graphs are cycle isomorphic.

Theorem 2.5 *For any connected signed graph $S = (G, \sigma)$, $\mathcal{BLFS}(S) \sim \mathcal{LBS}(S)$ if and only if G is a block.*

Proof Suppose $\mathcal{BLFS}(S) \sim \mathcal{LBS}(S)$. This implies, $\mathcal{BLFG}(G) \cong \mathcal{LBG}(G)$ and hence G is a block.

Conversely, suppose that G is a block. Then $\mathcal{BLFG}(G) \cong \mathcal{LBG}(G)$. Now, if S any signed graph with G is a block, By Theorem 2.1 and 2.4, $\mathcal{BLFS}(S)$ and $\mathcal{LBS}(S)$ are balanced and hence, the result follows from Theorem 1.2. This completes the proof. □

The following result characterize signed graphs which are block-line forest signed graphs.

Theorem 2.6 *A signed graph $S = (G, \sigma)$ is a block-line forest signed graph if, and only if, S is balanced signed graph and its underlying graph G is a block-line forest graph.*

Proof Suppose that S is balanced and G is a block-line forest graph. Then there exists a graph G' such that $\mathcal{BLFG}(G') \cong G$. Since S is balanced, by Theorem 1.1, there exists a marking ζ of G such that each edge uv in S satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. Now consider the signed graph $S' = (G', \sigma')$, where for any edge e in G', $\sigma'(e)$ is the marking of the corresponding vertex in G. Then clearly, $\mathcal{BLFS}(S') \cong S$. Hence S is a block-line forest signed graph.

Conversely, suppose that $S = (G, \sigma)$ is a block-line forest signed graph. Then there exists a signed graph $S' = (G', \sigma')$ such that $\mathcal{BLFS}(S') \cong S$. Hence, G is the block-line forest graph of G' and by Theorem 2.1, S is balanced. □

The notion of *negation* $\eta(S)$ of a given signed graph S defined in [3] as follows:

The negation $\eta(S)$ of a given signed graph S has the same underlying graph as that of S with the sign of each edge opposite to that given to it in S. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

For a signed graph $S = (G, \sigma)$, the $\mathcal{BLFS}(S)$ is balanced (Theorem 2.1). We now examine, the conditions under which negation $\eta(S)$ of $\mathcal{BLFS}(S)$ is balanced.

Proposition 2.7 *Let $S = (G, \sigma)$ be a signed graph. If $\mathcal{BLFG}(G)$ is bipartite then $\eta(\mathcal{BLFS}(S))$ is balanced.*

Proof Since, by Theorem 2.1, $\mathcal{BLFS}(S)$ is balanced, it follows that each cycle C in $\mathcal{BLFS}(S)$ contains even number of negative edges. Also, since $\mathcal{BLFG}(G)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $\mathcal{BLFS}(S)$ is also even. Hence $\eta(\mathcal{BLFS}(S))$ is balanced. □

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Study on Hausdorff Supra Fuzzy Bitopological Space — Approach of Quasi-Coincidence

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Abstract: In this paper, we have defined some notions of Hausdorff (T_2) separation on supra fuzzy bitopological spaces in the sense of quasi-coincidence. We have found the relations among the notions. We have shown that these notions satisfy good extension, hereditary, productive and projective properties. We have also shown that Hausdorff supra fuzzy bitopological space is preserved under one-one, onto fuzzy pairwise open and continuous mappings. Finally, we have discussed initial and final supra fuzzy T_2 bitopological spaces.

Key Words: Supra fuzzy bitopological space, quasi-coincidence, supra fuzzy Hausdorff bitopological space, good extension, mappings, initial and final fuzzy bitopology.

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§1. Introduction

The first concept of fuzzy sets proposed by Zadeh [24] in 1965. By using this concept Chang [2] defined fuzzy topological spaces in 1968. The concept of bitopological spaces was introduced by J.C. Kelly [5]. A set equipped with two topologies is called a bitopological spaces. The supra topological spaces have been introduced by A.S. Mashhour at [10] in 1983. In topological space, the arbitrary union condition is enough to have a supra topological space. Here every fuzzy topological space is a supra fuzzy bitopological space but the converse is not always true. Separation axioms [4,11,12, 14] are important parts in fuzzy topological spaces. Also many researchers have contributed various types of separation axioms [6,13,15] on fuzzy bitopological space which is introduced by Kandil and El-Shafee [6] in 1991. Among those axioms, T_2- type separation on fuzzy bitopological space is one and it has been introduced earlier by Abu Sufiya et al. [22], Nouh [20], Amin et al. [1] and others. The purpose of this paper is to further contribute to the development of supra fuzzy Hausdorff bitopological spaces especially on supra fuzzy bitopological spaces. In this paper, we define Hausdorff supra fuzzy bitopological space and showed that the definitions satisfy good extension property, hereditary property, order preserving, productive and projective properties hold on the new concepts, initial and final supra fuzzy bitopologies are discussed also.

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§2. Basic Notions and Preliminary Results

For the purpose of the main results, we need to introduce some definitions and notions.

Definition 2.1([24]) *For a set X, a function $u : X \to [0,1]$ is called a fuzzy set in X. For every $x \in X$, $u(x)$ represents the grade of membership of x in the fuzzy set u. Some authors say that u is a fuzzy subset of X.*

Definition 2.2([17]) *A fuzzy set $u \in X$ is called fuzzy singleton if and only if $u(x) = r, 0 < r \leq 1$ for a certain $x \in X$ an $u(y) = 0$ for all points y of X except x. The fuzzy singleton is denoted by x_r and x is its support. The class of all fuzzy singletons in x is denoted by $S(X)$. If $u \in I^X$ and $x_r \in S(X)$ then we say that $x_r \in u$ if and only if $r \leq u(x)$.*

Definition 2.3([2]) *A fuzzy singleton x_r is said to be quasi-coincidence with u denoted by $x_r qu$ if and only if $u(x) + r > 1$. If x_r is not quasi-coincidence with u we write $x_r \bar{q} u$ and defined as $u(x) + r \leq 1$.*

Definition 2.4([2]) *Let X and Y be two sets and $f : X \to Y$ be a function. For a fuzzy subset v of Y, the inverse image of v under f is the fuzzy subset $f^{-1}(v) = v_0 f$ in X and is defined by*

$$f^{-1}(v)(x) = v(f(x)), \; for \, x \in X.$$

Definition 2.5([2]) *Let X be a non empty set and t be the collection of fuzzy sets in I^X. Then t is called a fuzzy topology on X if it satisfies the following conditions:*

- *(i)* $1, 0 \in t$;
- *(ii)* *If* $u_i \in t$ *for each* $i \in \Lambda$, *then* $\cup_{i \in \Lambda} u_i \in t$;
- *(iii)* *If* $u_1, u_2 \in t$ *then* $u_1 \cap u_2 \in t$.

If t is a fuzzy topology on X, then the pair (X, t) is called a fuzzy topological space (fst, in short) and members of t are called t-open (or simply open) fuzzy sets. If u is open fuzzy set, then the fuzzy sets of the form 1-u are called t-closed (or simply closed) fuzzy sets.

Definition 2.6([10]) *Let X be a nonempty set. A subfamily t^* of I^X is said to be a supra fuzzy topology on X if and only if*

- *(i)* $1, 0 \in t^*$;
- *(ii) If* $u_i \in t^*$ *for each* $i \in \Lambda$, *then* $\cup_{i \in \Lambda} u_i \in t^*$,

Then the pair (X, t^*) is called a supra fuzzy topological spaces. The elements of t^* are called supra open sets in (X, t^*) and complement of supra open set is called supra closed set.

Definition 2.7([3]) *Let (X, t^*) and (Y, s^*) be two topological space. Let t^* and s^* are associated supra topological with t and s respectively and $f : (X, t^*) \to (Y, s^*)$ be a function. Then the function f is a supra fuzzy continuous if the inverse image of each i.e., if for any $v \in s^*, f^{-1}(v) \in t^*$. The function f is called supra fuzzy homeomorphic if and only if f is supra bijective and both f and f^{-1} are supra fuzzy continuous.*

Definition 2.8([3]) *The function $f : (X, t^*) \to (Y, s^*)$ is called supra fuzzy open if and only if for each supra open fuzzy set u in (X, t^*), f(u) is supra fuzzy set in (Y, s^*).*

Definition 2.9([3]) *The function $f : (X, t^*) \to (Y, s^*)$ is called supra fuzzy closed if and only if for each supra closed fuzzy set u in (X, t^*), f(u) is supra closed fuzzy set in (Y, s^*).*

Definition 2.10([8]) *Suppose $\{X_i, i \in \Lambda\}$, be any collection of sets and X denoted the Cartesian product of these sets, i.e., $X = \prod_{i\in\Lambda} X_i$. Here X consists of all points $p =< a_i, i \in \Lambda >$, where $a_i \in X_i$. For each $j_0 \in \Lambda$, we define the projection $\pi_{j_0} : X \longrightarrow X_{j_0}$ by $\pi_{j_0}(< a_i : i \in \Lambda >) = a_{j_0}$. These projection are used to defined the product supra topology.*

Definition 2.11([21]) *Let (X, t^*) be a topological space and t^* be associated supra topology with T. Then a function $f : X \longrightarrow I$ is lower semi continuous if and only if $\{x \in X : f(x) > \alpha\}$ is open for all $\alpha \in I$.*

Definition 2.12([3]) *Let (X, t^*) be a supra fuzzy topological space and t^* be associated supra topology with t. Then the lower semi continuous topology on X associated with t^* is $\omega(t^*) = \{\mu : X \to [0, 1], \mu\, is\, supra\, lsc\}$.*

Definition 2.13([17]) *The function $f : (X, t) \to (Y, s)$ is called fuzzy continuous if and only if for every $v \in s, f^{-1}(v) \in t$, the function f is called fuzzy homeomorphic if and only if f is bijective and both f and f^{-1} are fuzzy continuous.*

Definition 2.14([18]) *The function $f : (X, t) \to (Y, s)$ is called fuzzy open if and only if for every open fuzzy set u in $(X, t), f(u)$ is open fuzzy set in (Y, s).*

Definition 2.15([23]) *Let $\{(X_i, s_i, t_i) : i \in \Lambda\}$ is a family of fuzzy bitopological spaces. Then the space $(\prod X_i, \prod s_i, \prod t_i)$ is called the product fuzzy bitopological space of the family $\{(X_i, s_i, t_i) : i \in \Lambda\}$, where $\prod s_i$ and $\prod t_i$ denote the usual product fuzzy topologies of the families $\{\prod s_i : i \in \Lambda\}$ and $\{\prod t_i : i \in \Lambda\}$ of the fuzzy topologies respectively on X.*

A fuzzy bitopological property P is called productive if the product of fuzzy bitopological spaces of a family of fuzzy bitopological space, each having property P, has property P.

A fuzzy bitopological property P is called projective if for a family of fuzzy bitopological space $\{(X_i, s_i, t_i) : i \in \Lambda\}$, the product fuzzy bitopological space $(\prod X_i, \prod s_i, \prod t_i)$ has property P implies that each coordinate space has property P.

Definition 2.16([18]) *Let (X, T) be an ordinary topological space. The set of all lower semi continuous functions from (X, T) into the closed unit interval I equipped with the usual topology constitutive a fuzzy topology associated with (X, T) and is denoted by $(X, \omega(T))$.*

Definition 2.17([8]) *The initial fuzzy topology on a set X for the family of fuzzy topological spaces $\{(X_i, t_i)_{i\in\Lambda}\}$ and the family of functions $\{f_i : X \to (X_i, t_i)\}_{i\in\Lambda}$ is the smallest fuzzy topology on X making each f_i fuzzy continuous. It is easily seen that it is generated by the family $\{f_i^{-1}(u_i) : u_i \in t_i\}_{i\in\Lambda}$.*

Definition 2.18([8]) *The final fuzzy topology on a set X for the family of fuzzy topological spaces*

$\{(X_i,t_i)_{i\in\Lambda}\}$ and the family of functions $\{f_i : (X_i,t_i) \to X\}_{i\in\Lambda}$ is the finest fuzzy topology on X making each f_i fuzzy continuous.

Definition 2.19([19]) *A bijective mapping from an fts (X,t) to an fts (Y,s) preserves the value of a fuzzy singleton (fuzzy point).*

Note 2.1 The preimage of any fuzzy singleton (fuzzy point) under bijective mapping preserves its value.

§3. Definition and Properties of Supra Fuzzy T_2 Bi-Topological Spaces

We define our notions in Supra fuzzy T_2 bitopological spaces and show relations among the notions.

