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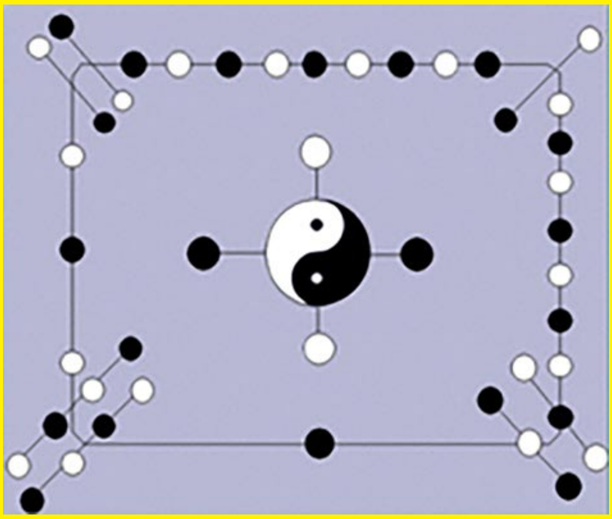

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Aims and Scope: The *mathematical combinatorics* is a subject that applying combinatorial notion to all mathematics and all sciences for understanding the reality of things in the universe, motivated by *CC Conjecture* of Dr.L.F. MAO on mathematical sciences. The **International J.Mathematical Combinatorics** (*ISSN 1937-1055*) is a fully refereed international journal, sponsored by the *MADIS of Chinese Academy of Sciences* and published in USA quarterly, which publishes original research papers and survey articles in all aspects of mathematical combinatorics, Smarandache multi-spaces, Smarandache geometries, non-Euclidean geometry, topology and their applications to other sciences. Topics in detail to be covered are:

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Topological graphs; Algebraic graphs; Random graphs; Combinatorial maps; Graph and map enumeration; Combinatorial designs; Combinatorial enumeration;

Differential Geometry; Geometry on manifolds; Low Dimensional Topology; Differential Topology; Topology of Manifolds;

Geometrical aspects of Mathematical Physics and Relations with Manifold Topology;

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Famous Words:

Among the natural enemy of mathematics, the most important thing is that how do we know something, rather than to know something.

By Pythagoras, a Greek philosopher and mathematician.

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Combinatorial Science –
How Science Leads Humans with the Nature in Harmony

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Abstract: Science has greatly improved the material civilization, also promoted the spiritual civilization of humans. Even so, *can we assert that science is consistent already with the sustainable developing of humans in 21st century?* The answer is certainly Not because science is itself only a conditional truth on the reality of things and it is verified by the disposal of wastes in industrial activities led by science over the past hundreds of years. Notice the sustainable developing of humans introduces that humans should live with the nature in harmony, namely all products in human activities must be properly disposed of and not disturb the nature but it is far from this objective until today. Actually, the universal connection between things implies the application of science should be a systemic or combinatorial one, not a solitary or fragmented one, namely it should be discovered the closed systems of substances produced in human activities with an inherited combinatorial structure G^L and then, applied it for benefiting humans without intruding to the nature. Such a pattern on science developing is essentially a pattern different from the traditional but a revolution on science, i.e., a biggest problem of science facing to promote human civilization in the 21st century.

Key Words: Combinatorial science, conditional truth, CC conjecture, Smarandache multispace, coexistence of humans with the nature in harmony, sustainable developing of humans, application rule of science, science revolution.

AMS(2010): 05C21, 91B76, 93A30.

§1. Introduction

As we all know, the nature is the condition for human survival and science is the recognition of humans ourselves on the laws of nature in order to continuously improve the ability of humans adapting to the nature. Certainly, all practices of humans showed that science has greatly improved the material civilization, also promoted the spiritual civilization of humans. Even so, *what is the relationship of humans with the nature and what is the significance of science in human developing?* Surely, the purpose of science is to serve our human society and it has been

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a great success already. For example, the power units such as the steam engines, hydraulic presses and internal combustion engines have promoted the development of textiles, mining, metallurgy and the transportation, given the birth to modern vehicles such as automobiles, trains, ships and airplanes, improved human's travelling and extended the range of human's activities, the replacing of manual labor with machines has greatly improved the productivity. And meanwhile, the development of science with technology such as the synthetic materials, aerospace technology, electronic computers, the internet and artificial intelligence as well as the wide application of a large number of robots in production and living have further reduced the labor intensity and danger of humans, improved the quality of human living which showed further the role of science in promoting the development of human society, and greatly satisfied human's material and spiritual needs. However, the more science develops, the greater the interference of humans activities led by it on the nature. *Is such a kind of development led by science really beneficial to humans ourselves?* A fundamental question for answering this is the recognition on the ultimate goal of science, i.e., *is science intended to promote humans becoming the master of universe?* In this regard, it is necessary to correctly understand the relationship of humans with the nature as well as the position and role of humans in the universe.

Clearly, the vastness of universe determines the insignificance of human's position and the limitation of recognition on things in the universe, namely we can only recognize the laws of nature within the recognizable range of humans under conditions, can not clearly recognize those of things too far away or too close to humans. For example, the observation law of a planet in 1500 light-years from the earth is its law in 1500 years ago, not the current but only a historical one. At the same time, the uncertainty principle of microscopic particles shows that the measuring on a microscopic particle by human is affected by the external fields including the body field of human itself. And so, we can only carry out the relative recognition under conditions within the range of knowability while we measures an objective thing.

So, *is there an interdependent relationship between humans and the nature?* The answer is certainly Not because human is the product of nature, namely the existing of human depends on the nature, following *the survival of the fittest* but the nature does not depend on humans because the nature objectively exists before humans, i.e., whether there is human or not, the nature is operating in its order. In such a situation, *is there an interaction between humans and the nature, and the nature can accommodate or not all the infestation or wastes of humans?* These questions seem simple but they are actually the crucial question in science, even humans developing ourselves because under the survival of the fittest, the change of nature may finally affects the human survival. For many years, lots of humans answered these questions only from the perspective of human interests and always believed that all activities that can bring benefits to humans can be carried out, ignoring the intrusion of human activities on the nature. For example, the *fire* originated in human civilization directly discharges carbon dioxide, carbon monoxide and other wastes produced by the combustion into the air. *How should we dispose of the wastes that accompany with the combustion?* If we stand only on the local interests or industrial benefits, we will not aware of the intrusion of humans to the nature but believe that the nature can indefinitely absorb all wastes produced by human activities led by science, accumulating to today. However, a large number of facts show that the cumulative effect of carbon in atmosphere is the reason of the destruction of ozone layer, rising temperatures, melting of Antarctic ice sheet, frequent droughts and extreme weather as well as the virus transmission and mutation, which have led to the deterioration of the human living environment, also harmful to human developing. Usually, such a cumulative effect can not be observed by one in a short period of time and may not be perceived by present humans for their limitation. And so, when we analyze the relationship of humans with the nature, we can not statically observe only in a specific time period or a short-term but should be historically measuring in a long-term after generations of work.

Generally, let humans (M) and the nature (U) be two independent systems and so, humans and the nature form an interacting binary system $K_2^L[M, U]$, as shown in Figure 1.

$$\mathrm{M} \longrightarrow A(M,U) \qquad\qquad A(U,M) \longleftarrow \mathrm{U}$$

Figure 1

Notice that the effect of nature on humans is immediate, instantaneous and visible to humans today, but the reaction of nature to humans caused by human's action on the nature is a delayed effect, which is an accumulated effect of human activities acting on the nature over a period of time t, i.e.,

$$A(M,U) \ \rightarrow \int\limits_{t=0}^{t} L(v,u)[s]ds = \int\limits_{t=0}^{t} (L_1(s), L_2(s), \cdots, L_n(s))\, ds = A(U,M), \qquad (1.1)$$

causing the human intrusion into the nature $A(M,U)$ from the quantitative to the qualitative change and the reaction $A(U,M)$ or the disasters to humans.

So, *how should we evaluate science with its role for promoting human civilization in the binary system $K_2^L[M,U]$ formed by humans with the nature?* We know that science is humans' own recognition on the laws of things, which is locally established by the human eyes, ears, nose, tongue, body and the mind, i.e., *six roots* on the present laws of things under conditions. Certainly, it is limited by human's own recognitive abilities and conditions, is not the essence of things, nor is it necessarily the whole pattern of the behavior of things ([4],[5],[12]), including the mathematical reality ([8]). And so, the application of any scientific conclusion is bound to be in the dilemma of that is both beneficial and harmful to humans, and the behavior of discarding directly to the nature is essentially forming $A(M,U)$ to invade the nature, and this infestation $A(M,U)$ accumulated over years will form $A(U,M)$ reaction of the nature to humans, i.e., the natural disasters or perceived by human's *six roots*. In this context, it really needs to review the intrusion of human activities led by science on the nature since the industrial revolution and repair or restore the coexistence of humans with the nature in harmony before the industrial revolution. Certainly, the main factor of which is the human activities ourselves, which is necessary to reflect on the ultimate goal of human, distinguish the relationship of humans with the nature and how science should develop with repairing the intrusion of previous scientific applications on the nature in order to achieve the ultimate goal of humans, lead human activities without intrusion to the nature under the allowable condition of nature and then, achieve the coexistence of humans with the nature in harmony, which is the biggest problems of science developing facing in 21st century. For this objective, this paper aims to use the combinatorial notion ([6], [12]), a systematization of science on the notion of coexistence of humans with the nature in harmony to hold on the intruding mechanism and the limitations of scientific recognition in promoting the material and spiritual civilization of humans, appeals for a revolution of science developing and the self-restraint of humans ourselves in steps of the coexistence of humans with the nature in harmony.

§2. **Scientific Life**

Certainly, the human living depends on the development and utilization of natural resources, which are divided into renewable and non-renewable resources. Here, the renewable resources can be further subdivided into two categories, i.e., short-term renewable such as years, months or days and long-term renewable. Generally, the resources that are beyond the limitation of life for generations, namely it requires thousands or even tens of thousands of years to regenerate are classified as non-renewable because once they are exhausted, they are no longer available to the present and serval subsequent generations. Among them, the living of humans in ancient times and the era of agricultural civilization depends on *grains* and the horses, cattle, pigs, sheep, etc., which are all renewable resources but the industrial and modern civilization led by science extended the living condition of human to non-renewable resources in a large number. So, *how are scientifically led changes in human living of natural resources intruding on the nature?* We analyze in the following.

2.1.Living Conditions. As a high-level animal on the earth, the birth of human is inseparable from the earth's environment with its providing of materials. Standing on the side of humans in $K_2^L[M, U$, we first look at the living conditions of other creatures on the earth. For example, ①Plant's living. Plant takes its continuation in cycle of the four seasons as the goal because there are environmental and material conditions on the earth adapting to the plant's living such as the sufficient rainwater, air, fertilizer, photosynthesis, etc., which can maintain their growth and reproduction; ②Animal's living. Animal is also maintaining its species going as the goal because there are environmental and material conditions on the earth that meet the survival and reproduction of the animal, namely a variety of foods that meet its survival and maintain the nutrients needed by its body, accompanying only by certain spiritual needs. Among them, the spiritual needs of animals are used as the condiments of material needs. For example, animals often fight when foraging or competing to be the leader of group, maintaining the social order of population, etc. Naturally, human's living is similar to that of the animal such as to meet the first need of survival and the species going, i.e., the necessary material conditions and the second, to pursue one's psychological satisfaction or happiness, i.e., to achieve human's spiritual needs. However, different from the ordinary animals, the human spiritual needs are mainly leaded by consciousness, which can be pursued separately from the material needs. This is the biggest difference of human to that of ordinary animal in behavior, also a main problem that we need to pay attention to in analyzing human activities led by science and achieve the coexistence of humans with the nature in harmony because all animals and plants are based on satisfying living conditions, i.e., the material needs on the earth but humans need both to meet their material and spiritual needs, and even to a certain extent, the spiritual needs are higher than that of the material needs ([2]).

So, *are there the material conditions of human living on the earth?* The answer is Yes! According to the principle of survival of the fittest, it is precisely because of the natural environment of living on the earth that led to the birth of human on the earth and then followed an era of agricultural civilization in about 7,000 years of humans. Conversely, if the changes of earth's environment are no longer suitable for human survival on a certain day, the earthlings will inevitably cease to exist. Notice that the human is born on the earth because it has the material conditions for human survival. So, *what material conditions are needed for human survival on the earth?* The answer is usually *the clothes, food, residence and traveling*, which is necessary for human survival. Here, the *clothes* refers to *dressing*, mainly reflecting the function of protecting one from cold, rain and shame; the *food* means the satiety of *diets*, i.e., from the living of picking wild fruits and hunting to imitating the growth law of grains, horses, cattle, pigs, sheep, etc., organizing cultivation and livestock feeding to obtain various nutrients for human body; the *residence* means the dwelling, i.e., shelter, house from the wind, rain and protecting one from other animals invasion; the *traveling* refers to the travel pattern, i.e., from walking, riding and other livestock traveling to horse-drawn carriage, ox cart traveling, etc., reflecting the function of shortening the traveling time by livestock in space. Notice that the agricultural civilization had also provided the material needs for human's living whether it is *grain* or horses, cattle, pigs, sheep, etc., meeting the needs of nutrition, clothing, dwelling, traveling and other materials of human body. However, they are all renewable ones on the earth, which achieved the coexistence of humans with the nature in harmony in about 7,000 years.

2.2.Scientific Mechanisms of Substance. Certainly, all substances are steady composites in the nature, not a pure molecule or atom with solid, liquid and gaseous states, which are the product of billions of years of natural mechanisms. The main way of science leading human activities and benefiting humans ourselves lies in designing the industrial production through by physical process and chemical reactions such as those of forging, decomposing or synthesizing substances that are beneficial to humans, developing and promoting the material and spiritual civilization of humans. Generally, a main way of scientific application for producing new substances needed by humans is to create artificial conditions, promote physical changes or chemical reactions between substances and achieve the change of shape, the decomposition or synthesis of substances. Usually, an industrial process contains multiple physical forging or chemical reactions with a chemical pattern

$$A_1 + A_2 + \cdots + A_m \ \rightarrow \ B_1 + B_2 + \cdots + B_n \tag{2.1}$$

in standard, where the symbol A_i denotes a molecule involved in the reaction, B_j is a produced molecule for integers $1 \leq i \leq m, 1 \leq j \leq n$, $m \geq 1, n \geq 1$ and there are two main matters that need special attention, namely ①$A_1, A_2, \cdots, A_m$ are m pure molecules involved in the chemical reaction which can be achieved only under the laboratory conditions. Usually, a compound not only contains molecules $A_1, A_2, \cdots, A_m$ but also accompany with other compounds or impurities in practice; ②All produced molecules $B_1, B_2, \cdots, B_n$ are not all beneficial to humans. Without lost of generality, assume that $B_1, B_2, \cdots, B_{s_0}$ are beneficial but $B_{s_0+1}, B_{s_0+2}, \cdots, B_n$ are harmful. Then, the outputs $B_{s_0+1}, B_{s_0+2}, \cdots, B_n$ are wastes, including the solid, liquid and gaseous in an industrial process. So, *what are the steps in the industrial process designed through the mechanism led by science and what are the substances that intrude to the nature?* Here, an analysis on the industrial processes is as follows.