Definition 3.1 *A supra fuzzy bitopological space (X,s^*,t^*) is called*

(a) $SFPT_2(i)$ if and only if for any pair x_m, $y_n \in S(X)$ for distinct x and y, there exist $\mu,\eta \in s^*\cup t^*$ such that $x_m q\mu, y_n q\eta$, and $\mu\cap\eta = 0$;

(b) $SFPT_2(ii)$ if and only if for any pair x_m, $y_n \in S(X)$ for distinct x and y, there exist $\mu,\eta \in s^*\cup t^*$ such that $x_m q\mu$, $y_n q\eta$ and $\mu\bar{q}\eta$;

(c) $SFPT_2(iii)$ if and only if any pair x_m, $y_n \in S(X)$ for distinct x and y, there exist $\mu,\eta \in s^*\cup t^*$ such that $x_m \in \mu, y_n \in \eta$ and $\mu\bar{q}\eta$;

(d) $SFPT_2(iv)$ if and only if any pair $x,y \in X$ for distinct x and y, there exist $\mu,\eta \in s^*\cup t^*$ such that $\mu(x) = 1, \eta(y) = 1$ and $\mu\cap\eta = 0$.

Lemma 3.1 *For a supra fuzzy bitopological space (X,s^*,t^*) the following implications are true:*

$$SFPT_2(i) \Rightarrow SFPT_2(ii),\ SFPT_2(iv) \Rightarrow SFPT_2(i),\ SFPT_2(iv) \Rightarrow SFPT_2(ii)$$

but in general, the converse is not true.

Proof For $SFPT_2(i) \Rightarrow SFPT_2(ii)$, let (X,s^*,t^*) be a supra fuzzy bitopological space and (X,s^*,t^*) is $SFPT_2(i)$. We have to prove that (X,s^*,t^*) is $SFPT_2(ii)$. Let x_m, y_n be fuzzy singletons in X for distinct x and y. Since (X,s^*,t^*) is $SFPT_2(i)$ fuzzy bitopological space, we have, there exist $\mu,\eta \in s^*\cup t^*$ such that $x_m q\mu, y_n q\eta$, and $\mu\cap\eta = 0$.

To prove (X,s^*,t^*) is $SFPT_2(ii)$, it is only needed to prove that $\mu\bar{q}\eta$. Now, $\mu\cap\eta = 0 \Rightarrow (\mu\cap\eta)(x) = 0 \Rightarrow \min(\mu(x),\eta(x)) = 0 \Rightarrow \mu(x) = 0\, or\, \eta(x) = 0 \Rightarrow \mu(x)+\eta(x) \le 1 \Rightarrow \mu\bar{q}\eta$. It follows that there exist $\mu,\eta \in s^*\cup t^*$ such that $x_m q\mu, y_n q\eta$, and $\mu\cap\eta = 0$. Hence it is clear that (X,s^*,t^*) is $SFPT_2(ii)$.

To show (X,s^*,t^*) is $SFPT_2(ii) \not\Longrightarrow (X,s^*,t^*)$ is $SFPT_2(i)$, we give a counterexample following.

Counterexample. Let $X = \{x,y\}$ and $\mu,\eta \in I^X$ be given by $\mu(x) = 1-\varepsilon$, $\mu(y) = 1-\frac{\varepsilon}{3}$ and $\eta(y) = 1-\varepsilon$, $\eta(x) = \frac{\varepsilon}{3}$, where $\varepsilon = \frac{m}{3}$ for $m \in (0,1]$. Consider the supra fuzzy topologies s^* and t^* on X generated by $\{0,\mu,\eta,1\}$. Then, $\mu(x) = 1-\frac{m}{3} \Rightarrow \mu(x)+\frac{m}{3} = 1 \Rightarrow \mu(x)+m > 1 \Rightarrow x_m q\mu$ also, $\eta(y) = 1-\frac{m}{3} \Rightarrow \eta(y)+\frac{m}{3} = 1$ $\eta(y)+m > 1 \Rightarrow y_m q\eta$ and, $\mu(x)+\eta(x) = 1-\varepsilon+\frac{\varepsilon}{3}$

$\Rightarrow \mu(x)+\eta(x)=1-\frac{\varepsilon}{3}\leq 1 \Rightarrow \mu(x)+\eta(x)\leq 1 \Rightarrow \mu\bar{q}\eta$. Hence, (X,s^*,t^*) is $SFPT_2(ii)$. But $\min(\mu(x),\eta(x))\neq 0 \Rightarrow \mu\cap\eta\neq 0$ Thus, (X,s^*,t^*) is not $SFPT_2(i)$.

For $SFPT_2(iv) \Rightarrow SFPT_2(i)$, let (X,s^*,t^*) be a supra fuzzy bitopological space and (X,s^*,t^*) is $SFPT_2(i)$. We have to prove that (X,s^*,t^*) is $SFPT_2(i)$. Let x_m, y_n be fuzzy singletons in X for distinct x and y. Since (X,s^*,t^*) is $SFPT_2(iv)$ fuzzy bitopological space, we have, there exist $\mu,\eta\in s^*\cup t^*$ such that $\mu(x)=1, \eta(y)=1$, and $\mu\cap\eta=0$.

To prove (X,s^*,t^*) is $SFPT_2(i)$, it is only needed to prove that $x_m q\mu, y_n q\eta$. Now, $\mu(x)=1\Rightarrow \mu(x)+m>1$, for any $m\in(0,1]\Rightarrow x_m q\mu$. Similarly, we can prove that $y_n q\eta$. It follows that there exist $\mu,\eta\in s^*\cup t^*$ such that $x_m q\mu, y_n q\eta$ and $\mu\cap\eta=0$. Hence it is clear that (X,s^*,t^*) is $SFPT_2(i)$.

To show (X,s^*,t^*) is $SFPT_2(i) \not\Longrightarrow (X,s^*,t^*)$ is $SFPT_2(iv)$, we give a counterexample following.

Counterexample. Let $X=\{x,y\}$ and $\mu,\eta\in I^X$ be given by $\mu(x)=1-\varepsilon$, $\mu(y)=0$ and $\eta(y)=1-\varepsilon$, $\eta(x)=0$, where $\varepsilon=\frac{m}{3}$ for $m\in(0,1]$. Consider the supra fuzzy topologies s^* and t^* on X generated by $\{0,\mu,\eta,1\}$. Then, $\mu(x)=1-\frac{m}{3}\Rightarrow \mu(x)+\frac{m}{3}=1\Rightarrow \mu(x)+m>1\Rightarrow x_m q\mu$. Similarly, we can prove that $y_n q\eta$. Also, $\min(\mu(x),\eta(x))=0\Rightarrow \mu\cap\eta=0$. Hence, (X,s^*,t^*) is $SFPT_2(i)$. But $\mu(x)\neq 1, \eta(y)\neq 1$ Thus, (X,s^*,t^*) is not $SFPT_2(iv)$.

For $SFPT_2(iv) \Rightarrow SFPT_2(ii)$, let (X,s^*,t^*) be a supra fuzzy bitopological space and (X,s^*,t^*) is $SFPT_2(iv)$. We have to prove that (X,s^*,t^*) is $SFPT_2(ii)$. Let x_m, y_n be fuzzy singletons in X for distinct x and y. Since (X,s^*,t^*) is $SFPT_2(iv)$ fuzzy bitopological space, we have, there exist $\mu,\eta\in s^*\cup t^*$ such that $\mu(x)=1, \eta(y)=1$, and $\mu\cap\eta=0$.

To prove (X,s^*,t^*) is $SFPT_2(ii)$, it is only needed to prove that $x_m q\mu, y_n q\eta$ and $\mu\bar{q}\eta$. Now, $\mu(x)=1\Rightarrow \mu(x)+m>1$, for any $m\in(0,1]\Rightarrow x_m q\mu$. Similarly, we can prove that $y_n q\eta$. Now, $\mu\cap\eta=0\Rightarrow(\mu\cap\eta)(x)=0\Rightarrow \min(\mu(x),\eta(x))=0\Rightarrow \mu(x)=0\, or\, \eta(x)=0$ $\Rightarrow \mu(x)+\eta(x)\leq 1\Rightarrow \mu\bar{q}\eta$. It follows that there exist $\mu,\eta\in s^*\cup t^*$ such that $x_m q\mu, y_n q\eta$ and $\mu\bar{q}\eta$. Hence it is clear that (X,s^*,t^*) is $SFPT_2(ii)$.

To show (X,s^*,t^*) is $SFPT_2(ii) \not\Longrightarrow (X,s^*,t^*)$ is $SFPT_2(iv)$, we give a counterexample following.

Counterexample. Let $X=\{x,y\}$ and $\mu,\eta\in I^X$ be given by $\mu(x)=1-\varepsilon$, $\mu(y)=\frac{\varepsilon}{3}$ and $\eta(y)=1-\varepsilon$, $\eta(x)=\frac{\varepsilon}{3}$, where $\varepsilon=\frac{m}{3}$ for $m\in(0,1]$. Consider the supra fuzzy topologies s^* and t^* on X generated by $\{0,\mu,\eta,1\}$. Then, $\mu(x)=1-\frac{m}{3}\Rightarrow \mu(x)+\frac{m}{3}=1$ $\Rightarrow \mu(x)+m>1\Rightarrow x_m q\mu$. Similarly, we can prove that $y_n q\eta$. Also, $\mu(x)+\eta(x)=1-\varepsilon+\frac{\varepsilon}{3}\Rightarrow$ $\mu(x)+\eta(x)=1-\frac{\varepsilon}{3}\leq 1\Rightarrow \mu(x)+\eta(x)\leq 1\Rightarrow \mu\bar{q}\eta$. Hence, (X,s^*,t^*) is $SFPT_2(ii)$. But $\mu(x)\neq 1, \eta(y)\neq 1, min(\mu(x),\eta(x))\neq 0\Rightarrow \mu\cap\eta\neq 0$. Thus, (X,s^*,t^*) is not $SFPT_2(iv)$. These complete the proof. □

§4. Good Extensions

In this section, we shall show that our notions satisfy good extension property.

Theorem 4.1 *Let (X,S^*,T^*) be a supra bitopological space. Consider the following statements:*

(1) (X, S^*, T^*) *be a* T_2 *supra bitopological space;*
(2) $(X, \omega(S^*), \omega(T^*))$ *be a* $SFPT_2(i)$ *bitopological space;*
(3) $(X, \omega(S^*), \omega(T^*))$ *be a* $SFPT_2(ii)$ *bitopological space;*
(4) $(X, \omega(S^*), \omega(T^*))$ *be a* $SFPT_2(iii)$ *bitopological space;*
(5) $(X, \omega(S^*), \omega(T^*))$ *be a* $SFPT_2(iv)$ *bitopological space.*

The following implications are true:

$$(1) \Longleftrightarrow (2),\ (1) \Longleftrightarrow (3),\ (1) \Longleftrightarrow (4),\ (1) \Longleftrightarrow (5).$$

Proof For $(1) \Longrightarrow (2)$, let (X, S^*, T^*) be a supra bitopological space and (X, S^*, T^*) is T_2. We have to prove that $(X, \omega(S^*), \omega(T^*))$ is $SFPT_2(i)$. Let x_m, y_n be fuzzy singletons in X with distinct x, y. Since (X, S^*, T^*) is T_2 supra bitopological space we have, there exist $U, V \in S^* \cup T^*$ such that $x \in U$, $y \in V$ and $U \cap V = 0$. From the definition of lower semi continuous we have $1_U, 1_V \in \omega(S^*) \cup \omega(T^*)$ and $1_U(x) = 1$, $1_V(y) = 1$. Then $1_U(x) + m > 1 \Rightarrow x_m q 1_U$. Similarly, $\Rightarrow y_n \bar{q} 1_V$.

Also, $1_U \cap 1_V = 0$. If $1_U \cap 1_V \neq 0$, then there exists $z \in X$ such that $(1_U \cap 1_V)(z) \neq 0 \Rightarrow 1_U(z) \neq 0, 1_V(z) \neq 0 \Rightarrow U(z) = 1, V(z) = 1 \Rightarrow z \in U, z \in V \Rightarrow z \in U \cap V \Rightarrow U \cap V \neq \phi$, a contradiction. So, $1_U \cap 1_V = 0$. Hence, $(X, \omega(S^*), \omega(T^*))$ is $SFPT_2(i)$. Thus $(1) \Longrightarrow (2)$ holds.