(1)Resource extraction. The resource extraction is a physical process, namely it does not change the molecular structure and properties of resources but only leaves a certain number of tunnels or holes on the earth. Generally, the measure will be taken accordingly on the extraction and will not affect the stability of the earth's structure, but it is not clear to what extent the holes will affect the moving of earth. At the same time, humans are not clear about the role of liquid and gaseous resources such as oil, shale gas in operating of the earth, but from the depletion of water in a region will lead to soil drought in this region and affect the growth of crops on it, the human exploitation of earth's resources must involve in operating mechanism of the earth, but humans can not understand clearly in a short period of time. For example, we can all sense that the lack of water will cause the earth surface to dry up, crack and plant on it deplete and die, etc. So, *what is the role of coal, oil, shale gas and other products of natural evolution over hundreds of millions of years in operating of the earth? Will it affect moving of the earth, will it cause frequent geological disasters such as earthquakes, mountain collapse and mudslides after large-scale mining? Does it have an impact on the operating environment of other stars?* Indeed, humans are there while exploiting these natural resources, they only see the useful value for humans but do not or can not see its final impact on operating of the earth, even the nature because the natural regulation on such an intrusion of humans is a long-term gradual process, its effect maybe perceptible decades or centuries later, namely its reacting on humans can be perceived by humans, not on the current but the future generations. Among them, not only the depletion of resources but also the natural reaction will cause more events that are relative to human disasters, may not be seen by current. So, when looking at the relationship of humans with the nature, we can not only stand on the human side in $K_2^L[M, U]$ and look at science to benefit humans at present and then, allow it to invade the nature without restraint because science to benefit humans needs to ensure first the sustainability of human reproduction, which can benefit not only the present but also the future generations.

(2)Energy production. Energy is the power source of industrial production and living of humans, including fossil energy such as the coal, oil, natural gas, nuclear materials as well as the thermal power, hydropower, nuclear power, wind power, solar energy, biomass energy, etc. Among them, the fossil energy is generated by natural mechanisms over tens of thousands of years and is the non-renewable resource on earth. For example, a thermal power plant consists of three main systems such as the combustion system, soda system and electrical system, and its waste generated includes sulfur dioxide, sulfur dioxide, dust, fly ash, waste water, etc.; A nuclear power plant consists of nuclear island and conventional island, and its waste generated includes the radioactive solid, liquid and gaseous substances, including the waste water and washing water, radioactive gaseous substances, etc. In contrast, a hydroelectric power plants produces fewer pollutants, and its impact on the nature mainly lies in the impact of dams on the ecological environment such as water resources and the barrier of fish migration channels.

(3)Waste disposal. All energy and industrial production, including the military and defense industries may cause certain disturbances to the nature. For example, a nuclear power plant has a special system for the disposal of radioactive waste because the direct discharge will cause severe pollution of soil, river and air. Generally, if the discharge from a nuclear reactor can not be reused, a storage measure needs to be taken. Usually, the raw materials used in industrial production also contain a certain proportion of impurities in addition to the main compounds. And meanwhile, the gaseous, liquid and solid wastes are directly discharged into the natural environment in most cases unless to centralized recovery and disposal in the sealed reaction device. So, *how do humans dispose of the waste, and under what conditions to do so?* Generally, humans dispose of waste according to their own standards and then, discharge it into the nature, namely after the disposal, it is discharged into air in a gaseous state, the liquid state into the river and the solid is placed in the field. Let's take the textile printing and dyeing as an example. Usually, each ton of textile printing and dyeing needs to consume 100-200 tons of water, resulting in 80%-90% of waste water, containing slurry and its decomposition products, fiber chips, enzyme pollutants, grease, nitrogenous compounds, bleach, sodium thiosulfate, sulfide alkali, aniline, copper sulfate, formaldehyde, terephthalic acid, ethylene glycol and other harmful substances. So, *can we discharge the printing and dyeing waste water directly into the river?* Of course Not! The discharging of printing and dyeing waste water directly into the river, even if it does not contain other impurities and meeting the condition ① in (2.1), it will still cause the river to be polluted and the aquatic organisms such as the fish and shrimp will gradually die. So, all industrial wastes $B_{s_0+1}, B_{s_0+2}, \cdots, B_n$ in (2.1) need to be disposed of according to the corresponding standards before they into the nature. After then, *does the disposed $B_{s_0+1}, B_{s_0+2}, \cdots, B_n$ according to standards do not pose an intrusion to the nature?* The answer is Not also because all wastes $B_{s_0+1}, B_{s_0+2}, \cdots, B_n$ are disposed of according to human standards set by ourselves and do not restore the natural composition to its original state. In fact, if all wastes are disposed of in its original state before the industrial production, it should be existed a reverse process

$$B_1 + B_2 + \cdots + B_n \ \rightarrow \ A_1 + A_2 + \cdots + A_m \tag{2.2}$$

of (2.1). Even if the inverse process (2.2) of (2.1) exists, its implementation mechanism with condition $\mathcal{C}^-$ needs to be researched further under the human ability.

2.3.Spiritual Need on Science. Generally, human needs are physiological needs in the first, including food, water, air, sexual desire, health, etc., and then follows by the spiritual needs, including personal safety, life stability, protection from physical pain, disease and other safety needs as well as the social needs such as the socialization, respect, self-realization, personal feelings of achievement or self-worth, recognition, respectful needs and the self-realization needs for the pursuit of ultimate goal of life. In fact, all human activities led by science are more around human's spiritual needs after the material need is satisfied. For example, Energy science has been built to meet the human needs for protection from the frost damage, cooking meals, industrial power and organizing the production, transmission of energy; Architectural science has been established and the light textile, building material production, construction technology have been developed in order to meet the comfort and safety of human's living, which constructs a large number of artificial buildings, commercial housing and the household industries; Nutrition and medical science have been established and the food processing, light industry, pharmaceutical industry have been established in order to ensure human food safety and disease prevention; The establishment of traffic science, the research and development of driving devices, the creation of trains, cars, planes and ships are shortening the time and space distance of human traveling. In order to meet the needs of human's information exchange, the information science has been established with the terminal facilities such as the internet, telephone, mobile phone and mobile communication. Similarly, in order to reduce the labor intensity and reduce the danger, the artificial intelligence production or auxiliary systems have been constructed, robots that can replace manual labor in whole or in part have been developed to achieve the automatic production and the automatic monitoring, alarm systems and the facilities have been developed to meet the needs of industrial production and social management; And also, the devices, facilities and weapons of mass destruction produced to meet the needs of politics, military and the national defense have increased, the excessive exploitation and consumption of natural resources to satisfy the spiritual desires of a small group or number of humans become the norm. Meanwhile, whether it is energy or industrial production, the output of waste and the intrusion to the nature are greatly increased compared with the era of agricultural civilization such as those of the nitrogen and hydrogen compounds, sulfur dioxide, nitric oxide, nitrogen dioxide, suspended particulate matter and other waste emissions, and the threat to human survival by reaction of the nature is forming in the extreme disasters.

2.4.Application Rule of Science. Usually, science is a local recognition of humans on the laws of objective things, i.e., it is the recognition under conditions, namely science is the local or conditional truth on things. Thus, its improper application will bring certain harms to humans, which forms a situation that science is both of beneficial and harmful, i.e., dilemma to humans. So, *what is the application rule of science?* The application rule of science means that the application of scientific conclusion should be in accordance with the principle that no intruding to the nature, namely they should be applied and benefit humans ourselves in the field and under the conditions in which their conclusions are established with all products properly disposed of. But in fact, most of the human activities led by science fail to strictly comply with this principle, which is reflected in the fact that the energy exploitation, preparation and industrial activities for promoting the material and spiritual civilization of humans are not carried out in a strictly closed container and most of the wastes generated in process, including the gaseous, liquid and solid are directly discharged into the nature, which forms a factual invading to the nature including the air, water and land around or on the earth.

Certainly, the abusive application of science to meet the needs of a few humans in certain extent such as the domination, resource plunder, capital profit-seeking, excessive pursuit of spiritual enjoyment in history resulted in the human activities led by science violating the application rule of science ([1], [9]) such as those of ①Applying the nuclear fission and nuclear fusion of physics to the manufacture of nuclear weapons such as atomic bombs and hydrogen bombs, destroying humans, buildings and structures of humans; ②Applying the bacteria, protoorganisms, viruses, etc. in biotechnology to the manufacture of biological weapons and carries out large-scale transmission in the population through poisonous gas or infected mosquitoes, lice, bed bugs and human breathing; ③Capturing asteroids or capturing precious metals enormously on other planets, affecting the natural operating; ④Causing a large number humans to lose their jobs by the internet and AI while reducing labor intensity; ⑤Research and developing various computer viruses spreading on the internet, destroying or interfering with various management systems of the commerce, finance, government and military on which humans depend; ⑥Violating the principle of survival of the fittest, artificially changing the gene structure formed naturally for capital profits and affecting the social order of creatures or humans. For example, the growing number of the internet, mobiles and commercial housing has blocked the communication in humans. AI, including the robot was originally developed to facilitate the production and life for those of intellectual disabilities or partial physical deficiency, which should not be widely applied to general humans. Otherwise, while it provides convenience for humans it will also weaken or replace the natural functions of human body adapting to the natural environment. Particularly, those AI robots with functions such as those of learning, dialogue and interaction further weaken the social structure and the function of human body, which will urge a few ones to control the robots or natural humans modified only by relying on keys in their hands, form a new domination or robot empires, endangering the survival of humans further, and so on.

Notice that there is a dilemma in the application of science. So, *how should science developing to promote the coexistence of humans with the nature in harmony?* The answer is that science should develop with a criteria that leads all human activities in harmony with the nature, i.e., promoting the coexistence of humans with the nature in harmony on the application rule of science by systematic or combined scientific conclusions rather than a partial or an isolated one, actively terminates those of science that only satisfies the needs of humans ourselves but intruding too much to the nature ([3]).

§3. **Combinatorial Science in Closed System of Substances**

The existence of universal connections between things in the universe and the local recognition of things by humans ourselves naturally lead to a conclusion that the reality of a thing is nothing else but a Smarandache multispace ([13]-[14]) in recognition, which provides the combinatorial notion on the recognition of things ([6]). Generally, disposing of the wastes $B_{s_0+1}, B_{s_0+2}, \cdots, B_n$ to their original states $A_1, A_2, \cdots, A_m$ in chemical reaction (2.1), an effective way is by the reverse process (2.2) of (2.1). However, by this way all products $B_1, B_2, \cdots, B_{s_0}$ that are beneficial to human needs should be readded to $B_{s_0+1}, B_{s_0+2}, \cdots, B_n$ and then, the wastes $B_{s_0+1}, B_{s_0+2}, \cdots, B_n$ can be synthesized from (2.2) into $A_1, A_2, \cdots, A_m$ under condition $\mathcal{C}^-$. Generally, the implement condition $\mathcal{C}^- \supset \mathcal{C}$, i.e. the realizing condition of inverse process (2.2) is usually stronger than that of (2.1). But then, humans will not be able to get $B_1, B_2, \cdots, B_{s_0}$ and benefit from (2.1) also. In other words, to obtain $B_1, B_2, \cdots, B_{s_0}$ from the industrial pattern (2.1) necessarily produces the wastes $B_{s_0+1}, B_{s_0+2}, \cdots, B_n$. Thus, if we have no new evolutionary mechanism for applying or properly disposing of outputs $B_{s_0+1}, B_{s_0+2}, \cdots, B_n$ in (2.1), it is inevitable to form an intrusion $A(M,U)$ of (1.1) into the nature and it is impossible to achieve the coexistence of humans with the nature in harmony, namely the disposal of wastes produced by (2.1) can not be carried out in its reverse process. Notice that all science conclusions are local or conditional truth of humans and the existence of universal connection between things. There must be an inherent combinatorial structure G^L in scientific conclusions on the reality of things ([12]), which implies that any scientific conclusion should not be applied in isolation but should follow the application rule of science on a systematic recognition. We should find the inherited structure G^L in those of related conclusions such as the pattern (2.1) produces an evolutionary combination of substances so as not to invade the nature, and actively sequester wastes generated by (2.1) rather than discarding them to the nature if necessary. This is the biggest problem of science developing facing in 21st century.

3.1.Evoluing System of Substance. After hundreds of thousands of years, all substances in the nature have formed a relatively stable evolutionary closed system, following the law of conservation of substance. And so, the science combinatorization is to discover the closed system of evolving around substances in human activities according to the rule of substance evolving so as to change the single or partial application mode of science and then, lead human activities to achieve the coexistence of humans with the nature in harmony and complying with the application rule of science. For a typical example, the molecules $A_1, A_2, \cdots, A_m$ in the pattern (2.1) are reorganized and evolved into new substances $B_1, B_2, \cdots, B_n$ under conditions with atoms as the basis, namely $A_1, A_2, \cdots, A_m$ is a permissible group $\mathcal{G}(A_, A_2, \cdots, A_m)$ of molecules in 1st level, and the output of $B_1, B_2, \cdots, B_n$ is an induced molecular group $\mathcal{G}(B_1, B_2, \cdots, B_n)$ in 2nd level. In this case, the pattern (2.1) is

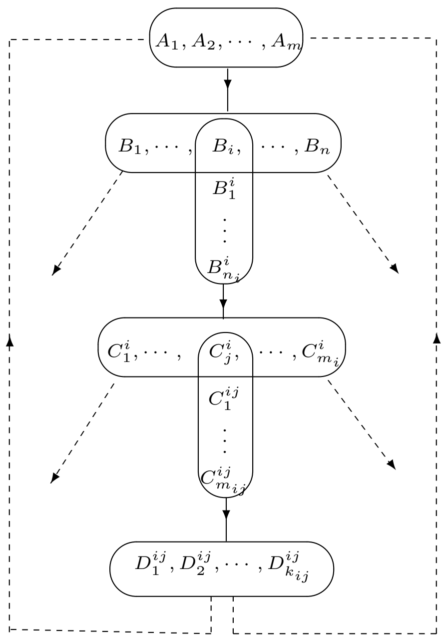

Figure 2

$$\mathcal{G}(A_, A_2, \cdots, A_m) \to \mathcal{G}(B_1, B_2, \cdots, B_n) \tag{3.1}$$

Among them, whether they are the beneficial substances $B_1, B_2, \cdots, B_{s_0}$ or the disposing of wastes $B_{s_0+1}, B_{s_0+2}, \cdots, B_n$ to humans, both of the cases can produce new substances, or other substances can be found to form new permissible groups with one or more molecules of $B_1, B_2, \cdots, B_n$. Without loss of generality, for any integer $1 \leq i \leq l$ assume that the molecules $B_1^i, B_2^i, \cdots, B_{n_i}^i$ with the molecule B_i constitute a permissible group $\mathcal{G}\left(B_i, B_1^i, B_2^i, \cdots, B_{n_i}^i\right)$ of molecules in 2nd level and there is

$$\mathcal{G}\left(B_i, B_1^i, B_2^i, \cdots, B_{n_i}^i\right) \to \mathcal{G}\left(C_1^i, C_2^i, \cdots, C_{m_i}^i\right), \quad 1 \leq i \leq n \tag{3.2}$$

Similarly, for any integer $1 \leq j \leq m_i$, assume that the molecules $C_1^{ij}, C_2^{ij}, \cdots, C_{m_{ij}}^{ij}$ with the molecule C_j^i form a permissible group $\mathcal{G}\left(C_j^i, C_1^{ij}, C_2^{ij}, \cdots, C_{m_{ij}}^{ij}\right)$ of molecules in 3rd level and there is

$$\mathcal{G}\left(C_j^i, C_1^{ij}, C_2^{ij}, \cdots, C_{m_{ij}}^{ij}\right) \to \mathcal{G}\left(D_1^{ij}, D_2^{ij}, \cdots, D_{k_{ij}}^{ij}\right), \quad 1 \leq i \leq n, \ 1 \leq j \leq m_i, \tag{3.3}$$

which results in an induced molecular group $\mathcal{G}\left(D_1^{ij}, D_2^{ij}, \cdots, D_{k_{ij}}^{ij}\right)$ in 4th level, $\cdots$.