For $(2) \Longrightarrow (1)$, let $(X, \omega(S^*), \omega(T^*))$ is $SFPT_2(i)$. We have to prove that (X, S^*, T^*) is T_2. Let x, y be distinct points in X. Since $(X, \omega(S^*), \omega(T^*))$ is $SFPT_2(i)$, we have, for any fuzzy singletons x_m, y_n in X, $\exists \mu, \eta \in \omega(S^*) \cup \omega(T^*)$ such that $x_m q \mu, y_n q \eta$ and $\mu \cap \eta = 0$. Now $x_m q \mu \Rightarrow \mu(x) + m > 1 \Rightarrow \mu(x) > 1 - m = \alpha \Rightarrow x \in \mu^{-1}(\alpha, 1]$ Similarly, $y \in \eta^{-1}(\alpha, 1]$. Also, $\mu^{-1}(\alpha, 1], \eta^{-1}(\alpha, 1) \in S^* \cup T^*$. Now, $\mu \cap \eta = 0 \Rightarrow \mu \cap \eta(z) = 0 \Rightarrow \min(\mu(z), \eta(z)) = 0$.

We claim that $\mu^{-1}(\alpha, 1] \cap \eta^{-1}(\alpha, 1] = \phi$. For, if $z \in \mu^{-1}(\alpha, 1] \cap \eta^{-1}(\alpha, 1]$, then $z \in \mu^{-1}(\alpha, 1]\, and\, z \in \eta^{-1}(\alpha, 1] \Rightarrow \mu(z) > \alpha\, and\, \eta(z) > \alpha \Rightarrow \min(\mu(z), \eta(z)) > \alpha$, a contradiction. Then $\mu^{-1}(\alpha, 1] \cap \eta^{-1}(\alpha, 1] = \phi$.

It follows that there exist $\mu^{-1}(\alpha, 1], \eta^{-1}(\alpha, 1] \in S^* \cup T^*$ such that $x \in \mu^{-1}(\alpha, 1], y \in \eta^{-1}(\alpha, 1]$ and $\mu^{-1}(\alpha, 1] \cap \eta^{-1}(\alpha, 1] = \phi$. Thus $(2) \Longrightarrow (1)$ holds.

Similarly, we can prove the other results. □

§5. Hereditary, Productivity and Projectivity in Supra Fuzzy T_2 Bitopological Spaces

In this section, we shall show that our notions satisfy hereditary, productive and projective properties.

Theorem 5.1 *Let* (X, s^*, t^*) *be a supra fuzzy bitopological space,* $A \subseteq X$, $s^*_A = \{\mu/A : \mu \in s^* \cup t^*\}$, $t^*_A = \{\eta/A : \eta \in s^* \cup t^*\}$, *then* (X, s^*, t^*) *is* $SFPT_2(j) \Longrightarrow (A, s^*_A, t^*_A)$ *is* $SFPT_2(j)$*; where* $j = i, ii, iii, iv$.

Proof Let (X, s^*, t^*) be a supra fuzzy bitopological space and (X, s^*, t^*) is $SFPT_2(i)$. We have to prove that (A, s^*_A, t^*_A) is $SFPT_2(i)$. Let x_m, y_n be fuzzy singletons in A for distinct

x and y. Since $A \subseteq X$, these fuzzy singletons are also fuzzy singletons in X. Also since (X, s^*, t^*) is $SFPT_2(i)$ supra fuzzy bitopological space we have, there exist $\mu, \eta \in s^* \cup t^*$ such that $x_m q\mu$, $y_n q\eta$ and $\mu \cap \eta = 0$. For $A \subseteq X$, we have $\mu/A, \eta/A \in s^*_A \cup t^*_A$. Now, $x_m q\mu \Rightarrow \mu(x) + m > 1, x \in X \Rightarrow (\mu/A)(x) + m > 1, x \in A \subseteq X \Rightarrow x_m q(\mu/A)$. and $y_n q\eta \Rightarrow \eta(y) + n > 1, y \in X \Rightarrow (\eta/A)(y) + n > 1, y \in A \subseteq X \Rightarrow y_n q(\eta/A)$.

Also,

$$\begin{aligned}
\mu \cap \eta = 0 \quad &\Rightarrow \quad (\mu \cap \eta)(x) = 0,\, x \in X \Rightarrow \min(\mu(x), \eta(x)) = 0,\, x \in X \\
&\Rightarrow \quad \min(\mu/A(x), \eta/A(x)) = 0,\, x \in A \subseteq X \\
&\Rightarrow \quad ((\mu/A) \cap (\eta/A)(x)) = 0 \Rightarrow (\mu/A) \cap (\eta/A) = 0.
\end{aligned}$$

It follows that there exist $\mu/A, \eta/A \in s^*_A \cup t^*_A$ such that $x_m q(\mu/A)$, $y_n q(\eta/A)$ and $(\mu/A) \cap (\eta/A) = 0$. Hence, (A, s^*_A, t^*_A) is $SFPT_2(i)$. The proof of others is of similar manner. □

Theorem 5.2 *Let (X_i, s^*_i, t^*_i), $i \in \Lambda$ be a supra fuzzy bitopological spaces and $(X = \prod_{i\in\Lambda} X_i, s^*, t^*)$ be the corresponding product bitopological space, then for all $i \in \Lambda$, (X_i, s^*_i, t^*_i) is $SFPT_2(j)$ if and only if (X, s^*, t^*) is $SFPT_2(j)$; where $j = i, ii, iii, iv$.*

Proof Let for all $i \in \Lambda$, (X_i, s^*_i, t^*_i) is $SFPT_2(iii)$ space. We have to prove that (X, s^*, t^*) is $SFPT_2(iii)$. Let x_m, y_n be fuzzy singletons in X for distinct x and y. Then $(x_i)_m$, $(y_i)_n$ are fuzzy singletons for distinct x_i and y_i for some $i \in \Lambda$. Since (X_i, s^*_i, t^*_i) is $SFPT_2(iii)$, there exist $\mu_i, \eta_i \in s^*_i \cup t^*_i$ such that $(x_i)_m \in \mu_i$, $(y_i)_n \in \eta_i$ and $\mu_i \bar{q} \eta_i$. Now, $(x_i)_m \in \mu_i$, $(y_i)_n \in \eta_i$. But we have $\pi_i(x) = x_i$ and $\pi_i(y) = y_i$. Now, $(x_i)_m \in \mu_i \Rightarrow \mu(x_i) \geq m \Rightarrow \mu_i(\pi_i(x)) \geq m \Rightarrow (\mu_i \circ \pi_i)(x) \geq m \Rightarrow x_m \in (\mu_i \circ \pi_i)$. Similarly, we can prove that $(y_i)_n \in \mu_i$. Now, $\Rightarrow \mu_i(x_i) + \eta_i(x_i) \leq 1 \Rightarrow \mu_i(\pi_i(x)) + \eta_i(\pi_i(x)) \leq 1 \Rightarrow (\mu_i \circ \pi_i)(x) + (\eta_i \circ \pi_i)(x) \leq 1 \Rightarrow (\mu_i \circ \pi_i) \bar{q} (\eta_i \circ \pi_i)$. It follows that there exist $(\mu_i \circ \pi_i), (\eta_i \circ \pi_i) \in s^*_i \cup t^*_i$ such that $x_m \in (\mu_i \circ \pi_i)$, $y_n \in (\eta_i \circ \pi_i)$ and $(\mu_i \circ \pi_i) \bar{q} (\eta_i \circ \pi_i)$. Hence, (X, s^*, t^*) is $SFPT_2(iii)$.

Conversely, Let (X, s^*, t^*) be a supra fuzzy bitopological space and (X, s^*, t^*) is $SFPT_2(iii)$. We have to prove that (X_i, s^*_i, t^*_i), $i \in \Lambda$ is $SFPT_2(iii)$. Let a_i be a fixed element in X_i. Let

$$A_i = \{x \in X = \prod_{i\in\Lambda} X_i : x_j = a_j \, for \, some \, i \neq j\}$$

.

Then A_i is a subset of X, and hence $(A_i, s^*_{A_i}, t^*_{A_i})$ is subspace of (X, s^*, t^*). Since (X, s^*, t^*) is $SFPT_2(iii)$, so $(A_i, s^*_{A_i}, t^*_{A_i})$ is $SFPT_2(iii)$. Now we have A_i is homeomorphic image of X_i. Hence it is clear that for all $i \in \Lambda$, (X_i, s^*_i, t^*_i) is $SFPT_2(iii)$ space. Similarly, other results can be proved. □

§6. Mappings in Supra Fuzzy T_2 Bitopological Spaces

In this section, we shall show that our notions are preserved under one-one, onto, fuzzy open and fuzzy continuous mappings.

Theorem 6.1 *Let* (X, s_1^*, t_1^*) *and* (Y, s_2^*, t_2^*) *be two supra fuzzy bitopological spaces and* $f : X \to Y$ *be a one-one, onto and fuzzy open map then,* (X, s_1^*, t_1^*) *is* $SFPT_2(j) \Longrightarrow (Y, s_2^*, t_2^*)$ *is* $SFPT_2(j)$*; where* $j = i, ii, iii, iv$.

Proof Let (X, s_1^*, t_1^*) be a supra fuzzy bitopological space and (X, s_1^*, t_1^*) is $SFPT_2(i)$. We have to prove that (Y, s_2^*, t_2^*) is $SFPT_2(i)$. Let x'_m, y'_n be fuzzy singletons in Y for distinct x' and y'. Since f is onto then there exist $x, y \in X$ with $f(x) = x'$, $f(y) = y'$ and x_m, y_n are fuzzy singletons in X with $x \neq y$ as f in one-one. Again since (X, s_1^*, t_1^*) is $SFPT_2(i)$ space, there exist $\mu, \eta \in s_1^* \cup t_1^*$ such that $x_m q\mu$, $y_n q\eta$ and $\mu \cap \eta$. Now, $x_m q\mu \Rightarrow \mu(x) + m > 1$ and, $y_n q\eta \Rightarrow \mu(y) + n > 1$.

Now, $f(\mu)(x') = \{\sup \mu(x) : f(x) = x'\} \Rightarrow f(\mu)(x') = \mu(x)$, for some x and $f(\eta)(y') = \{\sup \eta(y) : f(y) = y'\} \Rightarrow f(\eta)(y') = \eta(y)$ for some y. Also since, f is open map then $f(\mu), f(\eta) \in s_2^* \cup t_2^*$ as $\mu, \eta \in s_1^* \cup t_1^*$.

Again, $\Rightarrow \mu(x)+m > 1 \Rightarrow f(\mu)(x')+m > 1 \Rightarrow x'_m qf(\mu), \Rightarrow \eta(x)+n > 1 \Rightarrow f(\eta)(x')+n > 1$ $\Rightarrow y'_n qf(\eta)$ and $\mu \cap \eta = 0$. Here, $f(\mu \cap \eta)(x') = \{\sup(\mu \cap \eta)(x) : f(x) = x'\}$, $f(\mu \cap \eta)(x') = 0$ and $f(\mu \cap \eta)(y') = \{\sup(\mu \cap \eta)(y) : f(y) = y'\}$. Therefore, $f(\mu \cap \eta) = 0 \Rightarrow f(\mu) \cap f(\eta) = 0$. It follows that there exist $f(\mu), f(\eta) \in s_2^* \cup t_2^*$ such that $x'_m qf(\mu)$, $y'_n qf(\eta)$ and $f(\mu) \cap f(\eta) = 0$. Hence it is clear that (Y, s_2^*, t_2^*) is $SFT_2(i)$ space. Similarly, we can prove the remaining. □

Theorem 6.2 *Let* (X, s_1^*, t_1^*) *and* (Y, s_2^*, t_2^*) *be two supra fuzzy bitopological spaces and* $f : X \to Y$ *be a one-one and fuzzy continuous map then,* (Y, s_2^*, t_2^*) *is* $SFPT_2(j) \Longrightarrow (X, t^*)$ *is* $SFT_2(j)$*; where* $j = i, ii, iii, iv$.

Proof Let (Y, s_2^*, t_2^*) be a supra fuzzy topological space and (Y, s_2^*, t_2^*) is $SFPT_2(j)$. We have to prove that (X, s_1^*, t_1^*) is $SFPT_2(j)$. Let x_m, y_n be fuzzy singletons in X for distinct x and y. Then $(f(x))_m$, $(f(y))_n$ are fuzzy singletons in Y with $f(x) \neq f(y)$ as f is one-one. Again since, (Y, s_2^*, t_2^*) is $SFPT_2(j)$ space, there exist $\mu, \eta \in s_2^* \cup t_2^*$ such that $(f(x))_m \in \mu$, $(f(y))_n \in \eta$ and $\mu \bar{q} \eta$. Now, $(f(x))_m \in \mu \Rightarrow \mu(f(x)) \geq m \Rightarrow f^{-1}(\mu(x)) + m > 1 \Rightarrow (f^{-1}(\mu))(x) + m > 1$ $\Rightarrow x_m \in f^{-1}(\mu)$ and, $(f(x))_n \in \eta \Rightarrow \eta(f(x)) \geq n \Rightarrow f^{-1}(\eta(x)) \geq n \Rightarrow (f^{-1}(\eta))(x) \geq n$ $\Rightarrow y_n \in f^{-1}(\eta)$. Also, $\mu \bar{q} \eta \Rightarrow u_1(f(x)) + u_2(f(x)) \leq 1 \Rightarrow (f^{-1}(\eta))(x) + (f^{-1}(\eta))(x) \leq 1$ $\Rightarrow (f^{-1}(\mu)\bar{q}(f^{-1}(\eta))$. Now, since, f is continuous map and $\mu, \eta \in s_2^* \cup t_2^*$ then $f^{-1}(\mu), f^{-1}(\eta) \in s_1^* \cup t_1^*$. It follows that there exist $f^{-1}(\mu), f^{-1}(\eta) \in s_1^* \cup t_1^*$ such that $x_m \in f^{-1}(\mu)$, $y_n \in f^{-1}(\eta)$ and $f^{-1}(\mu)\bar{q}f^{-1}(\eta)$. Hence, (X, s_1^*, t_1^*) is $SFPT_2(iii)$ space. The proofs of others are of similar procedure. □

§7. Initial and Final Supra Fuzzy T_2 Bitopological Spaces

We discuss the initial and final fuzzy bitopologies in this section.

Definition 7.1([16]) *The initial fuzzy bitopology on a set* X *for the family of fuzzy bitopological spaces* $\{(X_i, s_i, t_i)\}_{i \in \Lambda}$ *and the family of functions* $\{f_i : X \to (X_i, s_i \cup t_i)\}_{i \in \Lambda}$ *is the smallest fuzzy bitopology on* X *making each* f_i *fuzzy continuous. It is easily seen that it is generated by the family* $\{f_i^{-1}(u_i) : u_i \in s_i \cup t_i\}_{i \in \Lambda}$.

Definition 7.2([16]) *The final fuzzy bitopology on a set X for the family of fuzzy bitopological spaces $\{(X_i, s_i, t_i)\}_{i\in\Lambda}$ and the family of functions $\{f_i : (X_i, s_i \cup t_i) \to X\}_{i\in\Lambda}$ is the finest fuzzy bitopology on X making each f_i fuzzy continuous.*

Theorem 7.1 *If $\{(X_i, s_i^*, t_i^*)\}_{i\in\Lambda}$ is a family of $SFPT_2(j)$ and $\{f_i : X \to (X_i, s_i^* \cup t_i^*)\}_{i\in\Lambda}$, a family of one-one and fuzzy continuous functions, then the initial supra fuzzy bitopology on X for the family $\{f_i\}_{i\in\Lambda}$ is $SFPT_2(j)$ for $j = i, ii, iii, iv$.*

Proof We shall prove the theorem for $j = i, ii$ and the remaining is similar. Let s^*, t^* be the initial supra fuzzy topologies on X for the family $\{f_i\}_{i\in\Lambda}$. Let x_m, y_n be fuzzy singletons in X for distinct x and y. Then $f_i(x)$, $f_i(y) \in X_i$ and $f_i(x) \neq f_i(y)$ as f_i is one-one. Since (X_i, s_i^*, t_i^*) is $SFPT_2(i)$, then for every two distinct fuzzy singletons $(f_i(x))_m$, $(f_i(y))_n$ in X_i, there exist fuzzy sets $\mu_i\, \eta_i \in s_i^* \cup t_i^*$ such that $(f_i(x))_m q\mu_i$, $(f_i(y))_n q\eta_i$ *and* $\mu_i \cap \eta_i = 0$. Now,

$$(f_i(x))_m q\mu_i \quad \text{and} \quad (f_i(y))_n q\eta_i,$$

i.e.,

$$\mu_i(f_i(x)) + m > 1 \quad \text{and} \quad \eta_i(f_i(y)) + n > 1.$$

That is

$$f_i^{-1}(\mu_i)(x) + m > 1 \quad \text{and} \quad f_i^{-1}(\eta_i)(y) + n > 1.$$

Also,

$$\mu_i \cap \eta_i \Rightarrow \mu_i(f_i(x)) + \eta_i(f_i(x)) \leq 1 \Rightarrow f_i^{-1}(\mu_i)(x) + f_i^{-1}(\eta_i)(x) \leq 1.$$

This is true for every $i \in \Lambda$. So,

$$\inf f_i^{-1}(\mu_i)(x) + m > 1, \quad \inf f_i^{-1}(\eta_i)(y) + n > 1 \quad \text{and} \quad \inf f_i^{-1}(\mu_i)(x) + \inf f_i^{-1}(\eta_i)(x) \leq 1.$$

Let $\mu = \inf f_i^{-1}(\mu_i)$ and $\eta = \inf f_i^{-1}(\eta_i)$. Then $\mu, \eta \in s^* \cup t^*$ as f_i is fuzzy continuous. So,

$$\mu(x) + m > 1, \quad \eta(y) + n > 1 \quad \text{and} \quad \mu(x) + \eta(x) \leq 1.$$

Hence, $x_m q\mu, y_n q\eta$ and $\mu \cap \eta = 0$. Therefore, (X, s^*, t^*) is $SFPT_2(i)$.

Again, Since (X_i, s_i^*, t_i^*) is $SFPT_2(ii)$, then for every two distinct fuzzy singletons $(f_i(x))_m$, $(f_i(y))_n$ in X_i, there exist fuzzy sets $\mu_i, \eta_i \in s_i^* \cup t_i^*$ such that

$$(f_i(x))_m q\mu_i,\ (f_i(y))_n q\eta_i \quad \text{and} \quad \mu_i \bar{q} \eta_i.$$

Now,

$$(f_i(x))_m q\mu_i \quad \text{and} \quad (f_i(y))_n q\eta_i,$$

i.e.,

$$\mu_i(f_i(x)) + m > 1 \quad \text{and} \quad \eta_i(f_i(y)) + n > 1.$$

That is

$$f_i^{-1}(\mu_i)(x)+m>1 \quad \text{and} \quad f_i^{-1}(\eta_i)(y)+n>1.$$

Also,

$$\mu_i\bar{q}\eta_i \Rightarrow \mu_i(f_i(x))+\eta_i(f_i(x))\leq 1 \Rightarrow f_i^{-1}(\mu_i)(x)+f_i^{-1}(\eta_i)(x)\leq 1.$$

This is true for every $i\in\Lambda$. So,

$$\inf f_i^{-1}(\mu_i)(x)+m>1, \quad \inf f_i^{-1}(\eta_i)(y)+n>1 \quad \text{and} \quad \inf f_i^{-1}(\mu_i)(x)+\inf f_i^{-1}(\eta_i)(x)\leq 1.$$

Let $\mu=\inf f_i^{-1}(\mu_i)$ and $\eta=\inf f_i^{-1}(\eta_i)$. Then $\mu,\eta\in s^*\cup t^*$ as f_i is fuzzy continuous. So,

$$\mu(x)+m>1, \quad \eta(y)+n>1 \quad \text{and} \quad \mu(x)+\eta(x)\leq 1.$$

Hence, $x_m q\mu, y_n q\eta$ and $\mu\bar{q}\eta$. Therefore, (X,s^*,t^*) is $SFPT_2(ii)$. □

Theorem 7.2 *If $\{(X_i,s_i^*,t_i^*)\}_{i\in\Lambda}$ is a family of $SFPT_2(j)$ and $\{f_i : X\to(X_i,s_i^*\cup t_i^*)\}_{i\in\Lambda}$, a family of fuzzy open and bijective functions, then the final supra fuzzy bitopology on X for the family $\{f_i\}_{i\in\Lambda}$ is $SFPT_2(j)$ for $j=i,ii,iii,iv$.*

Proof We shall prove the theorem for $j=ii$ and the remaining is similar. Let s^*,t^* be the final supra fuzzy topologies on X for the family $\{f_i\}_{i\in\Lambda}$. Let x_m, y_n be fuzzy singletons in X for distinct x and y. Then $f_i^{-1}(x)$, $f_i^{-1}(y)\in X_i$ and $f_i^{-1}(x)\neq f_i^{-1}(y)$ as f_i is bijective. Since (X_i,s_i^*,t_i^*) is $SFPT_2(ii)$, then for every two distinct fuzzy singletons $(f_i^{-1}(x))_m$, $(f_i^{-1}(y))_n$ in X_i, there exist fuzzy sets $\mu_i,\eta_i\in s_i^*\cup t_i^*$ such that $(f_i^{-1}(x))_m q\mu_i$, $(f_i^{-1}(y))_n q\eta_i$ *and* $\mu_i\bar{q}\eta_i$.

Now,

$$(f_i^{-1}(x))_m q\mu_i \quad \text{and} \quad (f_i^{-1}(y))_n q\eta_i,$$

i.e.,,

$$\mu_i(f_i^{-1}(x))+m>1 \quad \text{and} \quad \eta_i(f_i^{-1}(y))+n>1.$$

That is,

$$f_i(\mu_i)(x)+m>1 \quad \text{and} \quad f_i(\eta_i)(y)+n>1.$$

Also,

$$\mu_i\bar{q}\eta_i \Rightarrow \mu_i(f_i^{-1}(x))+\eta_i(f_i^{-1}(x))\leq 1 \Rightarrow (f_i(\mu_i(x)+f_i(\eta_i)(x)\leq 1.$$

This is true for every $i\in\Lambda$. So,

$$\inf f_i(\mu_i)(x)+m>1, \quad \inf f_i(\eta_i)(y)+n>1 \quad \text{and} \quad \inf f_i(\mu_i)(x)+\inf f_i(\eta_i)(x)\leq 1.$$

Let $\mu=\inf f_i(\mu_i)$ and $\eta=\inf f_i(\eta_i)$. Then $\mu,\eta\in s^*\cup t^*$ as f_i is fuzzy open. So,

$$\mu(x)+m>1, \eta(y)+n>1 \quad \text{and} \quad \mu(x)+\eta(x)\leq 1.$$

Hence, $x_m q\mu, y_n q\eta$ and $\mu\bar{q}\eta$. Therefore, (X,s^*,t^*) is $SPFT_2(ii)$. □

§8. Conclusion

The main result of this paper is introducing some new concepts of supra fuzzy T_2 bitopological spaces. We discuss some features of these concepts and present the hereditary, productive and projective properties. Also, we have observed that these notions are preserved under one-one, onto, supra fuzzy open and supra fuzzy continuous mappings. We think that this research work will contribute to the development of the field of modern mathematics. Initial and final bitopologies introduced in $SFPT_2$ spaces are interesting.

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4-Total Mean Cordial Labeling of Some Graphs Derived From H-Graph and Star

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Abstract: Let G be a graph. Let $f : V(G) \to \{0, 1, 2, \ldots, k-1\}$ be a function where $k \in \mathbb{N}$ and $k > 1$. For each edge uv, assign the label $f(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called k-total mean cordial labeling of G if $|t_{mf}(i) - t_{mf}(j)| \leq 1$ for all $i, j \in \{0, 1, 2, \cdots, k-1\}$, where $t_{mf}(x)$ denotes the total number of vertices and edges labelled with x, $x \in \{0, 1, 2, \cdots, k-1\}$. A graph with admit a k-total mean cordial labeling is called k-total mean cordial graph.

Key Words: Total mean cordial labeling, Smarandachely total mean cordial labeling, path, complete graph, corona, star.

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§1. Introduction

In this paper we consider simple, finite and undirected graphs only. Cordial labeling was introduced by Cahit [3] and cordial relation labeling technique was studied in [1, 2, 4, 5, 6, 9, 17, 18, 19, 20]. The notation of k-total mean cordial labeling has been introduced in [10]. We investigate the 4-total mean cordial labeling behaviour of several graphs like cycle, complete graph, star, bistar, comb and crown in [10, 11, 12, 13, 14, 15, 16].. Let x be any real number. Then $\lceil x \rceil$ stands for the smallest integer greater than or equal to x. Terms are not defined here follow from Harary [8] and Gallian [7]. In this paper we investigate the 4-total mean cordial labeling of some graphs derived from H- graph and star.

§2. k-Total Mean Cordial Graph

Definition 2.1 *Let G be a graph. Let $f : V(G) \to \{0, 1, 2, \ldots, k-1\}$ be a function where $k \in \mathbb{N}$ and $k > 1$. For each edge uv, assign the label $f(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called k-total mean cordial labeling of G if $|t_{mf}(i) - t_{mf}(j)| \leq 1$, for all $i, j \in \{0, 1, 2, \cdots, k-1\}$, where $t_{mf}(x)$ denotes the total number of vertices and edges labelled with x, $x \in \{0, 1, 2, \cdots, k-1\}$.*

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A graph with admit a k-total mean cordial labeling is called a k-total mean cordial graph.