Notice that this evolving process can go on forever, as shown in Figure 2. So, *under what conditions does such an evolving constitutes a closed system?* The answer is the new substance induced by the permissible groups of molecules have already appeared in permissible groups of molecule with lower level, namely the collection

$$\left\langle A_{i_0}; B_i, B_j^i; C_j^i, C^{ij}; D_k^{ij}, D_l^{ijk}, \cdots, 1 \leq i_0 \leq m, 1 \leq i \leq n, 1 \leq j \leq n_i, 1 \leq k \leq m_{ij}, \cdots \right\rangle \tag{3.4}$$

forms a closed set under the evolving rule of molecules. In this case, the molecular system (3.4) naturally forms a substance flow ([7],[15]) under the rules of material evolving and follows the conservation law of substance in the nature. Furthermore, let the vertex set $V(G)$ of graph G consists of each molecule in (3.4) and if the permissible groups of molecules $\mathcal{G}(\cdots, u, \cdots) \to \mathcal{G}(\cdots, v, \cdots)$, connect an edge (u, v) between u and v, and if a molecule v in an induced groups of molecule has appeared already in a permissible group of molecules, connect an edge (v, v) from v to v in the molecular group with lower level. Now, let the output of molecule v in permissible group of molecules $\mathcal{G}(\cdots, u, \cdots) \to \mathcal{G}(\cdots, v, \cdots)$ be $C(v)$ and label the edge (u, v) or (v, v) by $C(v)$. In this way, a mathematical model on the evolution of substance flow is obtained, called the *continuity flow* G^L with all edge-end operators 1 ([10]-[12]). At this time, the conservative law of substance in nature is shown by the conservative law of each vertex in G^L, which is a closed system of substances evolving in human activities.

Generally, by the coexistence of humans with the nature in harmony, we should discover the closed system similar to (3.4) on substance evolving and then, establish the continuity flow G^L on each of human activities for the sustainable developing of humans.

3.2. Allowable Contents of Living. Certainly, the construction of substance evolving system G^L of human activities is an ideal situation in which all human activities follow the natural laws and do not intrude with the nature. However, there are many important issues that need to be clarified. It should be pointed out that the discharge of wastes $B_{s_0+1}, B_{s_0+2}, \cdots, B_n$ into the nature for obtaining benefits $B_1, B_2, \cdots, B_{s_0}$ of (2.1) in case of difficult discovering the closed evolving system G^L of substances, is essentially an intruding on the nature and violating the application rule of science. Among them, the problem that the content of each substance suitable for human survival in the nature or its limitation has not yet been solved, which results in that all wastes in industrial production are disposed of according to the standards set by humans ourselves. It seems to be consistent with the laws of substance evolving but the impact may be on the descendants of humans because the nature's self-regulation or reaction to human intrusion is a cumulative effect, not necessarily on the present. In the case that the closed substance evolving system G^L of human activities can be difficult discovered in a short term, the coexistence of humans with the nature in harmony urgently needs to clarify the substance content or boundary suitable for human survival in the binary system $K_2^L[M,U]$, whose mechanism is that assume there are n substances suitable for human survival in the nature, expressed as a vector $(\omega_1,\omega_2,\cdots,\omega_n)$, $L(v,u)=(L_1(t),L_2(t),\cdots,L_n(t))$ is the influent vector of human activity on the n content of natural substance at the time t and $\mathcal{R}^P[t]=(\mathcal{R}_1[t],\mathcal{R}_2[t],\cdots,\mathcal{R}_n[t])$ is the corresponding natural absorbable vector. Then, the cumulative intrusion of human activities on the content of natural substances is an integral

$$A(M,U) \ = \ \int_{t=0}^{t} (L_1(s),L_2(s),\cdots,L_n(s))\,ds.$$

At this time, if the vector composed of maximum content of natural substances allowable for human survival is $\mathcal{T}_{\max}^P=(\mathcal{T}_1^{\max},\mathcal{T}_2^{\max},\cdots,\mathcal{T}_n^{\max})$, then the inequality

$$\int\limits_{t=0}^{t} \big(L(v,u)[s]-\mathcal{R}^P[s]\big)\,ds \ \leq \ \mathcal{T}_{\max}^P \tag{3.5}$$

is a natural allowable condition for human activities. Otherwise, it is an intrusion of humans into the nature, and may cause the nature's reaction to humans or natural disasters known by humans. Consequently, while discovering the closed substance evolving system G^L of human activities, it is necessary to find the range or limitation of natural substances $(\omega_1,\omega_2,\cdots,\omega_n)$ suitable for human survival, reduce the emission of human wastes until the zero emission of them to the nature and then, form the closed substance evolving system G^L. In this process, if the current disposal process or technology on wastes of humans do not meet the discharge of natural allowing, the waste should be effectively sealed for waiting the progress of disposal technology after it completely complies with the natural allowing.

3.3.Reflecting on Human Civilization. The coexistence of humans with the nature in harmony requires us to reflect on human activities led by science, actively take the corresponding measures for making up all mistakes in the past and adjust misconduct of humans ourselves.

The first is to review on scientific research and human activities led by science that excessive intrude to the nature, strictly restrict or end those of them, including ①Affecting the operating order of universe. Today, unless some hypotheses and phenomenology on the operating of universe, humans do not really know the internal rule or mechanism of universe. Until today, various aircrafts launched for space exploration have formed garbage in space after being scrapped, which increase the probability of collision with asteroids or human vehicles and should be properly handed. Furthermore, a few of ambitious humans plan to seize rare metals such as the gold and other precious metals on extraterrestrial planets or capture asteroids in order to pursue the economic interests, which will eventually affect the operating of universe or change the universe order and may have an immeasurable impact on the earth, including humans ourselves; ②Destroying the biodiversity. An outstanding manifestation of biodiversity is the survival of the fittest living in environment. And so, the destruction of biodiversity directly affects the food chain and climate on the earth. So far, humans have only a limited understanding on most other creatures, do not fully know their living conditions. In this case, the application of local scientific conclusion is easy to induce the destruction on the biodiversity. For example, the pesticides can effectively control pests and diseases, eliminate weeds, improve crop yields and quality. However, most pesticides can be not or difficult to be naturally degraded, which results in environmental pollution and pesticide residues on crops, destroys the soil structure, pollutes terrestrial water resources and kills the creatures or microorganisms in soil and water to a certain extent; ③Changing the natural operating order of human body. Notice that human is the product of the environment. But, *why was human born on the earth, not on other planets?* The answer is that there are environmental conditions on the earth that are suitable for human existence. In this case, the first should be to protect the earth's environment, including to restrict or end those behaviors that may destroy the earth's environment for capital profit and the second, is to protect the human body itself, limit or terminate those scientific research with applications that try to change the laws of reproduction and operating of human because whether it is the impact on the earth's environment or on human body, the impact of such an application of science on humans has a time delay effect, which will affect the descendants of humans for generations. Thus, under the coexistence of humans with the nature in harmony, those of scientific researches with applications, including genetic engineering, gene editing and the deep integration of AI with the human body must be strictly restricted or ended.

Secondly, it is to find scientific solutions for the coexistence of humans with the nature in harmony, correct and eliminate the harm caused by excessive intrusion of humans into the nature in the past. Certainly, the industrial civilization is too dependent on non-renewable resources of the earth, which throws waste and aggravates the intrusion of human activities to the nature. For example, there is a threshold R_{human} in air, by which humans can only survive on the earth if the amount R_{air} of carbon dioxide in air satisfies $R_{air} \leq R_{human}$. Thus, if the nature is not enough to absorb the carbon dioxide emitted by human activities, the redundant carbon dioxide will float in air, and it is necessary to artificially reduce the carbon dioxide emissions to meet the living condition $R_{air} \leq R_{human}$. And meanwhile, even if it holds with $R_{air} \leq R_{human}$ but the accumulation of carbon dioxide in air to a certain amount will cause also the destruction of atmospheric ozone layer, result in the natural phenomena such as those of the rising of global temperature, melting ice sheets, rising sea levels, extreme weather, drought, virus mutation and others, affect the human survival also. To this end, the first is to study the scientific mechanism of carbon neutrality, reduce or absorb carbon dioxide in air using for humans, restore the content of carbon dioxide in air under R_{human} below; Next is to improve the equipment and facilities in human production and living, collect and dispose of the carbon dioxide produced by each of them in a centralized manner, no longer directly discharge it into the nature; And again is to extend the cultivation of green plants, especially to those of occupied for production or living as the roofs, exterior walls of buildings and extending water areas further, etc.

Thirdly, it is the self-examination and self-discipline on behavior of humans. Indeed, science has promoted the civilization of humans in material and the spiritual. But how science should lead the coexistence of humans with the nature in harmony is a major problem that humans need to reflect on and solve. First of all, the coexistence of humans with the nature in harmony is an overall step of humans while science is human's local recognition or conditional truth on laws of things; Next, the only criterion for the promotion of human progress by science lies in the fact that science increases the well-being of humans but without intruding on the nature; And again, it should be the human self-discipline. Certainly, realizing the coexistence of humans with the nature in harmony lies in the correct understanding and human actions, i.e, depends on humans ourselves. For this objective, humans need to follow other creatures on the earth, consciously comply with the natural laws, quit human's subjective greed, abandon those of seemingly glamorous flashes with the philosophy that the clothes, food, residence and traveling are for living only, which needs one to consciously self-discipline on spiritual needs. And meanwhile, it needs to examine all of scientific researches with achievements, and all human activities led by science, actively ends those of actions that are too intrusive to the nature such as those of only for the temporary interests or luxury of a few humans as well as the resource plundering, harm to humans ourselves or the ecological environment, correctly understand and apply the conditions of scientific conclusions, promote the science combinatorization in closed systems, change the intrusion on nature in existing technological path and then, construct the substance evolving system G^L of human activities. This is a biggest problems of science for promoting human progress in 21st century, which is the foundation of coexistence of humans with the nature in harmony and so, needs to be faced by all humans on the earth.

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Some New Inequalities for
N-Times Differentiable Strongly Godunova-Levin Functions

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Abstract: In this manuscript, by using an integral identity together with the Hölder integral inequality and Hölder-İşcan integral inequality we establish several new inequalities for n-times differentiable strongly Godunova-Levin functions. In addition, the results obtained in this article coincide with those obtained previously in special cases.

Key Words: Convex function, Godunova-Levin function, Hölder integral inequality.

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§1. Introduction

For some inequalities, generalizations and applications about convexity theory and inequalities (see [6, 7, 8, 10]). Recently, in the literature there are so many studies about n-times differentiable functions on several kinds of convexities. In references [3, 4, 5, 15, 19], readers can find some results about this study. Many papers have been written by a number of mathematicians concerning inequalities for different classes of convex and Godunova-Levin functions see for instance the recent papers [13, 14, 16, 17, 20] and the references within these papers. Strongly convex functions play an important role in optimization theory, mathematical economics and some other branches of science. Since strongly convexity is a strengthening of the notion of convexity, some properties of strongly convex functions are just “stronger versions of known properties of convex functions.

Definition 1.1 *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality*

$$f\left(tx + (1-t)y\right) \leq tf\left(x\right) + (1-t)\, f\left(y\right)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$. This definition is well known in the literature.

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Definition 1.2 *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be Godunova-Levin function, if*

$$f\left(tx + (1-t)y\right) \leq \frac{f\left(x\right)}{t} + \frac{f(y)}{1-t}$$

where $\forall x, y \in I,\ t \in (0,1)$.

Definition 1.3([18]) *Let $I \subset \mathbb{R}$ be an interval and c be a positive number. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called strongly convex with modulus c if*

$$f\left(ta + (1-t)b\right) \leq tf\left(a\right) + (1-t)f(b) - ct\left(1-t\right)\left(b-a\right)^2$$

for all $a, b \in I$ and $t \in [0,1]$.

If a function $f : I \to \mathbb{R}$ is strongly convex with modulus c, then

$$f\left(\frac{a+b}{2}\right) + \frac{c}{12}\left(b-a\right)^2 \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} - \frac{c}{6}\left(b-a\right)^2$$

for all $a, b \in I,\ a < b$. In this definition, if we take $c = 0$, we get the definition of convexity in the classical sense.

Definition 1.4 *Let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \to \mathbb{R}$ is an h-convex function, or that f belongs to the class $SX\left(h, I\right)$, if f is non-negative and for all $x, y \in I$, $\alpha \in (0,1)$ we have*

$$f\left(\alpha x + (1-\alpha)y\right) \leq h(\alpha)f\left(x\right) + h(1-\alpha)f\left(y\right).$$

If this inequality is reversed, then f is said to be h-concave, i.e. $f \in SV\left(h, I\right)$. It is clear that, if we choose $h(\alpha) = \alpha$ and $h(\alpha) = 1$, then the h-convexity reduces to convexity and definition of P-function, respectively.

Readers can look at [2, 11] for studies on h-convexity.

Definition 1.5([1]) *Let $(X, \|.\|)$ be a real normed space, D stands for a convex subset of X, $h : (0,1) \to (0,\infty)$ is a given function and c is a positive constant. Then we say that a function $f : D \to \mathbb{R}$ is strongly h-convex with module c if*

$$f\left(tx + (1-t)y\right) \leq h(t)f(x) + h(1-t)f(y) - ct\left(1-t\right)\|x-y\|^2 \tag{1.1}$$

for all $x, y \in D$ and $t \in (0,1)$. In particular, if f satisfies (1.1) with $h(t) = t$, $h(t) = t^s$ $(s \in (0,1))$, $h(t) = \frac{1}{t}$, and $h(t) = 1$, then f is said to be strongly convex, strongly s-convex, strongly Godunova-Levin functions and strongly P-function, respectively. The notion of h-convex function corresponds to the case $c = 0$.

Theorem 1.1(Hölder-İşcan integral inequality, [9]) *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are* *real functions defined on interval* $[a,b]$ *and if* $|f|^p$, $|g|^q$ *are integrable functions on* $[a,b]$, *then*

$$\begin{aligned}\int_a^b |f(x)g(x)|\,dx &\leq \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right.\\ &\quad\left.+\left(\int_a^b (x-a)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}. \qquad (1.2)\end{aligned}$$

Theorem 1.2 *Let* $h:(0,1)\to(0,\infty)$ *be a given function. If a function* $f: I\subseteq\mathbb{R}\to\mathbb{R}$ *is Lebesgue integrable and strongly* h*-convex with module* $c>0$, *then*

$$\begin{aligned}\frac{1}{2h\left(\frac{1}{2}\right)}\left[f\left(\frac{a+b}{2}\right)+\frac{c}{12}(b-a)^2\right] &\leq \frac{1}{b-a}\int_a^b f(x)dx\\ &\leq (f(a)+f(b))\int_0^1 h(t)dt-\frac{c}{6}(b-a)^2\end{aligned}$$

for all $a,b\in I$, $a<b$.

In [1], the authors gave the following definition.

Throughout this paper we will use the following notations and conventions. Let $J=[0,\infty)\subset\mathbb{R}=(-\infty,+\infty)$, and $a,b\in J$ with $0<a<b$ and

$$\begin{aligned}A(a,b) &= \frac{a+b}{2},\\ L_p(a,b) &= \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}},\ a\neq b,\ p\in\mathbb{R},\ p\neq -1,0\end{aligned}$$

be the arithmetic, geometric, identic, harmonic, logarithmic, generalized logarithmic mean for $a,b>0$ respectively.

For we obtain the main results we will use the following Lemma [15].

Lemma 1.1 *Let* $f: I\subseteq\mathbb{R}\to\mathbb{R}$ *be* n*-times differentiable mapping on* I° *for* $n\in\mathbb{N}$ *and* $f^{(n)}\in L[a,b]$, *where* $a,b\in I^\circ$ *with* $a<b$, *we have the identity*

$$\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)\,dx=\frac{(-1)^{n+1}}{n!}\int_a^b x^n f^{(n)}(x)\,dx.$$

In [13], the authors proved the following theorems.

Theorem 1.3 ([13]) *For* $\forall n\in\mathbb{N}$, *let* $f: I\subset(0,\infty)\to\mathbb{R}$ *be* n*-times differentiable function on* I° *and* $a,b\in I^\circ$ *with* $a<b$. *If* $\left|f^{(n)}\right|^q$ *for* $q>1$ *is Godunova-Levin function on* $[a,b]$, *then the* *following inequality holds:*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|$$
$$\leq \frac{1}{n!}(b-a)^{\frac{3}{q}}C^{\frac{1}{p}}(a,b,n,p)A^{\frac{1}{q}}\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right),$$

where $\frac{1}{p}+\frac{1}{q}=1,\ 1<p<2$ *and* $C(a,b,n,p)=\int_a^b \frac{x^{np}}{(x-a)^{p-1}(b-x)^{p-1}}dx.$

Theorem 1.4([13]) *For* $n\in\mathbb{N}$*; let* $f:(0,\infty)\subset\mathbb{R}\to\mathbb{R}$ *be* n*-times differentiable function and* $0\leq a<b$*. If* $\left|f^{(n)}\right|^q\in L[a,b]$ *and* $\left|f^{(n)}\right|^q$ *for* $q>1$ *is Godunova-Levin function on* $[a,b]$*, then the following inequality*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|\leq\frac{1}{n!}(b-a)^{\frac{2}{q}}D^{\frac{1}{p}}(a,b,n,p)$$
$$\times\left[\left|f^{(n)}(b)\right|^q\left\{bL_n^n(a,b)-L_{n+1}^{n+1}(a,b)\right\}+\left|f^{(n)}(a)\right|^q\left\{L_{n+1}^{n+1}(a,b)-aL_n^n(a,b)\right\}\right]^{\frac{1}{q}},$$

holds, where $\frac{1}{p}+\frac{1}{q}=1,\ 1<p<2$ *and* $D(a,b,n,p)=\int_a^b \frac{x^{n}}{(x-a)^{p-1}(b-x)^{p-1}}dx.$

§2. Main Results

Theorem 2.1 *For* $\forall n\in\mathbb{N}$*, let* $f:I\subset(0,\infty)\to\mathbb{R}$ *be* n*-times differentiable function on* I° *and* $a,b\in I^\circ$ *with* $a<b$*. If* $\left|f^{(n)}\right|^q$ *for* $q>1$ *is a strongly Godunova-Levin function with modulus* c *on* $[a,b]$*, then the following inequalities*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|$$
$$\leq \frac{1}{n!}(b-a)^{\frac{3}{q}}C^{\frac{1}{p}}(a,b,n,p)\left[A\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right)-c\frac{(b-a)^2}{30}\right]^{\frac{1}{q}},$$

holds, where $\frac{1}{p}+\frac{1}{q}=1,\ 1<p<2$ *and* $C(a,b,n,p)=\int_a^b \frac{x^{np}}{(x-a)^{p-1}(b-x)^{p-1}}dx.$

Proof Firstly, let $x\in(a,b)$. Then, we can write the following inequalities

$$\left|f^{(n)}(x)\right|^q=\left|f^{(n)}\left(\frac{x-a}{b-a}b+\frac{b-x}{b-a}a\right)\right|^q\leq\frac{\left|f^{(n)}(b)\right|^q}{\frac{x-a}{b-a}}+\frac{\left|f^{(n)}(a)\right|^q}{\frac{b-x}{b-a}}-c\frac{x-a}{b-a}\frac{b-x}{b-a}(b-a)^2,$$

$$\left|f^{(n)}(x)\right|^q\leq\frac{b-a}{x-a}\left|f^{(n)}(b)\right|^q+\frac{b-a}{b-x}\left|f^{(n)}(a)\right|^q-c(x-a)(b-x)$$

$$(x-a)(b-x)\left|f^{(n)}(x)\right|^q\leq(b-a)(b-x)\left|f^{(n)}(b)\right|^q+(b-a)(x-a)\left|f^{(n)}(a)\right|^q-c(x-a)^2(b-x)^2. \tag{2.1}$$

The last inequality is also valid in case of $x \in [a,b]$. If $\left|f^{(n)}\right|^q$ for $q > 1$ is a strongly Godunova-Levin function on the interval $[a,b]$, using Lemma 1.1, the Hölder integral inequality and the inequality (2.1) we have

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right| \leq \frac{1}{n!}\int_a^b x^n\left|f^{(n)}(x)\right|dx$$

$$\begin{aligned}
&\leq \frac{1}{n!}\left(\int_a^b \frac{x^{np}}{(x-a)^{p-1}(b-x)^{p-1}}dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)(b-x)\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}\\
&\leq \frac{1}{n!}C^{\frac{1}{p}}(a,b,n,p)\left(\int_a^b\left\{\begin{array}{c}(b-a)(b-x)\left|f^{(n)}(b)\right|^q+(b-a)(x-a)\left|f^{(n)}(a)\right|^q\\ -c\,(x-a)^2\,(b-x)^2\end{array}\right\}dx\right)^{\frac{1}{q}}\\
&= \frac{1}{n!}C^{\frac{1}{p}}(a,b,n,p)\left(\begin{array}{c}(b-a)\left|f^{(n)}(b)\right|^q\int_a^b(b-x)dx+(b-a)\left|f^{(n)}(a)\right|^q\int_a^b(x-a)dx\\ -c\int_a^b(x-a)^2\,(b-x)^2\,dx\end{array}\right)^{\frac{1}{q}}\\
&= \frac{1}{n!}C^{\frac{1}{p}}(a,b,n,p)\left[(b-a)\left|f^{(n)}(b)\right|^q\frac{(b-a)^2}{2}+(b-a)\left|f^{(n)}(a)\right|^q\frac{(b-a)^2}{2}-c\frac{(b-a)^5}{30}\right]^{\frac{1}{q}}\\
&= \frac{1}{n!}C^{\frac{1}{p}}(a,b,n,p)\left[\left|f^{(n)}(b)\right|^q\frac{(b-a)^3}{2}+\left|f^{(n)}(a)\right|^q\frac{(b-a)^3}{2}-c\frac{(b-a)^5}{30}\right]^{\frac{1}{q}}\\
&= \frac{1}{n!}(b-a)^{\frac{3}{q}}C^{\frac{1}{p}}(a,b,n,p)\left[A\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right)-c\frac{(b-a)^2}{30}\right]^{\frac{1}{q}}
\end{aligned}$$

It is stated that the improper integral $C(a,b,n,p)$ is convergent for $1 < p < 2$. □

Corollary 2.1 *Under the conditions in Theorem* 1.3 *for* $c = 0$, *we have the following inequalities*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|\\
&\leq \frac{1}{n!}(b-a)^{\frac{3}{q}}C^{\frac{1}{p}}(a,b,n,p)A^{\frac{1}{q}}\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right).
\end{aligned}$$

The last inequality coincides with the inequality in Theorem 2.1 in [13].

Corollary 2.2 *Under the conditions in Theorem* 2.1 *for* $n = 1$, *we have the following inequality:*

$$\begin{aligned}
&\left|\frac{f(b)\,b-f(a)\,a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\leq (b-a)^{\frac{3}{q}-1}C^{\frac{1}{p}}(a,b,1,p)\left[A\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)-c\frac{(b-a)^2}{30}\right]^{\frac{1}{q}}.
\end{aligned}$$

Corollary 2.3 *Under the conditions in Theorem* 2.1 *for* $n=1$ *and* $c=0$*, we have the following inequality*

$$\left|\frac{f(b)\,b-f(a)\,a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq (b-a)^{\frac{3}{q}-1}C^{\frac{1}{p}}(a,b,1,p)A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right).$$

This inequality coincides with the inequality in [13].

Theorem 2.2 *For* $n\in\mathbb{N}$*; let* $f:(0,\infty)\subset\mathbb{R}\to\mathbb{R}$ *be n-times differentiable function and* $0\leq a<b$*. If* $\left|f^{(n)}\right|^q\in L[a,b]$ *and* $\left|f^{(n)}\right|^q$ *for* $q>1$ *is a strongly Godunova-Levin function with modulus c on the interval* $[a,b]$*, then the following inequality*

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|\\&\leq\frac{1}{n!}(b-a)^{\frac{2}{q}}D^{\frac{1}{p}}(a,b,n,p)\times\left[\left|f^{(n)}(b)\right|^q\left\{bL_n^n(a,b)-L_{n+1}^{n+1}(a,b)\right\}\right.\\&\quad\left.+\left|f^{(n)(a)}\right|^q\left\{L_{n+1}^{n+1}(a,b)-aL_n^n(a,b)\right\}-\frac{cE(a,b,n)}{(b-a)^2}\right]^{\frac{1}{q}}\end{aligned}$$

holds, where $\frac{1}{p}+\frac{1}{q}=1,\ 1<p<2,\ D(a,b,n,p)=\int_a^b\frac{x^n}{(x-a)^{p-1}(b-x)^{p-1}}dx$ *and* $E(a,b,n)=\int_a^b x^n(x-a)^2(b-x)^2\,dx$*.*

Proof Firstly, let $x\in(a,b)$ (Notices that this proof is also valid in case of $x\in[a,b]$). From Lemma 1.1, Hölder integral inequality and the inequality (2.1), we obtain

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|\\&\leq\frac{1}{n!}\int_a^b x^n\left|f^{(n)}(x)\right|dx\\&\leq\frac{1}{n!}\left(\int_a^b\frac{x^n}{(x-a)^{p-1}(b-x)^{p-1}}dx\right)^{\frac{1}{p}}\left(\int_a^b x^n(b-x)(x-a)\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}\\&\leq\frac{1}{n!}D^{\frac{1}{p}}(a,b,n,p)\left[\int_a^b x^n\left[\begin{array}{c}(b-a)(b-x)\left|f^{(n)}(b)\right|^q+(b-a)(x-a)\left|f^{(n)}(a)\right|^q\\-c\,(x-a)^2(b-x)^2\end{array}\right]dx\right]^{\frac{1}{q}}\\&\leq\frac{1}{n!}D^{\frac{1}{p}}(a,b,n,p)\end{aligned}$$

$$\times\left[\begin{array}{c}(b-a)\left|f^{(n)}(b)\right|^{q}\left\{b\left(\frac{b^{n+1}-a^{n+1}}{n+1}\right)-\left(\frac{b^{n+2}-a^{n+2}}{n+2}\right)\right\}\\+(b-a)\left|f^{(n)}(a)\right|^{q}\left\{\left(\frac{b^{n+2}-a^{n+2}}{n+2}\right)-a\left(\frac{b^{n+1}-a^{n+1}}{n+1}\right)\right\}-c\int_a^b x^n(x-a)^2(b-x)^2\,dx\end{array}\right]^{\frac{1}{q}}$$

$$\leq\frac{1}{n!}D^{\frac{1}{p}}(a,b,n,p)\left[\begin{array}{c}(b-a)^2\left|f^{(n)}(b)\right|^{q}\left\{b\left(\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}\right)-\left(\frac{b^{n+2}-a^{n+2}}{(b-a)(n+2)}\right)\right\}\\+(b-a)^2\left|f^{(n)}(a)\right|^{q}\left\{\left(\frac{b^{n+2}-a^{n+2}}{(b-a)(n+2)}\right)-a\left(\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}\right)\right\}-\frac{cE(a,b,n)}{(b-a)^2}\end{array}\right]^{\frac{1}{q}}$$

$$\leq\frac{1}{n!}(b-a)^{\frac{2}{q}}D^{\frac{1}{p}}(a,b,n,p)\left[\begin{array}{c}\left|f^{(n)}(b)\right|^{q}\left\{bL_n^n(a,b)-L_{n+1}^{n+1}(a,b)\right\}\\+\left|f^{(n)}(a)\right|^{q}\left\{L_{n+1}^{n+1}(a,b)-aL_n^n(a,b)\right\}-\frac{cE(a,b,n)}{(b-a)^2}\end{array}\right]^{\frac{1}{q}}$$

It is stated that the improper integral $D(a,b,n,p)$ is convergent for $1<p<2$. □

Corollary 2.4 *Under the conditions in Theorem* 2.2 *for* $c=0$, *we have the following inequality*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|\leq\frac{1}{n!}(b-a)^{\frac{2}{q}}D^{\frac{1}{p}}(a,b,n,p)$$

$$\times\left[\left|f^{(n)}(b)\right|^{q}\left\{bL_n^n(a,b)-L_{n+1}^{n+1}(a,b)\right\}+\left|f^{(n)}(a)\right|^{q}\left\{L_{n+1}^{n+1}(a,b)-aL_n^n(a,b)\right\}\right]^{\frac{1}{q}}.$$

This inequality coincides with the inequality in Theorem 2.2 in [13].

Corollary 2.5 *Under the conditions in Theorem* 2.2 *for* $n=1$ *we have the following inequality*

$$\left|\frac{f(b)\,b-f(a)\,a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq(b-a)^{\frac{3}{q}-1}D^{\frac{1}{p}}(a,b,1,p)\left[(b+2a)|f'(b)|^q+(2b+a)|f'(a)|^q-c\frac{(a+b)(b-a)^2}{10}\right]^{\frac{1}{q}}.$$

Corollary 2.6 *Under the conditions in Theorem* 2.2 *for* $n=1$ *and* $c=0$, *we have the following inequality*

$$\left|\frac{f(b)\,b-f(a)\,a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq(b-a)^{\frac{3}{q}-1}D^{\frac{1}{p}}(a,b,1,p)\left[(b+2a)|f'(b)|^q+(2b+a)|f'(a)|^q\right]^{\frac{1}{q}}.$$

This inequality coincides with the inequality in [13].

Theorem 2.3 *For* $\forall n\in\mathbb{N}$; *let* $f:I\subset(0,\infty)\to\mathbb{R}$ *be* n*-times differentiable function on* I° *and* $a,b\in I^{\circ}$ *with* $a<b$. *If* $\left|f^{(n)}\right|^q$ *for* $q>1$ *is a strongly Godunova-Levin function with modulu* c

on $[a,b]$*, then the following inequality*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|$$

$$\le\frac{(b-a)^{\frac{4}{q}}}{n!}\left(bC\left(a,b,n,p\right)-D\left(a,b,n,p\right)\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n)}(b)\right|^q}{3}+\frac{\left|f^{(n)}(a)\right|^q}{6}-c\frac{(b-a)^2}{60}\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)^{\frac{4}{q}}}{n!}\left(D\left(a,b,n,p\right)-aC\left(a,b,n,p\right)\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n)}(b)\right|^q}{6}+\frac{\left|f^{(n)}(a)\right|^q}{3}-c\frac{(b-a)^6}{60}\right)^{\frac{1}{q}},$$

holds, where $\frac{1}{p}+\frac{1}{q}=1,\ 1<p<2$ *and* $C\left(a,b,n,p\right)=\int_a^b\frac{x^{np}}{(x-a)^{p-1}(b-x)^{p-1}}dx$ *and* $D\left(a,b,n,p\right)=\int_a^b\frac{x^{np+1}}{(x-a)^{p-1}(b-x)^{p-1}}dx.$

Proof Firstly, let $x\in(a,b)$. If $\left|f^{(n)}\right|^q$ for $q>1$ is a strongly Godunova-Levin function with modulus c on the interval $[a,b]$, by using the Lemma 1.1, the Hölder-İşcan integral inequality and the inequality (2.1) we obtain