Such a labeling f is called a Smarandachely k-total mean cordial labeling of G if there are integers $i, j \in \{0, 1, 2, \cdots, k-1\}$ hold with $|t_{mf}(i) - t_{mf}(j)| \geq 2$ and G is called a Smarandachely k-total mean cordial graph.

§3. Preliminaries

Definition 3.1 *Let $P_n^{(1)} : u_1u_2 \ldots u_n$ and $P_n^{(2)} : v_1v_2 \cdots v_n$ be any two paths. We join the vertices $u_{\frac{n+1}{2}}$ and $v_{\frac{n+1}{2}}$ by an edge, if n is odd and join the vertices $u_{\frac{n}{2}}$ and $v_{\frac{n}{2}+1}$ by an edge, if n is even. Then the resulting graph is called a H-graph on $2n$ vertices. We denote it by $H(n)$.*

Definition 3.2 *If $e = uv$ is an edge of G then e is said to be subdivided when it is replaced by the edges uw and wv. The graph obtained by subdividing each edge of a graph G is called the subdivision graph of G and is denoted by $S(G)$.*

Definition 3.3 *The duplication of an edge $e = uv$ of a graph G is the graph G' obtained from G by adding a new vertex x to G such that x is adjacent to both u and v.*

Definition 3.4 *Let G_1, G_2 respectively be (p_1, q_1), (p_2, q_2) graphs. The corona of G_1 with G_2 is the graph $G_1 \odot G_2$ obtained by taking one copy of G_1 , p_1 copies of G_2 and joining the i^{th} vertex of G_1 by an edge to every vertex in the i^{th} copy of G_2 where $1 \leq i \leq p_1$.*

Definition 3.5 *The complement $\overline{G}$ of a graph G also has $V(G)$ as its vertex set, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.*

Definition 3.6 *The complete bipartite graph $K_{1,n}$ is called a star.*

Definition 3.7 *$K_{1,3} * K_{1,n}$ is the graph obtained from $K_{1,3}$ by attaching root of a star $K_{1,n}$ at each pendent vertex of $K_{1,3}$.*

Definition 3.8 *Consider two copies of graph G namely G_1 and G_2. Then the graph $G' = \langle G_1 \Delta G_2 \rangle$ is the graph obtained by joining the apex vertices of G_1 and G_2 by an edge as well as to a new vertex x.*

Definition 3.9 *A sparkler denoted as P_m^{+n} is a graph obtained from the path P_m and appending n edges to an end point. This is a special case of a caterpillar. We refer to the hub of P_m^{+n}, the sparkler as the vertex of degree $n + 1$.*

Definition 3.10 *Let $u_i^{(k)}$ and $v_i^{(k)}$ be the vertices in the k^{th} copy of H-graph, where $i = 1, 2, 3, \cdots, n$ and $k = 1, 2, 3, \cdots, r$. Join the vertices v_1^k and v_1^{k+1} for $k = 1, 2, 3, \ldots, r-1$. The resulting graph is denoted by $P(r, H(n))$.*

Theorem 3.1([10]) *Any path is k-total mean cordial.*

§4. Main Results

4.1 Graphs Derived From H-Graph

Theorem 4.1 *The graph $H(n)$ is a 4-total mean cordial for all values of $n \geq 2$.*

Proof Take the vertex set and edge set of $H(n)$ as in Definition 3.1. Clearly $|V(H(n))| + |E(H(n))| = 4n-1$. Obviously $H(2) \cong P_4$. Therefore $H(2)$ is 4-total mean cordial follow from Theorem 3.1.

Case 1. $n \equiv 1 \pmod 2$.

Let $n = 2r+1$, $r \in \mathbb{N}$. Assign the label 2 to the $r+1$ vertices $u_1, u_2, \ldots, u_{r+1}$. Next we assign the label 3 to the r vertices $u_{r+2}, u_{r+3}, \cdots, u_{2r+1}$. Now we assign the label 0 to the $r+1$ vertices $v_1, v_2, \cdots, v_{r+1}$. Finally we assign the label 1 to the r vertices $v_{r+2}, v_{r+3}, \cdots, v_{2r+1}$.

Case 2. $n \equiv 0 \pmod 2$.

Let $n = 2r$, $r \geq 2$. We assign the label 2 to the r vertices $u_1, u_2, \cdots, u_r$. Now we assign the label 3 to the r vertices $u_{r+1}, u_{r+2}, \cdots, u_{2r}$. Next we assign the label 0 to the r vertices $v_1, v_2, \cdots, v_r$. Finally we assign the label 1 to the r vertices $v_{r+1}, v_{r+2}, \cdots, v_{2r}$.

This shows that vertex labeling f is a 4-total mean cordial labeling follows from the Table 1.This completes the proof. □

n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 2r+1$	$2r+1$	$2r+1$	$2r+1$	$2r$
$n = 2r$	$2r-1$	$2r$	$2r$	$2r$

Table 1.

Theorem 4.2 *The subdivision of $H(n)$, $S(H(n))$ is a 4-total mean cordial for all values of $n \geq 2$.*

Proof Take the vertex set and edge set as in Definition 3.1. Let x_i $(1 \leq i \leq n-1)$ be the vertex which subdivide the edge u_iu_{i+1} $(1 \leq i \leq n-1)$ and y_i $(1 \leq i \leq n-1)$ be the vertex which subdivide the edge v_iv_{i+1} $(1 \leq i \leq n-1)$. Let w be the vertex which subdivide the edge $u_{\frac{n+1}{2}}v_{\frac{n+1}{2}}$, if n is odd and w be the vertex which subdivide the edge $u_{\frac{n}{2}}v_{\frac{n}{2}+1}$, if n is even.

Clearly, $\left|V\left(S\left(H_{(n)}\right)\right)\right| + \left|E\left(S\left(H_{(n)}\right)\right)\right| = 8n-3$.

Case 1. $n \equiv 0 \pmod 2$.

Let $n = 2r$, $r \in \mathbb{N}$. Assign the label 2 to the vertex w. We now assign the label 0 to the r vertices $u_1, u_2, \cdots, u_r$. Now we assign the label 1 to the r vertices $u_{r+1}, u_{r+2}, \cdots, u_{2r}$. Next we assign the label 0 to the r vertices $x_1, x_2, \cdots, x_r$. Now we assign the label 1 to the $r-1$ vertices $x_{r+1}, x_{r+2}, \cdots, x_{2r-1}$. Next we assign the label 3 to the r vertices $v_1, v_2, \ldots, v_r$. Now we assign the label 2 to the r vertices $v_{r+1}, v_{r+2}, \cdots, v_{2r}$. Next we assign the label 3 to the r vertices $y_1, y_2, \cdots, y_r$. Finally we assign the label 2 to the $r-1$ vertices $y_{r+1}, y_{r+2}, \cdots, y_{2r-1}$.

Case 2. $n \equiv 1 \pmod 2$.

Let $n = 2r+1$, $r \in \mathbb{N}$. Now we assign the label 1 to the vertex w. We now assign the label 0 to the $r+1$ vertices $u_1, u_2, \cdots, u_{r+1}$. Next we assign the label 2 to the r vertices $u_{r+2}, u_{r+3}, \cdots, u_{2r+1}$. We now assign the label 0 to the r vertices $x_1, x_2, \ldots, x_r$. Now we assign the label

2 to the r vertices x_{r+1}, x_{r+2}, $\cdots$, x_{2r}. Now we assign the label 3 to the $r+1$ vertices v_1, v_2, $\cdots$, v_{r+1}. Next we assign the label 1 to the r vertices v_{r+2}, v_{r+3}, $\cdots$, v_{2r+1}. Now we assign the label 3 to the r vertices y_1, y_2, $\cdots$, y_r. Finally we assign the label 1 to the r vertices y_{r+1}, y_{r+2}, $\cdots$, y_{2r}. This shows that the vertex labeling f is a 4-total mean cordial labeling follows from the Table 2.

n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n=2r$	$4r-1$	$4r-1$	$4r-1$	$4r$
$n=2r+1$	$4r+1$	$4r+2$	$4r+1$	$4r+1$

Table 2.

Theorem 4.3 *Duplication of all edges of H-graph $H(n)$ is a 4-total mean cordial labeling, if n is odd.*

Proof Take the vertex set and edge set of $H(n)$ as in Definition 3.1. Let $H^*(n)$ be the graph obtained by duplication of all edges u_1u_2,u_2u_3,$\cdots$,$u_{n-1}u_n$ and v_1v_2,v_2v_3,$\cdots$,$v_{n-1}v_n$ by a new vertices x_1,x_2,$\cdots$,x_{n-1} and y_1,y_2,$\cdots$,y_{n-1} respectively. Let w be a new vertex obtained by duplicating the edge $u_{\frac{n+1}{2}}v_{\frac{n+1}{2}}$. In graph $H^*(n)$, $|V(H^*(n))| + |E(H^*(n))| = 10n-4$.

Assign the label 3 to the vertex w. We now assign the label 2 to the $\frac{n+1}{2}$ vertices u_1, u_2, $\cdots$, $u_{\frac{n+1}{2}}$. Now we assign the label 3 to the $\frac{n-1}{2}$ vertices $u_{\frac{n+3}{2}}$, $u_{\frac{n+5}{2}}$, $\cdots$, u_n. Next we assign the label 2 to the $\frac{n-1}{2}$ vertices x_1, x_2, $\cdots$, $x_{\frac{n-1}{2}}$. Now we assign the label 3 to the $\frac{n-1}{2}$ vertices $x_{\frac{n+1}{2}}$, $x_{\frac{n+3}{2}}$, $\cdots$, x_{n-1}. We now assign the label 0 to the $\frac{n+1}{2}$ vertices v_1, v_2, $\cdots$, $v_{\frac{n+1}{2}}$. Next we assign the label 1 to the $\frac{n-1}{2}$ vertices $v_{\frac{n+3}{2}}$, $v_{\frac{n+5}{2}}$, $\cdots$, v_n. Now we assign the label 0 to the $\frac{n-1}{2}$ vertices y_1, y_2, $\cdots$, $y_{\frac{n-1}{2}}$. Finally we assign the label 1 to the $\frac{n-1}{2}$ vertices $y_{\frac{n+1}{2}}$, $y_{\frac{n+3}{2}}$, $\ldots$, y_{n-1}.

Clearly, $t_{mf}(0) = t_{mf}(1) = \frac{5n-3}{2}$; $t_{mf}(2) = t_{mf}(3) = \frac{5n-1}{2}$. This completes the proof. □

Theorem 4.4 *The graph $H(n) \odot K_1$ is a 4-total mean cordial for all values of $n \geq 2$.*

Proof Let $V(H(n)) = \{u_i, v_i : 1 \leq i \leq n\}$ and let x_1,x_2,$\cdots$, x_n be the pendent vertices connected to u_1,u_2,$\cdots$, u_n and y_1,y_2,$\cdots$, y_n be the pendent vertices connected to v_1,v_2,$\cdots$, v_n. Clearly, $|V(H(n) \odot K_1)| + |E(H(n) \odot K_1)| = 8n-1$.

Case 1. $n \equiv 1 \pmod 2$.

Let $n = 2r+1$, $r \in \mathbb{N}$. Assign the label 2 to the $2r+1$ vertices u_1, u_2, $\cdots$, u_{2r+1}. Now we assign the label 3 to the $2r+1$ vertices x_1,x_2,$\cdots$, x_{2r+1}. Next we assign the label 0 to the r vertices v_1, v_2, $\cdots$, v_r. We now assign the label 1 to the $r+1$ vertices v_{r+1},v_{r+2},$\cdots$, v_{2r+1}. Now we assign the label 0 to the $r+2$ vertices y_1, y_2, $\cdots$, y_{r+2}. Finally we assign the label 1 to the $r-1$ vertices y_{r+3},y_{r+4},$\cdots$, y_{2r+1}.

Case 2. $n \equiv 0 \pmod 2$.

Let $n = 2r$, $r \in \mathbb{N}$. We assign the label 2 to the $2r$ vertices u_1, u_2, $\cdots$, u_{2r}. Next we assign the label 3 to the $2r$ vertices x_1,x_2,$\cdots$, x_{2r}. Now we assign the label 0 to the r vertices v_1, v_2, $\cdots$, v_r. Next we assign the label 1 to the r vertices v_{r+1}, v_{r+2}, $\cdots$, v_{2r}. Now we assign the label 0 to the $r+1$ vertices y_1,y_2, $\cdots$, y_{r+1}. Finally we assign the label 1 to the $r-1$ vertices y_{r+2},y_{r+3},$\cdots$, y_{2r}. Thus, this vertex labeling f is a 4-total mean cordial labeling follows from the Table 3. □

n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n=2r+1$	$4r+1$	$4r+2$	$4r+2$	$4r+2$
$n=2r$	$4r$	$4r-1$	$4r$	$4r$

Table 3.