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|\le\frac{1}{n!}\int_a^b x^n\left|f^{(n)}(x)\right|dx$$

$$\le\frac{1}{n!}\left(\int_a^b\frac{(b-x)x^{np}}{(x-a)^{p-1}(b-x)^{p-1}}dx\right)^{\frac{1}{p}}\left(\int_a^b(x-a)(b-x)^2\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}$$

$$+\frac{1}{n!}\left(\int_a^b\frac{(x-a)x^{np}}{(x-a)^{p-1}(b-x)^{p-1}}dx\right)^{\frac{1}{p}}\left(\int_a^b(x-a)^2(b-x)\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}$$

$$\le\frac{1}{n!}\left(b\int_a^b\frac{x^{np}}{(x-a)^{p-1}(b-x)^{p-1}}dx-\int_a^b\frac{x^{np+1}}{(x-a)^{p-1}(b-x)^{p-1}}dx\right)^{\frac{1}{p}}$$

$$\times\left(\int_a^b(b-x)\left[(b-a)(b-x)\left|f^{(n)}(b)\right|^q+(b-a)(x-a)\left|f^{(n)}(a)\right|^q-c\,(x-a)^2\,(b-x)^2\right]dx\right)^{\frac{1}{q}}$$

$$+\frac{1}{n!}\left(\int_a^b\frac{x^{np+1}}{(x-a)^{p-1}(b-x)^{p-1}}dx-a\int_a^b\frac{x^{np}}{(x-a)^{p-1}(b-x)^{p-1}}dx\right)^{\frac{1}{p}}$$

$$\times\left(\int_a^b(x-a)\left[(b-a)(b-x)\left|f^{(n)}(b)\right|^q+(b-a)(x-a)\left|f^{(n)}(a)\right|^q-c\,(x-a)^2\,(b-x)^2\right]dx\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}\left(bC\left(a,b,n,p\right)-D\left(a,b,n,p\right)\right)^{\frac{1}{p}}$$

$$\times\left(\begin{array}{c}(b-a)\left|f^{(n)}(b)\right|^q\int_a^b(b-x)^2dx\\+(b-a)\left|f^{(n)}(a)\right|^q\int_a^b(x-a)(b-x)dx-c\int_a^b(x-a)^2\,(b-x)^3\,dx\end{array}\right)^{\frac{1}{q}}$$

$$+\frac{1}{n!}\left(D\left(a,b,n,p\right)-aC\left(a,b,n,p\right)\right)^{\frac{1}{p}}$$

$$\times\left(\begin{array}{c}(b-a)\left|f^{(n)}(b)\right|^{q}\int_a^b(x-a)(b-x)dx\\+(b-a)\left|f^{(n)}(a)\right|^{q}\int_a^b(x-a)^2dx-c\int_a^b(x-a)^3(b-x)^2dx\end{array}\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}\left(bC(a,b,n,p)-D(a,b,n,p)\right)^{\frac{1}{p}}\left(\frac{(b-a)^4}{3}\left|f^{(n)}(b)\right|^q+\frac{(b-a)^4}{6}\left|f^{(n)}(a)\right|^q-c\frac{(b-a)^6}{60}\right)^{\frac{1}{q}}$$

$$+\frac{1}{n!}\left(D(a,b,n,p)-aC(a,b,n,p)\right)^{\frac{1}{p}}\left(\frac{(b-a)^4}{6}\left|f^{(n)}(b)\right|^q+\frac{(b-a)^4}{3}\left|f^{(n)}(a)\right|^q-c\frac{(b-a)^6}{60}\right)^{\frac{1}{q}}$$

$$=\frac{(b-a)^{\frac{4}{q}}}{n!}\left(bC(a,b,n,p)-D(a,b,n,p)\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n)}(b)\right|^q}{3}+\frac{\left|f^{(n)}(a)\right|^q}{6}-c\frac{(b-a)^2}{60}\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)^{\frac{4}{q}}}{n!}\left(D(a,b,n,p)-aC(a,b,n,p)\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n)}(b)\right|^q}{6}+\frac{\left|f^{(n)}(a)\right|^q}{3}-c\frac{(b-a)^6}{60}\right)^{\frac{1}{q}}.$$

It is stated that the improper integral $C(a,b,n,p)$ and $D(a,b,n,p)$ are convergent for $1<p<2$. □

Corollary 2.7 *Under the conditions in Theorem* 2.3 *for* $c=0$, *we have the following inequality*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|$$

$$\le\frac{(b-a)^{\frac{4}{q}}}{n!}\left(bC(a,b,n,p)-D(a,b,n,p)\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n)}(b)\right|^q}{3}+\frac{\left|f^{(n)}(a)\right|^q}{6}\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)^{\frac{4}{q}}}{n!}\left(D(a,b,n,p)-aC(a,b,n,p)\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n)}(b)\right|^q}{6}+\frac{\left|f^{(n)}(a)\right|^q}{3}\right)^{\frac{1}{q}}.$$

Corollary 2.8 *Under the conditions in Theorem* 2.3 *for* $n=1$, *we have the following inequality*

$$\left|\frac{f(b)\,b-f(a)\,a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\le\frac{(b-a)^{\frac{4}{q}}}{n!}\left(bC(a,b,1,p)-D(a,b,1,p)\right)^{\frac{1}{p}}\left(\frac{|f'(b)|^q}{3}+\frac{|f'(a)|^q}{6}-c\frac{(b-a)^2}{60}\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)^{\frac{4}{q}}}{n!}\left(D(a,b,1,p)-aC(a,b,1,p)\right)^{\frac{1}{p}}\left(\frac{|f'(b)|^q}{6}+\frac{|f'(a)|^q}{3}-c\frac{(b-a)^6}{60}\right)^{\frac{1}{q}},$$

Corollary 2.9 *Under the conditions in Theorem* 2.3 *for* $n=1$ *and* $c=0$, *we have the following* *inequality*

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le \frac{(b-a)^{\frac{4}{q}}}{n!}\left(bC(a,b,1,p)-D(a,b,1,p)\right)^{\frac{1}{p}}\left(\frac{|f'(b)|^q}{3}+\frac{|f'(a)|^q}{6}\right)^{\frac{1}{q}}$$
$$+\frac{(b-a)^{\frac{4}{q}}}{n!}\left(D(a,b,1,p)-aC(a,b,1,p)\right)^{\frac{1}{p}}\left(\frac{|f'(b)|^q}{6}+\frac{|f'(a)|^q}{3}\right)^{\frac{1}{q}}.$$

Theorem 2.4 *For $n\in\mathbb{N}$; let $f:(0,\infty)\subset\mathbb{R}\to\mathbb{R}$ be n-times differentiable function and $0\le a<b$. If $\left|f^{(n)}\right|^q\in L[a,b]$ and $\left|f^{(n)}\right|^q$ for $q>1$ is strongly Godunova-Levin function with modulus c on the interval $[a,b]$, then the following inequality*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|$$
$$\le\frac{1}{n!}\left[aD(a,b,n,p)-E(a,b,n,p)\right]^{\frac{1}{p}}$$
$$\times\left[\begin{array}{c}(b-a)^2\left|f^{(n)}(b)\right|^q\left(b^2L_n^n(a,b)-2bL_{n+1}^{n+1}(a,b)+L_{n+2}^{n+2}(a,b)\right)\\+(b-a)^2\left|f^{(n)}(a)\right|^q\left(-abL_n^n(a,b)+(a+b)L_{n+1}^{n+1}(a,b)-L_{n+2}^{n+2}(a,b)\right)\\-c\frac{G(a,b,n)}{(b-a)^2}\end{array}\right]^{\frac{1}{q}}$$
$$+\frac{1}{n!}\left[E(a,b,n,p)-aD(a,b,n,p)\right]^{\frac{1}{p}}$$
$$\times\left[\begin{array}{c}(b-a)^2\left|f^{(n)}(b)\right|^q\left(-abL_n^n(a,b)+(a+b)L_{n+1}^{n+1}(a,b)-(b-a)L_{n+2}^{n+2}(a,b)\right)\\+(b-a)^2\left|f^{(n)}(a)\right|^q\left(a^2L_n^n(a,b)-2aL_{n+1}^{n+1}(a,b)+L_{n+2}^{n+2}(a,b)\right)\\-c\frac{H(a,b,n)}{(b-a)^2}\end{array}\right]^{\frac{1}{q}},$$

holds, where $\frac{1}{p}+\frac{1}{q}=1,\ 1<p<2$ and

$$\begin{aligned}D(a,b,n,p)&=\int_a^b\frac{x^n}{(x-a)^{p-1}(b-x)^{p-1}}dx, \qquad E(a,b,n,p)=\int_a^b\frac{x^{n+1}}{(x-a)^{p-1}(b-x)^{p-1}}dx,\\G(a,b,n)&=\int_a^b x^n(x-a)^2(b-x)^3dx, \qquad H(a,b,n)=\int_a^b x^n(x-a)^3(b-x)^2dx.\end{aligned}$$

Proof Firstly, let $x\in(a,b)$ (The proof is also valid in case of $x\in[a,b]$). By using the Lemma 1.1, well known Hölder integral inequality and the inequality

$$(x-a)(b-x)\left|f^{(n)}(x)\right|^q\le(b-a)(b-x)\left|f^{(n)}(b)\right|^q+(b-a)(x-a)\left|f^{(n)}(a)\right|^q-c(x-a)^2(b-x)^2,$$

we get that

$$
\begin{aligned}
&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|\le\frac{1}{n!}\int_a^b x^n\left|f^{(n)}(x)\right|dx\\
&\le\frac{1}{n!}\left(\int_a^b\frac{(b-x)x^n}{(x-a)^{p-1}(b-x)^{p-1}}dx\right)^{\frac1p}\left(\int_a^b x^n\,(b-x)^2(x-a)\left|f^{(n)}(x)\right|^q dx\right)^{\frac1q}\\
&\quad+\frac{1}{n!}\left(\int_a^b\frac{(x-a)x^n}{(x-a)^{p-1}(b-x)^{p-1}}dx\right)^{\frac1p}\left(\int_a^b x^n\,(b-x)\,(x-a)^2\left|f^{(n)}(x)\right|^q dx\right)^{\frac1q}\\
&\le\frac{1}{n!}\left(b\int_a^b\frac{x^n}{(x-a)^{p-1}(b-x)^{p-1}}dx-\int_a^b\frac{x^{n+1}}{(x-a)^{p-1}(b-x)^{p-1}}dx\right)^{\frac1p}\\
&\quad\times\left(\int_a^b x^n\,(b-x)\left[(b-a)(b-x)\left|f^{(n)}(b)\right|^q+(b-a)(x-a)\left|f^{(n)}(a)\right|^q-c\,(x-a)^2\,(b-x)^2\right]dx\right)^{\frac1q}\\
&\quad+\frac{1}{n!}\left(\int_a^b\frac{x^{n+1}}{(x-a)^{p-1}(b-x)^{p-1}}dx-a\int_a^b\frac{x^n}{(x-a)^{p-1}(b-x)^{p-1}}dx\right)^{\frac1p}\\
&\quad\times\left(\int_a^b x^n\,(x-a)\left[(b-a)(b-x)\left|f^{(n)}(b)\right|^q+(b-a)(x-a)\left|f^{(n)}(a)\right|^q-c\,(x-a)^2\,(b-x)^2\right]dx\right)^{\frac1q}\\
&=\frac{1}{n!}\left[bD\,(a,b,n,p)-E\,(a,b,n,p)\right]^{\frac1p}\\
&\quad\times\left[\begin{array}{c}(b-a)\left|f^{(n)}(b)\right|^q\int_a^b x^n(b-x)^2dx\\+(b-a)\left|f^{(n)}(a)\right|^q\int_a^b x^n(b-x)(x-a)dx-c\int_a^b x^n\,(x-a)^2\,(b-x)^3\,dx\end{array}\right]^{\frac1q}\\
&\quad+\frac{1}{n!}\left[E\,(a,b,n,p)-aD\,(a,b,n,p)\right]^{\frac1p}\\
&\quad\times\left[\begin{array}{c}(b-a)\left|f^{(n)}(b)\right|^q\int_a^b x^n(x-a)(b-x)dx\\+(b-a)\left|f^{(n)}(a)\right|^q\int_a^b x^n\,(x-a)^2dx-c\int_a^b x^n\,(x-a)^3\,(b-x)^2\,dx\end{array}\right]^{\frac1q}\\
&=\frac{1}{n!}\left[aD\,(a,b,n,p)-E\,(a,b,n,p)\right]^{\frac1p}\\
&\quad\times\left[\begin{array}{c}(b-a)^2\left|f^{(n)}(b)\right|^q\left(b^2L_n^n(a,b)-2bL_{n+1}^{n+1}(a,b)+L_{n+2}^{n+2}(a,b)\right)\\+(b-a)^2\left|f^{(n)}(a)\right|^q\left(-abL_n^n(a,b)+(a+b)L_{n+1}^{n+1}(a,b)-L_{n+2}^{n+2}(a,b)\right)\\-c\frac{G(a,b,n)}{(b-a)^2}\end{array}\right]^{\frac1q}\\
&\quad+\frac{1}{n!}\left[E\,(a,b,n,p)-aD\,(a,b,n,p)\right]^{\frac1p}\\
&\quad\times\left[\begin{array}{c}(b-a)^2\left|f^{(n)}(b)\right|^q\left(-abL_n^n(a,b)+(a+b)L_{n+1}^{n+1}(a,b)-(b-a)L_{n+2}^{n+2}(a,b)\right)\\+(b-a)^2\left|f^{(n)}(a)\right|^q\left(a^2L_n^n(a,b)-2aL_{n+1}^{n+1}(a,b)+L_{n+2}^{n+2}(a,b)\right)\\-c\frac{H(a,b,n)}{(b-a)^2}\end{array}\right]^{\frac1q}\\
&\quad\times\left[\begin{array}{c}(b-a)\left|f^{(n)}(b)\right|^q\left\{b\left(\frac{b^{n+1}-a^{n+1}}{n+1}\right)-\left(\frac{b^{n+2}-a^{n+2}}{n+2}\right)\right\}\\+(b-a)\left|f^{(n)}(a)\right|^q\left\{\left(\frac{b^{n+2}-a^{n+2}}{n+2}\right)-a\left(\frac{b^{n+1}-a^{n+1}}{n+1}\right)\right\}-c\int_a^b x^n\,(x-a)^2\,(b-x)^2\,dx\end{array}\right]^{\frac1q}
\end{aligned}
$$

It is stated that the improper integral $D(a,b,n,p)$, $E(a,b,n,p)$ are convergent for $1<p<2$. □

Corollary 2.10 *Under the conditions in Theorem* 2.4 *for* $c = 0$*, we have the following inequality*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right| \\
&\leq \frac{1}{n!}\left[aD(a,b,n,p)-E(a,b,n,p)\right]^{\frac{1}{p}} \\
&\quad\times\left[\begin{array}{c}(b-a)^2\left|f^{(n)}(b)\right|^q\left(b^2L_n^n(a,b)-2bL_{n+1}^{n+1}(a,b)+L_{n+2}^{n+2}(a,b)\right)\\ +(b-a)^2\left|f^{(n)}(a)\right|^q\left(-abL_n^n(a,b)+(a+b)L_{n+1}^{n+1}(a,b)-L_{n+2}^{n+2}(a,b)\right)\end{array}\right]^{\frac{1}{q}} \\
&\quad+\frac{1}{n!}\left[E(a,b,n,p)-aD(a,b,n,p)\right]^{\frac{1}{p}} \\
&\quad\times\left[\begin{array}{c}(b-a)^2\left|f^{(n)}(b)\right|^q\left(-abL_n^n(a,b)+(a+b)L_{n+1}^{n+1}(a,b)-(b-a)L_{n+2}^{n+2}(a,b)\right)\\ +(b-a)^2\left|f^{(n)}(a)\right|^q\left(a^2L_n^n(a,b)-2aL_{n+1}^{n+1}(a,b)+L_{n+2}^{n+2}(a,b)\right)\end{array}\right]^{\frac{1}{q}}.
\end{aligned}$$

Corollary 2.11 *Under the conditions in Theorem* 2.4 *for* $n = 1$ *we have the following inequality*

$$\begin{aligned}
&\left|\frac{f(b)\,b-f(a)\,a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{1}{n!}\left[aD(a,b,1,p)-E(a,b,1,p)\right]^{\frac{1}{p}} \\
&\quad\times\left[\begin{array}{c}(b-a)^2|f'(b)|^q\left(b^2L_1^1(a,b)-2bL_2^2(a,b)+L_3^3(a,b)\right)\\ +(b-a)^2|f'(a)|^q\left(-abL_1^1(a,b)+(a+b)L_2^2(a,b)-L_3^3(a,b)\right)\\ -c\frac{G(a,b,1)}{(b-a)^2}\end{array}\right]^{\frac{1}{q}} \\
&\quad+\frac{1}{n!}\left[E(a,b,1,p)-aD(a,b,1,p)\right]^{\frac{1}{p}} \\
&\quad\times\left[\begin{array}{c}(b-a)^2|f'(b)|^q\left(-abL_1^1(a,b)+(a+b)L_2^2(a,b)-(b-a)L_3^3(a,b)\right)\\ +(b-a)^2|f'(a)|^q\left(a^2L_1^1(a,b)-2aL_2^2(a,b)+L_3^3(a,b)\right)\\ -c\frac{H(a,b,1)}{(b-a)^2}\end{array}\right]^{\frac{1}{q}}.
\end{aligned}$$

Corollary 2.12 *Under the conditions in Theorem* 2.4 *for* $n = 1$ *and* $c = 0$*, we have the following inequality*

$$\begin{aligned}
&\left|\frac{f(b)\,b-f(a)\,a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \left[aD(a,b,1,p)-E(a,b,1,p)\right]^{\frac{1}{p}} \\
&\quad\times\left[\begin{array}{c}(b-a)^2|f'(b)|^q\left(b^2L_1^1(a,b)-2bL_2^2(a,b)+L_3^3(a,b)\right)\\ +(b-a)^2|f'(a)|^q\left(-abL_1^1(a,b)+(a+b)L_2^2(a,b)-L_3^3(a,b)\right)\end{array}\right]^{\frac{1}{q}} \\
&\quad+\left[E(a,b,1,p)-aD(a,b,1,p)\right]^{\frac{1}{p}} \\
&\quad\times\left[\begin{array}{c}(b-a)^2|f'(b)|^q\left(-abL_1^1(a,b)+(a+b)L_2^2(a,b)-(b-a)L_3^3(a,b)\right)\\ +(b-a)^2|f'(a)|^q\left(a^2L_1^1(a,b)-2aL_2^2(a,b)+L_3^3(a,b)\right)\end{array}\right]^{\frac{1}{q}}.
\end{aligned}$$

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On Lorentzian Sasakian Space Form with Respect to Generalized Tanaka Connection and Some Solitons

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Abstract: The object of the present paper is to study several type of symmetricness (semi-symmetric, Ricci semi-symmetric) of Lorentzian Sasakian space forms with respect to generalized Tanaka connection and nature of $*$-Ricci soliton, $*$-conformal Ricci soliton, $*-$conformal η-Ricci soliton, generalized Ricci soliton, generalized conformal Ricci soliton of this type of space forms with respect to generalized Tanaka connection.