Theorem 4.5 *The graph $H(n)\odot\overline{K_2}$ is a 4-total mean cordial for all values of $n\geq 2$.*

Proof Let

$$V\left(H(n)\odot\overline{K_2}\right)=\{u_i,v_i,x_i,y_i,p_i,q_i:1\leq i\leq n\},$$
$$E\left(H(n)\odot\overline{K_2}\right)=\{u_iu_{i-1},v_iv_{i-1}:1\leq i\leq n-1\}\bigcup\{u_ix_i,u_iy_i,v_ip_i,v_iq_i:1\leq i\leq n\}.$$

Clearly, $\left|V\left(H(n)\odot\overline{K_2}\right)\right|+\left|E\left(H(n)\odot\overline{K_2}\right)\right|=12n-1$. Assign the label 1 to the n vertices u_1, u_2, $\ldots$, u_n. Now we assign the label 3 to the n vertices x_1,x_2,$\ldots$, x_n. We now assign the label 0 to the n vertices y_1,y_2,$\ldots$, y_n. Next we assign the label 0 to the n vertices v_1, v_2, $\ldots$, v_n. We now assign the label 3 to the n vertices p_1,p_2,$\ldots$, p_n. Finally we assign the label 3 to the n vertices q_1, q_2, $\ldots$, q_n. Thus $t_{mf}(0)=3n-1$; $t_{mf}(1)=t_{mf}(2)=t_{mf}(3)=3n$. □

Theorem 4.6 *The graph $P(r,H(n))$ is a 4-total mean cordial for all values of $n\geq 2$.*

Proof Take the vertex set and edge set of $P(r,H(n))$ as in Definition 3.10. In the graph $P(r,H(n))$, $|V(P(r,H(n)))|+|E(P(r,H(n)))|=4nr-1$.

Case 1. $n\equiv 1 \pmod 2$.

Let $n=2t+1$, $t\in\mathbb{N}$. Assign the label 1 to the t vertices u_1^k, u_2^k, $\cdots$, u_t^k. Now we assign the label 0 to the $t+1$ vertices u_{t+1}^k, u_{t+2}^k, $\cdots$, u_{2t+1}^k. Next we assign the label 3 to the t vertices v_1^k, v_2^k, $\cdots$, v_t^k. Finally we assign the label 2 to the $t+1$ vertices v_{t+1}^k, v_{t+2}^k, $\cdots$, v_{2t+1}^k.

Case 2. $n\equiv 0 \pmod 2$.

Let $n=2t$, $t\in\mathbb{N}$. We assign the label 2 to the t vertices u_1^k, u_2^k, $\cdots$, u_t^k. Next we assign the label 3 to the t vertices u_{t+1}^k, u_{t+2}^k, $\cdots$, u_{2t}^k. Now we assign the label 0 to the t vertices v_1^k, v_2^k, $\cdots$, v_t^k. Finally we assign the label 1 to the t vertices v_{t+1}^k, v_{t+2}^k, $\cdots$, v_{2t}^k. This shows that vertex labeling f is a 4-total mean cordial labeling follows from the Table 4. □

n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
n is odd	$(2t+1)r$	$(2t+1)r$	$(2t+1)r$	$(2t+1)r-1$
n is even	$2tr$	$2tr$	$2tr$	$2tr-1$

Table 4.

4.2 Graphs Derived From Stars

Theorem 4.7 *The graph $K_{1,3} * K_{1,n}$ is 4-total mean cordial for all values of n.*

Proof Let $V(K_{1,3} * K_{1,n}) = \{u, u_1, u_2, u_3, x_i, y_i, z_i : 1 \le i \le n\}$ and $E(K_{1,3} * K_{1,n}) = \{uu_1, uu_2, uu_3\} \cup \{u_1x_i, u_2y_i, u_3z_i : 1 \le i \le n\}$. Note that $|V(K_{1,3} * K_{1,n})| + |E(K_{1,3} * K_{1,n})| = 6n + 7$. Assign the labels 0,0,3,1 to the vertices u,u_1,u_2,u_3.

Case 1. $n \equiv 0 \pmod 2$.

Let $n = 2r$, $r \in \mathbb{N}$. Assign the label 0 to the $r-1$ vertices $x_1, x_2, \cdots, x_{r-1}$. Now we assign the label 1 to the vertex x_r. Now we assign the label 2 to the r vertices $x_{r+1}, x_{r+2}, \cdots, x_{2r}$. Next we assign the label 0 to the r vertices $y_1, y_2, \cdots, y_r$. We now assign the label 3 to the r vertices y_{r+1},y_{r+2},$\cdots$, y_{2r}. Now we assign the label 1 to the $r-1$ vertices $z_1, z_2, \cdots, z_{r-1}$. Finally we assign the label 3 to the $r+1$ vertices z_r,z_{r+1},$\cdots$, z_{2r}.

Case 2. $n \equiv 1 \pmod 2$.

Let $n = 2r+1$, $r \in \mathbb{N}$. We now assign the label 0 to the r vertices $x_1, x_2, \cdots, x_r$. Now we assign the label 1 to the two vertices x_{r+1}, x_{r+2}. Next we assign the label 2 to the $r-1$ vertices $x_{r+3}, x_{r+4}, \cdots, x_{2r+1}$. Now we assign the label 0 to the $r+1$ vertices $y_1, y_2, \cdots, y_{r+1}$. We now assign the label 3 to the r vertices y_{r+2},y_{r+3},$\cdots$, y_{2r+1}. Now we assign the label 1 to the $r-1$ vertices $z_1, z_2, \cdots, z_{r-1}$. Finally we assign the label 3 to the $r+2$ vertices z_r,z_{r+1},$\cdots$, z_{2r+1}. Thus, this vertex labeling f is a 4-total mean cordial labeling follows from the Table 5. This completes the proof. □

n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 2r+1$	$3r+4$	$3r+3$	$3r+3$	$3r+3$
$n = 2r$	$3r+1$	$3r+2$	$3r+2$	$3r+2$

Table 5.

Example 4.1 A 4-total mean cordial labeling of $K_{1,3} * K_{1,5}$ is given in Figure 1.

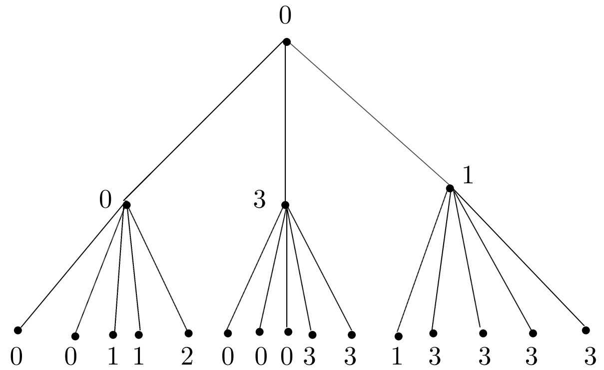

Figure 1

Theorem 4.8 *The graph $\left\langle K_{1,n}^{(1)} \Delta K_{1,n}^{(2)} \right\rangle$ is 4-total mean cordial for all values of n.*

Proof Let u_1,u_2,$\cdots$, u_n be the pendent vertices of $K_{1,n}^{(1)}$ and v_1,v_2,$\cdots$, v_n be the pendent vertices of $K_{1,n}^{(2)}$. Let u and v be the vertices of $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$ which adjacent to u_i $(1 \leq i \leq n)$ and v_i $(1 \leq i \leq n)$ respectively. Let u and v are adjacent to a new common vertex x. Note that $\left|V\left(\left\langle K_{1,n}^{(1)} \Delta K_{1,n}^{(2)}\right\rangle\right)\right| + \left|E\left(\left\langle K_{1,n}^{(1)} \Delta K_{1,n}^{(2)}\right\rangle\right)\right| = 4n+6$. Assign the labels 0,1,3 to the vertices x,u,v. Consider the vertices u_1, u_2, $\cdots$, u_n. Now we assign the label 0 to the n vertices u_1, u_2, $\cdots$, u_n. We now consider the vertices v_1, v_2, $\cdots$, v_n. Finally we assign the label 2 to the n vertices v_1, v_2, $\cdots$, v_n. Obviously $t_{mf}(0) = t_{mf}(3) = n+1$; $t_{mf}(1) = t_{mf}(2) = n+2$. □

Theorem 4.8 *The graph P_n^{+n} is a 4-total mean cordial for all values of n.*

Proof Let $u_1\ u_2 \cdots u_n$ be the path P_n. Then $V(P_n^{+n}) = V(P_n) \cup \{v_j : 1 \leq j \leq n\}$ and $E(P_n^{+n}) = E(P_n) \cup \{u_i v_j : 1 \leq j \leq n\}$. Note that $|V(P_n^{+n})| + |E(P_n^{+n})| = 4n-1$.

Case 1. $n \equiv 1 \pmod 2$.

Let $n = 2r+1$, $r \in \mathbb{N}$. Assign the label 0 to the $r+1$ vertices u_1, u_2, $\cdots$, u_{r+1}. Next we assign the label 1 to the r vertices u_{r+2}, u_{r+3}, $\cdots$, u_{2r+1}. Now we assign the label 3 to the $2r+1$ vertices v_1, v_2, $\cdots$, v_{2r+1}.

Case 2. $n \equiv 0 \pmod 2$.

Let $n = 2r$, $r \in \mathbb{N}$. We now assign the label 0 to the r vertices u_1, u_2, $\cdots$, u_r. Now we assign the label 1 to the r vertices u_{r+1}, u_{r+2}, $\cdots$, u_{2r}. Finally we assign the label 3 to the $2r$ vertices v_1, v_2, $\cdots$, v_{2r}.

Thus this vertex labeling f is a 4-total mean cordial labeling follows from the Table 6. □

n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
n is odd	$2r+1$	$2r$	$2r+1$	$2r+1$
n is even	$2r-1$	$2r$	$2r$	$2r$

Table 6.

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Enlightenment of the Combinatorial Notion

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Abstract: As is known to all, the science's function lies in guiding human activities and promotes the human civilization. However, science is the knowledge of humans ourselves on things in the universe. While it benefits humans, the accumulation of humans intrusion on the nature is increased year by year, causes the reaction of nature on humans such as the global warming, ice caps melt, sea level rise, extreme weather, drought, earthquakes, tsunamis and other natural disasters in the eyes of humans and affects human activities to some extent also. This fact awakes up humans to look at science with its leading to human activities and reflects the nature of science again. Among them, a most important question is *whether science is absolutely true or only local and conditional true?* Different answers to this question determine the attitude towards science with its application, namely living in harmony with or govern the nature. In fact, one's recognition on things is carried out by the "*six sense organs*", namely, the eyes, ears, nose, tongue, body and mind of human, which have certain limitations by their working mechanism. Accordingly, science is only a local recognition or conditional true of things. In this case, how to form a combined recognition based on their inherited topological structure of things and then to hold on the reality of things by local recognition is an important work in the development of science. In fact, this is nothing else but the combinatorial notion on recognition of things in the universe, explained detail in my book *Combinatorial Theory on the Universe* (in Chinese).

Key Words: Local recognition, recognitive limitation, scientific limitation, Smarandache multispace, mathematical combinatorics, combinatorial reality, harmonious coexistence of humans with the nature.

AMS(2010): 00A30, 00A35.

§1. Introduction

Essentially, the science lies in the recognition, holds on things in the universe and so as to promote human's living ability and benefit to humans ourselves.

Recently, I have finished the book *Combinatorial Theory on the Universe*. It systematically presents a combinatorial notion on scientific recognition, i.e., the combinatorial conjecture

[1]Reported at the book launch of *Combinatorial Theory on the Universe* (in Chinese), September 27, 2022, Beijing, P.R.China.

[2]Received August 2, 2022, Accepted September 27, 2022.

for mathematical sciences of mine proposed in 2006 to guide the recognition of things in the universe, which asserts that *any mathematical science can be reconstructed from or made by combinatorialization*[4-5], first reported by me at the *2nd Conference on Combinatorics and Graph Theory of China*, Tianjing, August 16-19, 2006. In fact, this conjecture is a kind of philosophical thought that reconstructs and develops science furthermore, and greatly promotes the human recognition of things in the universe ([9]).

The book *Combinatorial Theory on the Universe* (in Chinese)[16] including scientific recognition, mathematical sciences and philosophy was published by the Chinese Branch Xiquan House, USA in August, 2022 and distributed globally by the Global Knowledge – Publishing House. After publication of the book, the *First Construction Media of China* organized a book launch of *Combinatorial Theory on the Universe* (in Chinese) to recommend this book and spread my philosophical thoughts in Beijing on September 27, 2022. A group photo of this launch is shown in Figure 1. It is a little surprising that the participants of this launch reached more than 22,000 online and most guests gave a high evaluation on this book.