Key Words: Semi-symmetry, Ricci semi-symmetry, pseudo-symmetry, Ricci-pseudo-symmetry, $*$-Ricci soliton, $*$-conformal Ricci soliton, generalized Ricci soliton.

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§1. Introduction

In 1996, the authors [28] first time studied Sasakian manifold with Lorentzian metric i.e., a metric compatible Sasakiann manifold (M, η, ξ, φ) with Lorentzian metric g (a symmetric non-degenerated (0,2) tensor field of index 1) which we called Lorentzian Sasakiann manifold. A Lorentzian Sasakian manifold with φ-holomorphic sectional curvature, we called Lorentzian Sasakian space form. The curvature tensor and Ricci tensor of a Lorentzian Sasakiann space form with constant φ-holomorphic sectional curvature c proved by the authors [23] as follows:

$$\begin{aligned}R(X,Y)Z = &\frac{c-3}{4}[g(Y,Z)X - g(X,Z)Y]\\ &+\frac{c+1}{4}\{\eta(Z)[\eta(Y)X - \eta(X)Y] + [\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]\xi\\ &+g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z\},\end{aligned} \tag{1}$$

and

$$S(X,Y) = \frac{n(c-3)+4}{2}g(X,Y) + \frac{n(c+1)}{2}\eta(X)\eta(Y), \tag{2}$$

Tanaka [16] and, independently Webster [29] defined the canonical affine connection on

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a nondegenerate, integrable CR manifold. Tanno [26] generalized this connection extending its definition to the general contact metric manifold which called generalized TanakaWebster connection or generalized Tanaka connection.

A manifold M is said to be locally symmetric (see [25]) if we have $\nabla_X R = 0$ for all $X \in \mathfrak{X}(M)$, where R is curvature tensor. A locally symmetric Riemannian manifold satisfies $R(X,Y)\cdot R = 0$ for all tangent vectors X and Y, where the linear endomorphism $R(X,Y)$ acts on R as a derivation. The spaces with $R(X,Y)\cdot R = 0$ have been investigated first by E. Cartan [5] which directly generalizes the notion of symmetric spaces. Conversely, does the condition $R(X,Y)\cdot R = 0$ imply the manifold M is locally symmetric? K. Nomizu [15] conjectured that an irreducible, complete Riemannian space with dim≥ 3 and with the above symmetric property of the curvature tensor is always a locally symmetric space. But this conjecture was refuted by H. Takagi [7] who constructed 3-dimensional complete irreducible nonlocally-symmetric hypersurfaces with $R(X,Y)\cdot R = 0$. According to Szabó [30], we call a space satisfying $R(X,Y)\cdot R = 0$ is a semi-symmetric space. Okumura [14] proved that a Sasakian manifold which is at the same time a locally symmetric space is a space of constant curvature. This fact means that a symmetric space condition is too strong for a Sasakian manifold. In a semi-symmetric manifold, the condition $R(X,Y)\cdot S = 0$ satisfies for all $X, Y \in \mathfrak{X}(M)$, where S is the Ricci tensor. But the converse statement is however not true. These two conditions $R(X,Y)\cdot R = 0$ and $R(X,Y)\cdot S = 0$ are equivalent for hypersurfaces of Euclidean spaces proved by P.J. Ryan [18]. The spaces which satisfies $R(X,Y)\cdot S = 0$ we called Ricci-semisymmetric spaces. Thus, every semisymmetric space is Ricci-semisymmetric. The generalized condition $R\cdot R = LQ(g,R)$, where L is a non zero function and $Q(g,R)$ is defined in [1] of the conditions $\nabla_X R = 0$ and $R(X,Y)\cdot R = 0$ (symmetric and semi-symmetric) was introduced by R. Deszcz [8] and if a manifold satisfies this condition then it called pseudo-symmetric. On the other hand M.C. Chaki [11] introduced a different definition of pseudo-symmetric manifold. In this paper we approach the Deszcz's definition. Deszcz also defined Ricci-pseudo-symmetric manifold [19] by the condition $R\cdot S = LQ(g,S)$.

In 1982 Hamilton [21] introduced the concept of Ricci flow and proved its existence. The Ricci flow equation is given by

$$\frac{\partial g}{\partial t} = -2S \tag{3}$$

on a compact Riemannian manifold M with Riemannian metric g, where S is the Ricci tensor. A self-similar solution to the Ricci flow (3) is called a Ricci soliton which moves under the Ricci flow simply by diffeomorphisms of the initial metric, that is, they are stationary points of the Ricci flow in space of metrics on M. A Ricci soliton is a generalization of an Einstein metric. The Ricci soliton equation is given by

$$\mathcal{L}_X g + 2S = 2\lambda g \tag{4}$$

where $\mathcal{L}$ is the Lie derivative, S is the Ricci tensor, g is Riemannian metric, X is a vector field and λ is a scalar. The Ricci soliton is said to be shrinking, steady, and expanding according as λ is positive, zero and negetive respectively.

Fischer during 2003-2004 developed the concept of conformal Ricci flow [3] which is a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The conformal Ricci flow on M is defined by [24]

$$\frac{\partial g}{\partial t} + 2\left(S + \frac{g}{n}\right) = -pg \tag{5}$$

where $R(g) = -1$ and p is a non-dynamical scalar field(time dependent scalar field), $R(g)$ is the scalar curvature of the n-dimessional manifold M.

In 2015, N. Basu and A. Bhattacharyya [2] introduced the notion of conformal Ricci soliton and the equation is as follows

$$\mathcal{L}_X g + 2S = \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g \tag{6}$$

where λ is a scalar.

Cho and Kimura [9] introduced the notion of η-Ricci soliton in 2009, as follows

$$\mathcal{L}_\xi g + 2S = 2\lambda g + 2\mu\eta \otimes \eta \tag{7}$$

for some constants λ and μ, where ξ is a soliton vector field and η is an 1-form on M.

In 2018, Siddiqi [13] introduced the notion of conformal η-Ricci soliton, given by

$$\mathcal{L}_\xi g + 2S + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g + 2\mu\eta \otimes \eta = 0 \tag{8}$$

for some constants λ and μ, where ξ is a soliton vector field and η is an 1-form on M. where $\mathcal{L}_\xi$ is the Lie derivative along the vector field ξ , S is the Ricci tensor, λ, μ are constants, p is a scalar non-dynamical field (time dependent scalar field) and n is the dimension of manifold.

Tachibana [22] and Hamada [27] introduced the notion of $*$-Ricci tensor on almost Hermitian manifolds and on real hypersurfaces in non-flat complex space respectively and then in 2014, Kaimakamis and Panagiotidou [6] introduced the notion of $*$-Ricci soliton on non-flat complex space forms and the equation as

$$\mathcal{L}_V g + 2S^* + 2\lambda g = 0, \tag{9}$$

where $S^*(X,Y) = \frac{1}{2}\left[trace\{\varphi \circ R(X, \varphi Y)\}\right]$ for all vector fields X, Y on M and φ is a (1,1)-tensor field.

In 2022, the authors [24] have defined the $*$-conformal η-Ricci soliton on a Riemannian manifold as

$$\mathcal{L}_\xi g + 2S^* + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g + 2\mu\eta \otimes \eta = 0, \tag{10}$$

where $\mathcal{L}_\xi$ is the Lie derivative along the vector field ξ , S^* is the $*$-Ricci tensor, λ, μ are constants, p is a scalar non-dynamical field (time dependent scalar field) and n is the dimension of manifold.

In 2016, Nurowski and Randall [17] introduced the concept of generalized Ricci soliton as a class of over determined system of equations

$$\mathcal{L}_V g = -2aV^{\#} \odot V^{\#} + 2bS + 2\lambda g, \tag{11}$$

where $\mathcal{L}_V g$ and $V^{\#}$ denote, respectively, the Lie derivative of the metric g in the directions of vector field V and the canonical one-form associated to V, and some real constants a, b, and λ. Levy [10] acquired the necessary and sufficient conditions for the existence of such tensors. In 2018 M.D. Siddiqi [12] have studied generalized Ricci solitons on trans-Sasakian manifolds.

In this paper, we consider generalized Tanaka connection on Lorentzian Sasakian Space form and studied various symmetric properties of Lorentzian Sasakian space form with generalized Tanaka connection and solitons. After preliminaries in section-3, we consider consider generalized Tanaka connection on Lorentzian Sasakian space form, state and proved some results, finding curvature tensor and Ricci curvature tensor. In section-4, we study the semi-symmetric, Ricci semi-symmetric properties of Lorentzian Sasakian Space form. From section-5 to 7 $*$-Ricci soliton, $*$-conformal Ricci soliton, $*$-conformal η-Ricci soliton, generalized Ricci soliton, generalized conformal Ricci soliton have been studied on Lorentzian Sasakian space form with generalized Tanaka connection and obtained the values of the scalar λ of these solitons on which nature of solitons depend, whether it is shrinking, steady or expanding.

§2. **Preliminaries**

Let M be a $(2n+1)$ dimensional (denoted by M^{2n+1}) having almost contact structure (φ, ξ, η, g) i.e.,

$$\eta(\xi) = 1, \ \varphi^2 = -I + \eta \otimes \xi, \ \varphi(\xi) = 0, \ \eta \circ \varphi = 0, \tag{12}$$

where φ is a (1,1)-tensor field, ξ a contravariant vector field, η a covariant vector field.

A Lorentzian metric g is said to be compatible with the structure (φ, ξ, η, g) if

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y). \tag{13}$$

If the manifold M^{2n+1} equipped with an almost contact structure (φ, ξ, η, g) and a compatible Lorentzian metric g, is called an almost contact Lorentzian manifold.

Note that equations (12) and (13) imply

$$g(X, \xi) = -\eta(X) \quad \text{and} \quad g(\xi, \xi) = -1. \tag{14}$$

Also, equations (13) implies

$$g(X, \varphi Y) = -g(\varphi X, Y). \tag{15}$$

In almost contact Lorentzian manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$, the fundamental 2-form Φ is defined as

$$\Phi(X, Y) = g(X, \varphi Y) \quad \text{for all} \quad X, Y \in \mathfrak{X}(M).$$

An almost contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is Sasakian [4] if and only if it is normal and

$$d\eta = 0. \tag{16}$$

In Lorentzian Sasakian manifold, the following properties [23] hold good:

$$(\nabla_X \varphi)Y = \eta(Y)X + g(X,Y)\xi, \tag{17}$$

$$\nabla_X \xi = \varphi X, \tag{18}$$

$$(\nabla_X \eta)Y = g(X, \varphi Y). \tag{19}$$

Let (M,g) be an n-dimensional Riemannian manifold $n > 2$, its curvature tensor defined by

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

and let T be $(0,k)$-tensor, define a $(0,2+k)$-tensor field $R \cdot T$ by

$$(R \cdot T)(X_1, X_2, \cdots, X_k, X, Y) = R(X,Y)(T(X_1, X_2, \cdots, X_k)) - T(R(X,Y)X_1, X_2, \cdots, X_k)$$
$$- T(X_1, R(X,Y)X_2, \cdots, X_k) - \cdots - T(X_1, X_2, \cdots, R(X,Y)X_k).$$

One has

$$R(X,Y) \cdot T = \nabla_X(\nabla_Y T) - \nabla_Y(\nabla_X T) - \nabla_{[X,Y]}T.$$

When $T = R$, then we have a $(0,6)$-tensor $R \cdot R$.

Also, we can determine a $(0,k+2)$-tensor field $Q(A,T)$, associated with any $(0,k)$-tensor field T and any symmetric $(0,2)$-tensor field A by

$$\begin{aligned} Q(A,T)(X_1, X_2, \cdots, X_k, X, Y) &= ((X \wedge_A Y) \cdot T)(X_1, X_2, \cdots, X_k) \\ &= -T((X \wedge_A Y)X_1, X_2, \cdots, X_k) - T(X_1, (X \wedge_A Y)X_2, \cdots, X_k) \\ &\quad - \cdots - T(X_1, X_2, \cdots, (X \wedge_A Y)X_k), \end{aligned}$$

where $(X \wedge_A Y)$ is the endomorphism given by

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y. \tag{20}$$

Particularly, if we put $A = g$ we get

$$(X \wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{21}$$

and we will write $(X \wedge_g Y)$ as $(X \wedge Y)$ in General.