Figure 1

In this book launch, Mr.Qingzhong Ping, a professor of the *Academy of Internet Industry of Tsinghua University* evaluated this book as a universal instruction, comparable to *Critique of Pure Reason* of Kant, which aims to reveal the truth of human recognition and it has established the mathematical foundation for the digital economy, meta-universe to a certain extent; Mr.Dezhong Wang, the president of *Zhong Guan Cun Public Resources Optimal Allocation Association* evaluated this book as a "*Universal Book*" and said it is a yellow book that can accompany one's lifelong learning, has profound effect at first reading, interesting effect once again and insight effect at close reading. Many scholars in this launch claim that this book will have a place in the academic history of humans, which motivates me to write this paper for further spreading the academic notion that I explained in the book of *Combinatorial Theory on the Universe* (in Chinese).

§2. Science: Local or Conditional Reality

Certainly, science leads human activities and its function is to promote the material and spiritual civilization of humans, improve humans ability for surviving and benefit humans ourselves. In this way, believing in science and acting according to scientific laws is a basic principle of human's conduct. But *how many humans can understand this principle correctly?* The answer is unclear because most humans are standing on the humans side, understand this principle simply just for human benefits and ignore the intrusion of human activities on the nature. This is a narrow understanding of science because in the binary system consisting of humans and nature, the effect of nature on humans is immediate, visible by humans at once. However, the reaction of nature on humans caused by human's action on the nature is a delayed effect. It will appear only if the disturbance of nature caused by human activities accumulated to a certain amount, which will forms a disaster reaction on humans, i.e., produces the effect "*from quantitative change to qualitative change*" such as those destruction of the ozone layer, temperature rise, ice caps melt, extreme weather, drought and virus mutation, etc., also harmful to human development. It should be noted that this cumulative effects alone may not be visible to contemporary ones. Whence, one can not standing only on the humans side in response to scientific functions, can not only see the benefits of science to humans in present and allow the intrusion on the naturae without limitation. They should put the science with its application in the harmonious coexistence of humans with the nature and discuss its contribution and the harm to humans because science's benefits to humans should first guarantee the sustainability of human reproduction, i.e., not only the benefits to the present generation but also to the benefits of future generations. This ruler should be the basis or scientific motivation for the continuation of human civilization.

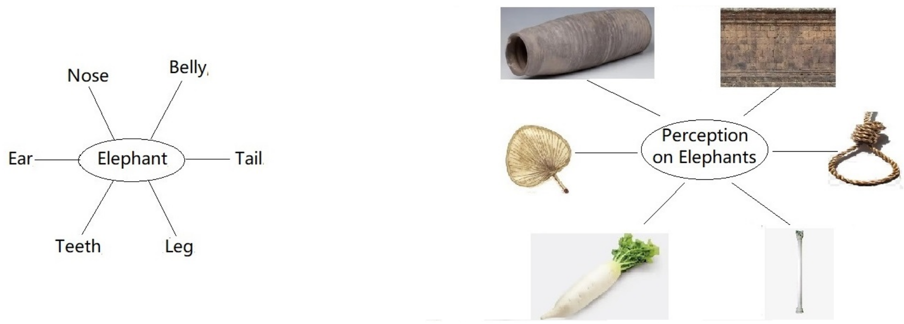


Figure 2

We should answer a basic question for recognition before exploring the science's function, i.e., *whether science is absolutely true or only local and conditional true?* The answer lies in a famous fable, i.e., the blind men with an elephant. In this fable, *why did the blind men respectively perceived the shape of an elephant as a pillar, a rope, a radish, a big fan, a wall or a pipe such as those shown in Figure 2?* Their answers are so different from the shape of an elephant in the eyes of an ordinary man. Why do they so answer is because of the blind men lack of vision. They can only perceive the shape of an elephant by touching parts of the elephant's body with their hands and different parts of an elephant are bound to be different perception. Indeed, the blind men touch different parts of the elephant's body for perceiving the shape of an elephant. Similarly, science is the human recognition of things in the universe, which is similar to the situation of the blind men in this fable. It is human's local recognition of unknown things with known characteristics. Consequently, the scientific recognition is not the real face of things but a local or conditional knowing of things. This recognitive limitation comes from the limitation of "*six sense organs*", i.e., the eyes, ears, nose, tongue, body and mind of human in perceiving things [2-3]. Compared with an ordinary man, a blind man is lack of vision, only with five or less sense organs in the perception of things. This is the reason why it results in the different recognition of blind men on an elephant shape. So, how to solve the recognitive limitations of humans is an important question. The answer is what the sophist said to the blind men in the fable, namely, "*You are all right about the elephant! The reason why you think the elephant's shape different is because each of you touches the different part of the elephant's body. In fact, an elephant has those all characteristics that you are talking about!*" Notice that the sophist uses the "*six sense organs*" of human to arouse the recognition of the blind men only with five or less sense organs on the shape of an elephant, which is also applicable to the perception of things by the "*six sense organs*" of human. That is, the reality of a thing T should be the combination of all local recognitions on T, i.e, the *combinatorial reality* and we should hold on the reality of things in the universe by the combinatorial notion. Certainly, the combinatorial notion on the reality of things is really a philosophical thought that humans follow the guidance of the sophist in the fable of the blind men with an elephant to solve the recognitive limitations of humans and then hold on the reality of things. A further generalization of this recognitive way that the sophist told the blind men is called the *Smarandache multispace* or *multisystem*([6-7, 17-18]). For example, the unified field theory, gauge field theory, electroweak theory and the standard model of particles are all Smarandache multisystems. Furthermore, a combinatorial model for the recognition of a thing T can be established on its Smarandache multispace or multisystem by considering the intersection of spaces or systems to form a combinatorial structure, a combinatorial notion also. For instance, the twelve meridians on which the vital energy and blood are moving is such an example of combinatorial model for recognizing the body of human in the traditional Chinese medicine, which are correspondent to the viscera organs of human body.

Usually, one needs to find the "*cause*" of a thing for its appearance "*effect*" and holds on the causal relationship of things. Among them, a systemic way is to decompose a thing by its appearance into the smallest units called *elements*, including the molecules, atoms, nuclei, leptons and quarks in material composition as well as the cells and genes in biological composition. Then, a behavior of the thing is assumed can be understood by its element behaviors, and the "*effect*" of its appearance is traced by the "*cause*" of the elements. This is the recognitive thought of reductionism. However, *is it possible to hold on anything in this way?* Certainly, it is an ideal model for systematically understand the causality of a thing. The difficulty lies in how to determine the elements from its appearance of things and characterize the action of elements. In this process, *how to determine the elements and how to characterize the behavior or the action of elements?* Such questions may become also the obstacle for holding on the truth of things. It is for this reason that our science is still a local knowledge of things rather than the reality of things. In this case, science with its applications needs to be evaluated under the harmonious coexistence of humans with the nature once again, to verify whether it is promoting human civilization rather than harming humans or excessive intrusion into the nature which will affects humans finally.

Realized this point, taking the quantitative recognition of things as the main line constraint on the ruler of harmonious coexistence of humans with the nature and using the combinatorial notion as the recognitive thought of things, it is a meaningful thing to systematically review and reflect on mathematical science and philosophy for the recognition of humans. In fact, I have been most concerned about the relationship between mathematical science and the recognition of things in the past ten years. Most of the topics that I reported in some academic conferences are related to the word *reality* (see [8-15] for details) because I think it is the most important that science needs to solve and it is necessary coming back to the philosophical thought of the combinatorial notion on the reality of things. Generally, a thing inherits a topological structure in space. We should establish an envelope mathematics, i.e., the "*mathematical combinatorics*" ([8-9]) for solving the limitations of scientific recognition and gradually tend to the reality of things. This is the initial intention that I wrote the book *Combinatorial Theory on the Universe* (in Chinese). For this objective, I choose the contents and arrange the order of chapters. In its expression, I apply the dialogue of a father with his daughter in sections, also with some vivid images to help the reader understand easily this book. In fact, if one removing the mathematical formulas and deduction, this book can be used as a popular scientific book. Notice that the contents containing mathematical formulas with deduction, including the last two chapters on the philosophy of science in this book is to guide those who are interested in mathematical combinatorics and aim to help them further study on the literatures for related topics and then, we can realize the harmonious coexistence of humans with the nature.

§3. What is the Book of Combinatorial Theory on the Universe About

There are 12 chapters in the book of *Combinatorial Theory on the Universe* (in Chinese), arranged in order [16]. Some main contents of each chapter are mentioned in the following.

Chapter 1 – Chapter 2 are an introduction to scientific recognition. Among them, Chapter 1 "*Ultimate Questions on the Universe*" presents the ultimate questions of the universe in the voice of a schoolboy, namely *where do we come from and where we will go?* including the celestial bodies, the earth, plants and animals and aims to briefly introduce scientific hypotheses or answers to such questions for children. Usually, a scientific answer often do not satisfy students in primary school because they do not believe in authority and like to repeatedly ask "*why*" for their frank desire of knowledge, which often leads to the disappearance of an adult's answer. But this frank attitude to study is exactly the quality of one engaged in scientific research. This chapter also introduces the answers to such ultimate problems in religion and legends in Chinese culture. It is not to propagate the superstition but in comparison because the answers in religious or the cultural legends are more vivid than scientific explanations and more accessible to primary school students. This is the object that the school education should pays attention to. Chapter 2 "*Perceptible Limitation on the Universe*" explains the limitation of human recognition of things, including the origin of human evolution and the legend of religion or god creation. Certainly, the "six sense organs" of human impact on the recognition. And in recognition, how to understand that "being out of non-being" and how to construct phenomenological theories by recognition of things such as grain cultivation and livestock rearing in the early period of humans are introduced. As an example, this chapter takes the double-slit experiment of physics in Figure 3 to summarize the limitations of human recognition, including the structural limitations of human eyes, ears, nose, tongue, body and mind which form three cases of perceptible unknown, unknowable and conditional limitation of humans. This chapter also introduces a few legends such as the Nuwa creation, Shennong taste herbs, the story of Adam and Eve ate forbidden fruits, the phenomenological theories such as the traditional Chinese medicine, the Kepler three laws of planets, etc.

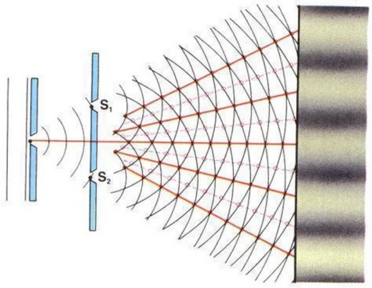
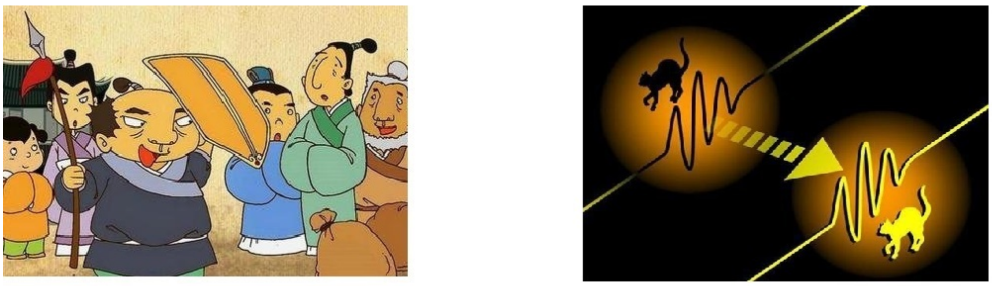


Figure 3

Chapter 3 "*Combinatorial Notion on the Universe*" explains the combinatorial notion of systematic recognition of things on local recognitions and verifies the combinatorial notion by summarizing the methods of recognition in physics, chemistry and biology as examples. It begins with the famous fable of the blind men with an elephant as an introduction with graphs, labeled graphs and topological graphs in space and then, shows the reductionism establishes local sciences such as it had done in physics, chemistry or biology but we can establish an envelope science by the combinatorial notion, i.e., the science over inherited topological structures of things. For counting elements in a Smarandache multispace or multisystem, this chapter introduces the *inclusion-exclusion principle* and *pigeonhole principle*, also introduces a generalization of the pigeonhole principle, i.e., the existence of substructure with certain relationships among individuals if the number of individuals large enough, which affirmatively answers the Ramsey problem in combinatorics. Certainly, the combinatorial notion of things derives easily the "technological combination" [1] in theory. However, we can get this conclusion by simulating the decomposition of matter. In fact, the social development is reflected by the improvement of human's ability adapting to the nature such as those of the improvement and innovation of artificial appliances, devices and facilities. Among them, a technology is a kind of methods, techniques and means to make such appliances, devices or facilities possible. Similar to the structure of matter, a technology can be viewed as a combination of blocks which have their own combinatorial structure and the block is a combination of sub-blocks with a specific function, is the next level of technology. Similarly, the function sub-block is also a combination of next level sub-blocks of functions, which is the next level technology. In this way, the decomposition of a technology level by level will eventually reaches to the elementary components in the technical composition, likewise the elementary particles. Indeed, different technologies are all combination of the elementary components of "*function*" or "*effect*".

Chapter 4 – Chapter 6 are the basis of systematic recognition of things by reductionism. Among them, Chapter 4 "*Characterizing the Universe*" introduces the characterizing methods of things in a reference frame, including the reference frame to determine the position of a thing with change characterizing, Einstein's relativity principle, vector algebra, linear space with basis, Newton mechanics, n-body problem, and the application of Newton's law of universal gravitation in determining the first, second and third velocity of universe, Lorenz transform in Einstein's special relativity theory, etc., points out that the essence of Einstein's principle of general relativity is the mathematical display of the philosophical thought that objective things are not transferred by human's will. Chapter 5 "*Systemic Recognizing the Universe*" is designed to the combinatorial notion of systemic recognition and methods of things, including the system structure, combinatorial characteristics of system and the motifs in the systemic recognizing of things. This chapter introduces physical dimension and the measuring methods of distance objects and micro particles, involved such as the quality, time, system, also the state equation of a system, the solution and the solving methods, etc. Particularly, the stability of system and the Lyapunov's direct judgment, linear and hyperbolic nonlinear systems are discussed. And corresponding to the recognitive thought of combinatorial notion of things, a philosophical thought on mathematics is introduced also. That is the combinatorial conjecture for mathematics, i.e., *mathematical combinatorics* which extends mathematics over topological structures in space. Chapter 6 "*System Synchronization*" aims to introduce an interesting synchronization phenomenon in the nature and explain the method of determining, regulating of system synchronization. It is pointed out that up to now, humans benefited from all systems simulating animal behavior such as automobile, train, ship, aircraft and other mechanical movements are based on the system synchronization and regulation of system elements. To this end, this chapter presents the major methods for determining the system synchronization, including the master function method, graph criteria as well as the introduction of error term converts a synchronization problem into the stability of system and the control of the system synchronization, explains the 2-matrix norm and Lyapunov index often used for determining system stability, system synchronization, etc.



(a) (b)

Figure 4

Chapter 7 "*Contradictory Systems*" applies the combinatorial notion of things to explain and characterize contradictory phenomena of things in the eyes of human, which is extremely different from the textbook. It should be noted that the contradiction is caused by one's recognition, implied in the definition or named process of Laozi explained in his *Tao Te Ching*, i.e., replaces the reality of things by local recognitions, not the real face of things because he said that "*the heaven and the earth view all things as straw dogs*" [2], i.e., all things are fair in the universe and the recognitive principle with "*logic consistency*" should be followed in this case. In this point, the allegory of Hanfeizi's contradiction of ancient China in Figure 4(a) is essentially consistent with the living-death state or quantum collapse of the Schrodinger's cat in Figure 4(b). The problem lies in how to describe the living-death state or contradiction of Schrodinger's cat. For resolving the contradiction, the parallel space or 2-branch tree was introduced by H.Everett to explain the living-death of Schrodinger's cat covering both the living and the death states of the cat, which is in fact a special case of Smarandache contradictory systems, Smarandache multispaces or multisystems. To this end, this chapter explains the Smarandache contradictory systems, Smarandache geometries, Smarandache multispace or multisystem and the relationship of Smarandache denied axiom with them [15]. On this basis, the application of Schrodinger cat's living-death state or quantum entanglement, quantum teleportation and the disentangling Smarandache multispaces and multisystems in the field of communication are discussed. Notice that the expression of Laozi's "*Name named not the eternal Name*" [2] in the symbol deduction of mathematics is the limitation of mathematics, including the limitations of mathematical abstracting and deduction, which usually appears as a non-harmonious group or system of non-solvable equations, i.e., a system is unsolvable with contradictions but its each equation is solvable. It is worth noting that different from the equations in classical mathematics, the combined solution of a non-harmonious group always exists, which provides the condition for characterizing such groups, including the sum stability and the product stability of non-harmonious groups.

Chapter 8 "*Complex Networks*" introduces the by-product in studying social phenomena, i.e., complex network which characterizes the social behavior of humans with certain randomness. Of course, a human's behavior is not completely random because he or she has a brain. So, it is only an assumption that human social behavior can be characterized by randomness. This chapter introduces some common random distribution, the law of large numbers, the central limit theorem and the network indexes. On this basis, the complete stochastic model introduced by Eröds and Rényi in the 1960s, the related network index and properties are introduced in details. In this field, the WS small-world network represents a breakthrough in the use of randomness to simulate social behavior, whose randomness is between the regular networks and the completely random models. At the same time, the BA scale-free network describes the connection of new sites with existing sites on the internet, corresponds to the phenomenon of "*the richer is more and more rich, the poorer is more and more poor*" in a society leading by the capitals. An extension of the BA scale-free network is the local world network. Different from the simple construction of differential equations to describe the spread of disease, the real spread of disease is carried out on the social network, which is related to one's social circle. Based on this situation, this chapter introduces also the application of various complex networks to simulate community networks, analyzes the SI, SIS, SIR models and describes the law of the spread of diseases on the social network.

Chapter 9 – Chapter 10 apply the combinatorial notion of things to generalize network flow to continuity flow, regard it as a new mathematical element for mathematics, which can be used as the mathematical model of things under the combinatorial notion. Among them, Chapter 9 "*Network Arithmetic*" begins with an introduction on some optimal problems in network and the methods operation on graphs, analyzes the possibility of simulating the behavior of things by network and constructs the operation system on network under the condition that the combinatorial structure of the network is unchanged. At this time, the network operations is similar to the operation of vectors, only need to keep the structure remains the same its in evolution, on which the metric can be introduced similar to that of the linear space and applying both the discrete or continuous model for simulating thing behaviors, namely the network sequences and the continuous networks, construct the algebraic operation, differential and integral operation on networks. In this way, the Newton-Leibniz theorem of integral operation on networks is generalized as an example. Chapter 10 "*Combinatorial Reality*" introduces the mathematical model that simulates the evolution of things under the guidance of the combinatorial notion of things, namely continuity flow. Certainly, the continuity flow is a generalization of network. That is, the labels of network vertices and edges are no longer limited to real numbers but vectors in Banach space. At the same time, operators in Banach space can be introduced on edge flows with requiring them to obey the law of flow conservation on vertices. This chapter begins with an introduction to Banach space, Hilbert space and some important results as well as three hypotheses of quantum behavior in quantum mechanics. Similar to network, continuity flow can also be regarded as a kind of mathematical element on which the operations such as the addition, subtraction, number multiplication and the Hadamard product can be defined, and the Banach space and Hilbert space on continuity flow, i.e., Banach or Hilbert flow space can be constructed and applied to the stability of continuity flow also. For describing the dynamic behavior of continuity flows, the Lagrange equation of continuity flows is obtained by the principle of least action. It should be noted that some important conclusions in functional analysis can be generalized on Banach flow spaces by using G-isomorphism operators. Particularly, the theorem of Fréchet and Riesz representation on Banach flow space holds which implies that the assumption of quantum behavior in quantum mechanics holds also. That is, whether a quantum has intrinsic structure or not it will not affects the conclusion of quantum mechanics.

Figure 5

Chapter 11 – Chapter 12 belong to the philosophy of science and discuss how science can promote human civilization in the combinatorial notion of things. Among them, Chapter 11 "*Chinese Recognizing the Universe*" aims to take Chinese civilization as an example to explain how the ancient Chinese perceived things and how the Chinese civilization formed under the thought of "*one union of the heaven and humans*". That is, the principle of the harmonious coexistence of humans with the nature. This chapter also compares some scientific achievements of the Chinese with the western. Particularly, Laozi's explaining on the creation of the universe, the relationship between the heaven, the earth and humans in his *Tao Te Ching* is compared with the theory of big bang. It points out that the western science is a local recognition of the law of things, i.e, "*Tao*" while the ancient Chinese were a recognition on the whole life cycle of things and behaviors. At the same time, this chapter introduces two typical examples for applications of continuity flows. One is the surprised theory of the 12 meridians and the relationship with the viscera organs of human body established on the Yin-Yang theory by the ancient Chinese such as the Hand Yang Ming large intestine meridian (LI), Foot Shao Yin kidney meridian(KI) and the Foot Shao Yang gallbladder meridian (GB) shown in Figure 5. Another is the corresponding of the 64 hexagrams in the *Change Book* to the continuous flow over cycles of order 6 such as those shown in Figure 6, which is essentially the soul of Chinese science and a scientific method for understanding objective things rather than a superstition.

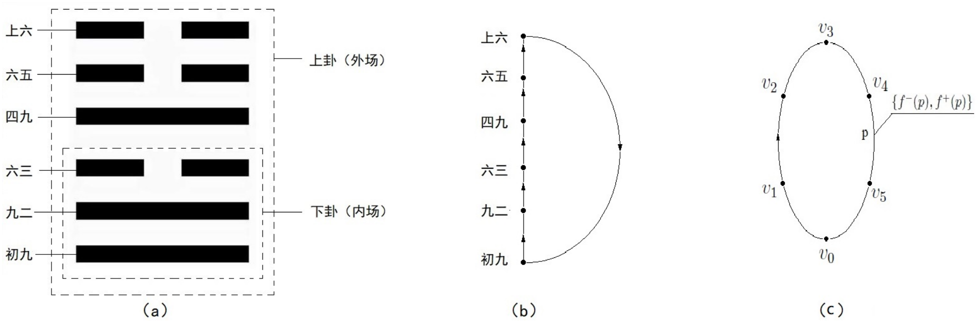


Figure 6

Chapter 12 "*Philosophy of Science*" aims to reaffirm that science is a kind of local recognition or conditional reality of things and to discuss how science promotes human civilization under the harmonious coexistence of humans with the nature. This chapter takes the *Theory of Everything* as an example, outlines the application of Smarandache multispace or multisystem in the development of science by combinatorial notion and points out also that humans should take the initiative to limit or end some fields or directions in science development. That is, science needs to have a limiting scale or standard while recognizing things in the universe for conducting the behaviour of humans and promotes humans harmonious coexistence with the nature, including those that affects the order of universe, destroys the biological diversity or affects the behavior of humans ourselves so as to realize the human activities guiding by science do not disturb the nature, which is the fundamental principle of human development.

§4. **A Most Important Objective of Science**

A central issues of philosophy of science is to discuss the ultimate goal of science or how science should develops. Certainly, science serves the ultimate goal of humans. So, *what is the ultimate goal of humans, to dominate the earth or the universe?* Of course Not! In the times substance shortage, the leader leads all humans of the clan to compete for resources or war with other clans or groups for the survival and continuation of the clan. In the times of organized production and material prosperity, humans are more greedy for spiritual enjoyment. Whence, the ultimate goal of humans is to satisfy the material needs first and then, to realize the spiritual needs of humans. At the same time, there are no a individual or population in the universe can dominates the earth or the universe, and the humans are no exception also. In this way, the ultimate goal of science in the service to humans is to promote the progress of human civilization and living in harmony with the nature. Since science is the local recognition of things by the "*six sense organs*" of human, its improper application in the benefit of humans is surely bound to disturb the nature because its effects will inevitably come back to humans after accumulated to a certain extent, resulting in a dilemma of science. However, the initiative to get out of this dilemma is in the hands of humans ourselves. That is, science should study how not to disturb the nature and realize the harmonious coexistence of humans with the nature while it benefits humans. This is the biggest challenge that science faces in guiding human activities.

Personally, I believe that the harmonious coexistence of humans with the nature is the most important objective in the development of science with the promotion of human civilization in the 21st century ([11-12]). In this process, the initiative to realize the harmonious coexistence of humans with the nature lies in humans ourselves. For this objective, the first is necessary to reflect on the immoral behavior in the past that humans excessively intrude on the nature, the second is to study the scientific programme of harmonious coexistence with the nature, to correct and eliminate the harm caused by human's excessive intrusion on the nature in the past and the third is the review and restraint of humans ourselves, including the immoral behavior in previous human activities and consciously harmonious coexistence with the nature for everyone. In this way and only in this way, the ultimate goal of our humans will comes true.

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Famous Words

If I have seen further, it is because I stand on the shoulders of giants.

By Isaac Newton, a British physicist, mathematicians and astronomers

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