§3. Generalized Tanaka Connection on Lorentzian Sasakian Space Form

For an $(2n+1)$-dimensional Lorentzian Sasakian manifold M with almost contact structure (φ, ξ, η, g), the relation between generalized Tanaka connection $\mathring{\nabla}$ and Levi-Civita connection

∇ is given by

$$\mathring{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi. \tag{22}$$

By (18) and (19),

$$\mathring{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + g(X, \varphi Y)\xi - \eta(Y)\varphi X. \tag{23}$$

Putting $Y = \xi$ in (22),

$$\mathring{\nabla}_X \xi = \nabla_X \xi + \eta(X)\varphi\xi + g(X, \varphi\xi)\xi - \eta(\xi)\nabla_X \xi$$

By (12),

$$\mathring{\nabla}_X \xi = 0. \tag{24}$$

$$(\mathring{\nabla}_X \eta)Y = \mathring{\nabla}_X \eta(Y) - \eta(\mathring{\nabla}_X Y)$$

From (23),

$$(\mathring{\nabla}_X \eta)Y = 0, \tag{25}$$

$$(\mathring{\nabla}_X g)(Y, Z) = 0. \tag{26}$$

Thus, we can state

Theorem 3.1 *In a Lorentzian Sasakian manifold ξ, η, g are parallel with respect to the generalized Tanaka connection.*

Now,

$$(\mathring{\nabla}_X \varphi)Y = \mathring{\nabla}_X \varphi Y - \varphi(\mathring{\nabla}_X Y).$$

Using (22),

$$(\mathring{\nabla}_X \varphi)Y = 0. \tag{27}$$

The curvature tensor of Lorentzian Sasakian manifold with respect to the generalized Tanaka connection is given by

$$\begin{aligned}
\mathring{R}(X,Y)Z &= \mathring{\nabla}_X\mathring{\nabla}_Y Z - \mathring{\nabla}_Y\mathring{\nabla}_X Z - \mathring{\nabla}_{[X,Y]}Z \\
&= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z + [\eta(X)\varphi(\nabla_Y Z) - \eta(Y)\varphi(\nabla_X Z)] \\
&\quad + [g(X, \varphi(\nabla_Y Z))\xi - g(Y, \varphi(\nabla_X Z))\xi] - [\eta((\nabla_Y Z))\varphi X - \eta((\nabla_X Z))\varphi Y] \\
&\quad + [\eta(\nabla_X Y)\varphi Z - \eta(\nabla_Y X)\varphi Z] + [g(X, \varphi Y)\varphi Z - g(Y, \varphi X)\varphi Z] \\
&\quad + [\eta(Y)\varphi(\nabla_X Z) - \eta(X)\varphi(\nabla_Y Z)] - [\eta(X)\eta(Y)Z - \eta(Y)\eta(X)Z] \\
&\quad + [g(\nabla_X Y, \varphi Z) - g(\nabla_Y X, \varphi Z)]\,\xi - [\eta(Y)g(\varphi X, \varphi Z)\xi - \eta(X)g(\varphi Y, \varphi Z)\xi] \\
&\quad + [g(Y, \varphi(\nabla_X Z))\xi - g(X, \varphi(\nabla_Y Z))\xi] + [\eta(Z)g(\varphi Y, \varphi X)\xi - \eta(Z)g(\varphi X, \varphi Y)\xi] \\
&\quad - [\eta(\nabla_X Z)\varphi Y - \eta(\nabla_Y Z)\varphi X] - [g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X] \\
&\quad - [\eta(Z)\varphi(\nabla_X Y) - \eta(Z)\varphi(\nabla_Y X)] + [\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X] \\
&\quad - \eta([X,Y])\varphi Z - g([X,Y], \varphi Z)\xi + \eta(Z)\varphi[X,Y].
\end{aligned}$$

So,

$$\mathring{R}(X,Y)Z = R(X,Y)Z + 2g(X,\varphi Y)\varphi Z - [\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]\,\xi$$
$$- [g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X] + \eta(Z)\,[\eta(X)Y - \eta(Y)X] \qquad (28)$$

and

$$\mathring{S}(X,Y) = S(X,Y) + 2g(X,Y) - 2(n-1)\eta(X)\eta(Y)$$

If M has constant φ-holomorphic sectional curvature c, then by (1) and (28) we get

$$\mathring{R}(X,Y)Z = \frac{c-3}{4}\{[g(Y,Z)X - g(X,Z)Y] + \eta(Z)[\eta(Y)X - \eta(X)Y]$$
$$+[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]\xi + [g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X]\}$$
$$+\frac{c+5}{2}g(X,\varphi Y)\varphi Z. \qquad (29)$$

and

$$\mathring{S}(X,Y) = \frac{n(c-3)+4}{2}g(X,Y)\frac{n(c+1)}{2}\eta(X)\eta(Y) + 2g(X,Y) - 2(n-1)\eta(X)\eta(Y)$$

Or,

$$\mathring{S}(X,Y) = \frac{n(c-3)+8}{2}g(\varphi X,\varphi Y). \qquad (30)$$

§4. Semi-symmetry and Ricci-semisymmetry on Lorentzian Sasakian Space Form with Respect to Generalized Tanaka Connection

Applying (21) in (29), we get

$$\mathring{R}(X,Y)Z = \frac{c-3}{4}\{(\varphi X \wedge \varphi Y)Z + (\varphi^2 X \wedge \varphi^2 Y)Z\} + \frac{c+5}{2}g(X,\varphi Y)\varphi Z. \qquad (31)$$

Lemma 4.1 *Let $M^{2n+1}(c)$ be a Lorentzian Sasakian space form with generalized Tanaka connection and $X,Y \in \mathfrak{X}(M)$, then the following properties hold:*

(*a*) $\varphi \cdot \mathring{S} = 0$;
(*b*) $(X \wedge Y)\cdot \mathring{S} = 0$ *if and only if* $c = \frac{3n-8}{n}$;
(*c*) $(\varphi X \wedge \varphi Y)\cdot \mathring{S} = 0$;
(*d*) $(\varphi^2 X \wedge \varphi^2 Y)\cdot \mathring{S} = 0$.

Proof (*a*) Since φ is a tensor field, we have

$$(\varphi\cdot \mathring{S})(U,V) = -\mathring{S}(\varphi U,V) - \mathring{S}(U,\varphi V) = \frac{n(c-3)}{2}\,[g(\varphi U,V) + g(U,\varphi V)]$$
$$+\frac{n(c+5)-4}{2}\,[\eta(\varphi U)\eta(V) + \eta(U)\eta(\varphi V)]$$
$$= \frac{n(c-3)}{2}\,[g(\varphi U,V) - g(\varphi U,V)] = 0.$$

Thus $(\varphi\cdot\mathring{S})(U,V)=0$ for any $U,V\in\mathfrak{X}(M)$.

(b) For any $U,V\in\mathfrak{X}(M)$, we have

$$\begin{aligned}((X\wedge Y).\mathring{S})(U,V) &= -\mathring{S}((X\wedge Y)U,V)-\mathring{S}(U,(X\wedge Y)V)\\ &= -g(Y,U)\mathring{S}(X,V)+g(X,U)\mathring{S}(Y,V)\\ &\quad -g(Y,V)\mathring{S}(U,X)+g(X,V)S(U,Y)\\ &= -\frac{n(c-3)+8}{2}[g(Y,U)\eta(X)\eta(V)-g(X,U)\eta(Y)\eta(V)\\ &\quad +g(Y,V)\eta(X)\eta(U)-g(X,V)\eta(Y)\eta(U)].\end{aligned}$$

Since, $-g(Y,U)\eta(X)\eta(V)+g(X,U)\eta(Y)\eta(V)-g(Y,V)\eta(U)\eta(X)$ $+\,g(X,V)\eta(U)\eta(Y)\neq 0$ always. Therefore

$$((X\wedge Y).\mathring{S})(U,V)=0 \ \text{ if and only if } \ n(c-3)+8=0, \quad i.e.,\ c=\frac{3n-8}{n}.$$

(c) For any $U,V\in\mathfrak{X}(M)$, we have

$$\begin{aligned}((\varphi X\wedge\varphi Y).\mathring{S})(U,V) &= -\mathring{S}((\varphi X\wedge\varphi Y)U,V)-\mathring{S}(U,(\varphi X\wedge\varphi Y)V)\\ &= -g(\varphi Y,U)\mathring{S}(\varphi X,V)+g(\varphi X,U)\mathring{S}(\varphi Y,V)\\ &\quad -g(\varphi Y,V)\mathring{S}(U,\varphi X)+g(\varphi X,V)\mathring{S}(U,\varphi Y).\end{aligned}$$

Using (12) and (21), we get the result.

(d) The proof is similar to (c). □

Theorem 4.2 *A Lorentzian Sasakian space form $M^{2n+1}(c)$ is Ricci-semi-symmetric with respect to generalized Tanaka connection.*

Proof In the equation (31), we see that the curvature tensor is of the form

$$\mathring{R}(X,Y)=\frac{c-3}{4}(\varphi X\wedge\varphi Y)+\frac{c-3}{4}(\varphi^2X\wedge\varphi^2Y)+\frac{c+5}{2}g(X,\varphi Y)\varphi.$$

So,

$$\mathring{R}(X,Y)\cdot\mathring{S}=\frac{c-3}{4}(\varphi X\wedge\varphi Y)\cdot\mathring{S}+\frac{c-3}{4}(\varphi^2X\wedge\varphi^2Y)\cdot\mathring{S}+\frac{c+5}{2}g(X,\varphi Y)\varphi\cdot\mathring{S}.$$

By the Lemma 4.1, we have $\mathring{R}\cdot\mathring{S}=0$. □

Lemma 4.3 *In a Lorentzian Sasakian space form $M^{2n+1}(c)$ with generalized Tanaka connection the following properties hold for all $X,Y\in\mathfrak{X}(M)$:*

(a) $\varphi\cdot\mathring{R}=0$;

(b) $(\varphi^2X\wedge\varphi^2Y)\cdot\mathring{R}=-(\varphi X\wedge\varphi Y)\cdot\mathring{R}$;

(c) $(X\wedge_S Y).\mathring{R}=0$.

Proof (a) For any $X, Y, U, V \in \mathfrak{X}(M)$

$$\begin{aligned}
(\varphi \cdot \mathring{R})(X,Y,U,V) &= -\mathring{R}(\varphi X,Y,U,V) - \mathring{R}(X,\varphi Y,U,V) \\
&\quad -\mathring{R}(X,Y,\varphi U,V) - \mathring{R}(X,Y,U,\varphi V) \\
&= -2[g((\varphi X \wedge Y)U,V) + g((X \wedge \varphi Y)U,V) + g((X \wedge Y)\varphi U,V) + g((X \wedge Y)U,\varphi V)] \\
&\quad +\frac{c+5}{4}[g((\varphi^2 X \wedge \varphi Y)U,V) + g((\varphi X \wedge \varphi^2 Y)U,V) + g((\varphi X \wedge \varphi Y)\varphi U,V) \\
&\quad +g((\varphi X \wedge \varphi Y)U,\varphi V)] + \frac{c+5}{4}[g((\varphi^3 X \wedge \varphi^2 Y)U,V) + g((\varphi^2 X \wedge \varphi^3 Y)U,V) \\
&\quad +g((\varphi^2 X \wedge \varphi^2 Y)\varphi U,V) + g((\varphi^2 X \wedge \varphi^2 Y)U,\varphi V)] - \frac{c-3}{2}[g(\varphi^2 X,Y)g(\varphi U,V) \\
&\quad +g(\varphi X,\varphi Y)g(\varphi U,V) + g(\varphi X,Y)g(\varphi^2 U,V) + g(\varphi X,Y)g(\varphi U,\varphi V)].
\end{aligned}$$

Using (12) and (21) we get result.

(b) For any $X, Y, Z, U, V, W \in \mathfrak{X}(M)$,

$$\begin{aligned}
((\varphi X \wedge \varphi Y)\cdot \mathring{R})(Z,U,V,W) &= -g(\varphi Y,Z)\mathring{R}(\varphi X,U,V,W) + g(\varphi X,Z)\mathring{R}(\varphi Y,U,V,W) \\
&\quad -g(\varphi Y,U)\mathring{R}(Z,\varphi X,V,W) + g(\varphi X,U)\mathring{R}(Z,\varphi Y,V,W) \\
&\quad -g(\varphi Y,V)\mathring{R}(Z,U,\varphi X,W) + g(\varphi X,V)\mathring{R}(Z,U,\varphi Y,W) \\
&\quad -g(\varphi Y,W)\mathring{R}(Z,U,V,\varphi X) + g(\varphi X,W)\mathring{R}(Z,U,V,\varphi Y) \\
&= \frac{c-3}{4}\{-g(\varphi Y,Z)g(\varphi U,W)g(\varphi X,\varphi V) - g(\varphi Y,Z)g(U,\varphi V)g(\varphi X,\varphi W) \\
&\quad +g(\varphi U,W)g(\varphi X,Z)g(\varphi Y,\varphi V) + g(U,\varphi V)g(\varphi X,Z)g(\varphi Y,\varphi W) \\
&\quad +g(\varphi Y,U)g(Z,\varphi V)g(\varphi X,\varphi W) + g(\varphi Y,U)g(\varphi Z,W)g(\varphi X,\varphi V) \\
&\quad -g(\varphi X,U)g(Z,\varphi V)g(\varphi Y,\varphi W) - g(\varphi X,U)g(\varphi Z,W)g(\varphi Y,\varphi V) \\
&\quad +g(\varphi Y,V)g(\varphi U,W)g(\varphi Z,\varphi X) - g(\varphi Y,V)g(\varphi Z,W)g(\varphi U,\varphi X) \\
&\quad -g(\varphi X,V)g(\varphi U,W)g(\varphi Z,\varphi Y) + g(\varphi X,V)g(\varphi Z,W)g(\varphi U,\varphi Y) \\
&\quad -g(\varphi Y,W)g(Z,\varphi V)g(\varphi U,\varphi X) + g(\varphi Y,W)g(U,\varphi V)g(\varphi Z,\varphi X) \\
&\quad +g(\varphi X,W)g(Z,\varphi V)g(\varphi U,\varphi Y) - g(\varphi X,W)g(U,\varphi V)g(\varphi Z,\varphi Y) \\
&\quad +\frac{c+5}{2}\{-g(\varphi Y,Z)g(\varphi X,\varphi U)g(\varphi V,W) + g(\varphi X,Z)g(\varphi Y,\varphi U)g(\varphi V,W) \\
&\quad +g(\varphi Y,U)g(\varphi Z,\varphi X)g(\varphi V,W) - g(\varphi X,U)g(\varphi Z,\varphi Y)g(\varphi V,W) \\
&\quad +g(\varphi Y,V)g(Z,\varphi U)g(\varphi X,\varphi W) - g(\varphi X,V)g(Z,\varphi U)g(\varphi Y,\varphi W) \\
&\quad -g(\varphi Y,W)g(Z,\varphi U)g(\varphi V,\varphi X) + g(\varphi X,W)g(Z,\varphi U)g(\varphi V,\varphi Y)\}. \qquad (32)
\end{aligned}$$

Now,

$$
\begin{aligned}
&((\varphi^2 X \wedge \varphi^2 Y)\cdot \mathring{R})(Z,U,V,W)\\
&= -g(\varphi^2 Y, Z)\mathring{R}(\varphi^2 X, U, V, W) + g(\varphi^2 X, Z)\mathring{R}(\varphi^2 Y, U, V, W)\\
&\quad -g(\varphi^2 Y, U)\mathring{R}(Z, \varphi^2 X, V, W) + g(\varphi^2 X, U)\mathring{R}(Z, \varphi^2 Y, V, W)\\
&\quad -g(\varphi^2 Y, V)\mathring{R}(Z, U, \varphi^2 X, W) + g(\varphi^2 X, V)\mathring{R}(Z, U, \varphi^2 Y, W)\\
&\quad -g(\varphi^2 Y, W)\mathring{R}(Z, U, V, \varphi^2 X) + g(\varphi^2 X, W)\mathring{R}(Z, U, V, \varphi^2 Y)\\
&= \frac{c-3}{4}\{g(\varphi Y, \varphi Z)g(\varphi X, V)g(\varphi U, W) + g(\varphi Y, \varphi Z)g(U, \varphi V)g(\varphi X, W)\\
&\quad -g(\varphi Y, V)g(\varphi U, W)g(\varphi X, \varphi Z) - g(U, \varphi V)g(\varphi Y, W)g(\varphi X, \varphi Z)\\
&\quad -g(\varphi Y, \varphi U)g(Z, \varphi V)g(\varphi X, W) - g(\varphi Y, \varphi U)g(\varphi X, V)g(\varphi Z, W)\\
&\quad +g(\varphi X, \varphi U)g(Z, \varphi V)g(\varphi Y, W) + g(\varphi X, \varphi U)g(\varphi Y, V)g(\varphi Z, W)\\
&\quad -g(\varphi Y, \varphi V)g(Z, \varphi X)g(\varphi U, W) + g(\varphi Y, \varphi V)g(U, \varphi X)g(\varphi Z, W)\\
&\quad +g(\varphi X, \varphi V)g(Z, \varphi Y)g(\varphi U, W) - g(\varphi X, \varphi V)g(U, \varphi Y)g(\varphi Z, W)\\
&\quad +g(\varphi Y, \varphi W)g(Z, \varphi V)g(U, \varphi X) - g(\varphi Y, \varphi W)g(U, \varphi V)g(Z, \varphi X)\\
&\quad -g(\varphi X, \varphi W)g(Z, \varphi V)g(U, \varphi Y) + g(\varphi X, \varphi W)g(U, \varphi V)g(Z, \varphi Y)\}\\
&\quad +\frac{c+5}{2}\{g(\varphi Y, \varphi Z)g(\varphi X, U)g(\varphi V, W) - g(\varphi X, \varphi Z)g(\varphi Y, U)g(\varphi V, W)\\
&\quad -g(\varphi Y, \varphi U)g(Z, \varphi X)g(\varphi V, W) + g(\varphi X, \varphi U)g(Z, \varphi Y)g(\varphi V, W)\\
&\quad -g(\varphi Y, \varphi V)g(Z, \varphi U)g(\varphi X, W) + g(\varphi X, \varphi V)g(Z, \varphi U)g(\varphi Y, W)\\
&\quad +g(\varphi Y, \varphi W)g(Z, \varphi U)g(V, \varphi X) - g(\varphi X, \varphi W)g(Z, \varphi U)g(V, \varphi Y)\}. \qquad (33)
\end{aligned}
$$

From (32) and (33), we see that

$$((\varphi^2 X \wedge \varphi^2 Y)\cdot \mathring{R})(Z,U,V,W) = -((\varphi X \wedge \varphi Y)\cdot \mathring{R})(Z,U,V,W).$$

(c) The Ricci curvature tensor can be written as

$$\mathring{S}(X,Y) = \frac{n(c-3)+8}{2} g(\varphi X, \varphi Y).$$

So, we have

$$
\begin{aligned}
(X \wedge_{\mathring{S}} Y)Z &= \mathring{S}(Y,Z)X - \mathring{S}(X,Z)Y\\
&= \frac{n(c-3)+8}{2}\{g(\varphi Y, \varphi Z)X - g(\varphi X, \varphi Z)Y\}.
\end{aligned}
$$

Replacing Z by $\mathring{R}$, we obtain

$$(X \wedge_{\mathring{S}} Y)\cdot \mathring{R} = \frac{n(c-3)+8}{2}\{g(\varphi Y, \varphi \cdot \mathring{R})X - g(\varphi X, \varphi \cdot \mathring{R})Y\}.$$

Using (a), $\varphi \cdot \mathring{R} = 0$, we get the result

$$(X \wedge_{\mathring{S}} Y)\cdot \mathring{R} = 0. \qquad \square$$

Theorem 4.4 *A Lorentzian Sasakian space form $M^{2n+1}(c)$ is semi-symmetric with respect to generalized Tanaka connection.*

Proof From (31), the curvature tensor is of the form

$$\mathring{R}(X,Y) = \frac{c-3}{4}(\varphi X \wedge \varphi Y) + \frac{c-3}{4}(\varphi^2 X \wedge \varphi^2 Y) + \frac{c+5}{2} g(X,\varphi Y)\varphi.$$

So,

$$\mathring{R}(X,Y)\cdot \mathring{R} = \frac{c-3}{4}(\varphi X \wedge \varphi Y)\cdot \mathring{R} + \frac{c-3}{4}(\varphi^2 X \wedge \varphi^2 Y)\cdot \mathring{R} + \frac{c+5}{2} g(X,\varphi Y)\varphi \cdot \mathring{R}.$$

Using (a) and (b) of Lemma 4.3, we have

$$\mathring{R}(X,Y)\cdot \mathring{R} = \frac{c-3}{4}(\varphi X \wedge \varphi Y)\cdot \mathring{R} - \frac{c-3}{4}(\varphi X \wedge \varphi Y)\cdot \mathring{R} = 0. \qquad \square$$

§5. *-Ricci Soliton on Lorentzian Sasakian Space Form with Respect to Generalized Tanaka Connection

In this section we first derived the $*$-Ricci tensor in Lorentzion Sasakian space form. The $*$-Ricci tensor first introduced by Kaimakamis and Panagiotidou [6] and given by

$$\mathring{S}^*(X,Y) = \frac{1}{2}\left[trace\{\varphi \circ R(X,\varphi Y)\}\right] \tag{34}$$

for all vector fields X, Y on M and φ is a (1,1)-tensor field.

Theorem 5.1 *In a Lorentzion Sasakian space form with generalized Tanaka connection, the $*$-Ricci tensor*

$$\mathring{S}^*(X,Y) = -\frac{n(c-3)+8}{4} g(\varphi X,\varphi Y). \tag{35}$$

Proof Replacing Z by φZ in (28), we get

$$\begin{aligned}\mathring{R}(X,Y)\varphi Z \;\; = \;\; & \frac{c-3}{4}\{[g(Y,\varphi Z)X - g(X,\varphi Z)Y] + \eta(\varphi Z)[\eta(Y)X - \eta(X)Y] \\ & +[\eta(Y)g(X,\varphi Z) - \eta(X)g(Y,\varphi Z)]\xi + [g(X,\varphi^2 Z)\varphi Y - g(Y,\varphi^2 Z)\varphi X]\} \\ & +\frac{c+5}{2} g(X,\varphi Y)\varphi^2 Z.\end{aligned}$$

Or,

$$\begin{aligned}\mathring{R}(X,Y)\varphi Z \;\; = \;\; & \frac{c-3}{4}\{[g(Y,\varphi Z)X - g(X,\varphi Z)Y] + [\eta(Y)g(X,\varphi Z) - \eta(X)g(Y,\varphi Z)]\xi \\ & +[g(X,\varphi^2 Z)\varphi Y - g(Y,\varphi^2 Z)\varphi X]\} + \frac{c+5}{2} g(X,\varphi Y)\varphi^2 Z.\end{aligned}$$

Taking inner product of the preceding equation with φW, we get

$$\begin{aligned} g(\mathring{R}(X,Y)\varphi Z,\varphi W) &= \frac{c-3}{4}\{[g(Y,\varphi Z)g(X,\varphi W)-g(X,\varphi Z)g(Y,\varphi W)] \\ &\quad +[\eta(Y)g(X,\varphi Z)-\eta(X)g(Y,\varphi Z)]g(\xi,\varphi W) \\ &\quad +[g(X,\varphi^2 Z)g(\varphi Y,\varphi W)-g(Y,\varphi^Z)g(\varphi X,\varphi W)]\} \\ &\quad +\frac{c+5}{2}g(X,\varphi Y)g(\varphi^2 Z,\varphi W). \end{aligned}$$

Using (12), we get

$$\begin{aligned} -g(\varphi\mathring{R}(X,Y)\varphi Z,W) = \frac{c-3}{4}&\{[g(Y,\varphi Z)g(X,\varphi W)+g(X,\varphi Z)g(\varphi Y,W)] \\ &-[g(X,\varphi^2 Z)g(\varphi^2 Y,W)-g(\varphi Y,\varphi Z)g(\varphi X,\varphi W)]\} \\ &+\frac{c+5}{2}g(X,\varphi Y)g(\varphi Z,W). \end{aligned}$$

Contracting X and W and using definition, we get

$$-2\mathring{S}^*(Y,Z) = \frac{c-3}{4}\{g(\varphi Y,\varphi Z)-[g(\varphi Y,\varphi Z)-g(\varphi Y,\varphi Z)2(n-1)]\}+\frac{c+5}{2}g(\varphi Y,\varphi Z).$$

Or,

$$-2\mathring{S}^*(Y,Z) = \frac{(c-3)(n-1)}{2}g(\varphi Y,\varphi Z)+\frac{c+5}{2}g(\varphi Y,\varphi Z).$$

Or,

$$\mathring{S}^*(Y,Z) = -\frac{n(c-3)+8}{4}g(\varphi Y,\varphi Z).$$

Replacing Y,Z by X,Y respectively we get the result. □

Corollary 5.2 *In a Lorentzion Sasakian space form,*

$$\mathring{S}^*(X,\xi)=0. \tag{36}$$

Theorem 5.3 *If $M(c)$ is a Lorentzian Sasakian space form with generalized Tanaka connection and (M,V,g) a $*$-Ricci soliton, where V is a pointwise collinear vector field with ξ. Then V is a constant multiple of ξ and the soliton is steady.*

Proof Let V be pointwise collinear vector field with ξ i.e. $V=f\xi$, where f is a function on the Lorentzian Sasakian manifold M. Then $(\mathcal{L}_V g+2\mathring{S}^*+2\lambda g)(X,Y)=0$, implies

$$g(\mathring{\nabla}_X f\xi,Y)+g(\mathring{\nabla}_Y f\xi,X)+2S^*(X,Y)+2\lambda g(X,Y)=0.$$

By (24),

$$-(Xf)\eta(Y)-(Yf)\eta(X)+2\mathring{S}^*(X,Y)+2\lambda g(X,Y)=0. \tag{37}$$

Replacing Y by ξ in (37) it follows that

$$-(Xf) - (\xi f)\eta(X) + 2\mathring{S}^*(X,\xi) + 2\lambda\eta(X) = 0.$$

Using (36),

$$Xf + (\xi f)\eta(X) - 2\lambda\eta(X) = 0. \tag{38}$$

Put $X = \xi$,

$$\xi f = \lambda.$$

From (38),

$$Xf = \lambda\eta(X).$$

Or,

$$df = \lambda\eta. \tag{39}$$

Applying (d) in (39),

$$\lambda d\eta = 0.$$

Since $d\eta \neq 0$ for Lorentzian Sasakian manifold, we have $\lambda = 0$. So by (39), V is constant multiple of ξ and as $\lambda = 0$, the soliton is steady. □

§6. *-Conformal Ricci Soliton on Lorentzian Sasakian Space Form with Respect to Generalized Tanaka Connection

Theorem 6.1 *If $M(c)$ is a Lorentzian Sasakian space form with generalized Tanaka connection and (M,V,g) a $*$-conformal Ricci soliton, where V is a pointwise collinear vector field with ξ. Then V is a constant multiple of ξ and the soliton is expanding or steady or shrinking according as $p < -\frac{2}{2n+1}$, $p = -\frac{2}{2n+1}$ or $p > -\frac{2}{2n+1}$.*

Proof Let V be pointwise collinear vector field with ξ i.e. $V = h\xi$, where h is a function on the Lorentzian Sasakian manifold M. Then $\left(\mathcal{L}_V g + 2\mathring{S}^* + \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right]\right)(X,Y) = 0$, implies

$$g(\mathring{\nabla}_X h\xi, Y) + g(\mathring{\nabla}_Y h\xi, X) + 2S^*(X,Y) + \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y) = 0.$$

By (24),

$$-(Xh)\eta(Y) - (Yh)\eta(X) + 2\mathring{S}^*(X,Y) + \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y) = 0. \tag{40}$$

Replacing Y by ξ in (40) it follows that

$$-(Xh) - (\xi h)\eta(X) + 2\mathring{S}^*(X,\xi) + \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] \eta(X) = 0.$$

Using (36),

$$Xb + (\xi h)\eta(X) - \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right]\eta(X) = 0. \tag{41}$$

Puting $X = \xi$,

$$\xi h = \lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right).$$

From (41),

$$Xh = \left(\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right)\eta(X).$$

Or,

$$dh = \left(\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right)\eta. \tag{42}$$

Applying (d) in (42),

$$\left(\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right)d\eta = 0.$$

Since $d\eta \neq 0$, we have $\lambda = \frac{1}{2}\left(p + \frac{2}{2n+1}\right)$. So by (42), V is constant multiple of ξ. Also we see that the soliton is expanding or steady or shrinking according as $p < -\frac{2}{2n+1}$, $p = -\frac{2}{2n+1}$ or $p > -\frac{2}{2n+1}$. □

Theorem 6.2 *If $M(c)$ is a Lorentzian Sasakian space form with generalized Tanaka connection and (M, V, g) a $*$-conformal $\eta-$Ricci soliton, where V is a pointwise collinear vector field with ξ. Then V is a constant multiple of ξ and the soliton is expanding or steady or shrinking according as $p < 2\mu - \frac{2}{2n+1}$ or $p = 2\mu - \frac{2}{2n+1}$ or $p > 2\mu - \frac{2}{2n+1}$.*

Proof Let V be pointwise co-linear vector field with ξ i.e. $V = \rho\xi$, where ρ is a function on the Lorentzian Sasakian manifold M. Then

$$\left(\mathcal{L}_V g + 2\mathring{S}^* + \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] + +2\mu\eta\otimes\eta\right)(X,Y) = 0,$$

which implies

$$g(\mathring{\nabla}_X \rho\xi, Y) + g(\mathring{\nabla}_Y \rho\xi, X) + 2S^*(X,Y) \\ + \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right]g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

By (24),

$$-(X\rho)\eta(Y) - (Y\rho)\eta(X) + 2\mathring{S}^*(X,Y) + \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right]g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \tag{43}$$

Replacing Y by ξ in (43) it follows that

$$-(X\rho) - (\xi\rho)\eta(X) + 2\mathring{S}^*(X,\xi) + \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right]\eta(X) + 2\mu\eta(X) = 0.$$

Using (36),

$$X\rho + (\xi\rho)\eta(X) - \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right]\eta(X) - 2\mu\eta(X) = 0. \tag{44}$$

Put $X = \xi$,

$$\xi\rho = \lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right) + \mu.$$

From (44),

$$X\rho = \left(\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right) + \mu\right)\eta(X).$$

Or,

$$d\rho = \left(\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right) + \mu\right)\eta. \tag{45}$$

Applying (d) in (45),

$$\left(\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right) + \mu\right)d\eta = 0.$$

Since $d\eta \neq 0$, we have $\lambda = \frac{1}{2}\left(p + \frac{2}{2n+1}\right) - \mu$. So by (45), V is constant multiple of ξ. Also we see that the soliton is expanding or steady or shrinking according as $p < 2\mu - \frac{2}{2n+1}$ or $p = 2\mu - \frac{2}{2n+1}$ or $p > 2\mu - \frac{2}{2n+1}$. □

§7. Generalized Ricci Soliton on Lorentzian Sasakian Space Form with Respect to Generalized Tanaka Connection

We defined $V^{\#}$ in the equation (11) by

$$V^{\#}(X) = g(V, X).$$

Replaced S by $\mathring{S}$, then (11) becomes

$$\mathcal{L}_V g = -2aV^{\#} \odot V^{\#} + 2b\mathring{S} + 2\lambda g. \tag{46}$$

Theorem 7.1 *If a Lorentzian Sasakian space form $M(c)$ with generalized Tanaka connection is a generalized Ricci soliton. Then*

$$\lambda = \frac{b[n(c-3)+8](n-1) - a}{2n-1}.$$

Proof The equation $\mathcal{L}_V g = -2aV^{\#} \odot V^{\#} + 2b\mathring{S} + 2\lambda g$, implies

$$g(\mathring{\nabla}_X\xi, Y) + g(X, \mathring{\nabla}_Y\xi) = -2a\eta(X)\eta(Y) + 2b\mathring{S}(X, Y) + 2\lambda g(X, Y).$$

Using (24), we get

$$a\eta(X)\eta(Y) - b\mathring{S}(X,Y) - \lambda g(X,Y) = 0. \tag{47}$$

Using (30), we have

$$a\eta(X)\eta(Y) - b\frac{n(c-3)+8}{2}g(\varphi X, \varphi Y) - \lambda g(X,Y) = 0.$$

Contracting X and Y, we get

$$-a + b[n(c-3)+8](n-1) - \lambda(2n-1) = 0.$$

Therefore, this implies

$$\lambda = \frac{b[n(c-3)+8](n-1) - a}{2n-1}. \tag{48}$$

This completes the proof. □

We introduce the generalized conformal Ricci soliton equation on a manifold of dimension n as

$$\mathcal{L}_V g = \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g - 2aV^{\#} \odot V^{\#} + 2bS. \tag{49}$$

where $V \in \Gamma(TM)$ and $\mathcal{L}_V g$ is the Lie-derivative of g along V and $V^{\#}$ the canonical one-form associated to V and a, b, λ some constants. Taking $V^{\#}(X) = g(V,X)$, and replace S by $\mathring{S}$. Then, (49) becomes

$$\mathcal{L}_V g(X,Y) = \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y) - 2aV^{\#}(X) \odot V^{\#}(Y) + 2b\mathring{S}(X,Y). \tag{50}$$

Theorem 7.2 *If a Lorentzian Sasakian space form $M(c)$ with generalized Tanaka connection is a generalized conformal Ricci soliton. Then the soliton is expanding or steady or shrinking according as $p < 2a\mu^2 - \frac{2}{2n+1}$ or $p = 2a - \frac{2}{2n+1}$ or $p > 2a\mu^2 - \frac{2}{2n+1}$.*

Proof The equation (50) implies

$$g(\mathring{\nabla}_X \xi, Y) + g(X, \mathring{\nabla}_Y \xi) = \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y) - 2a\mu^2\eta(X)\eta(Y) + 2b\mathring{S}(X,Y).$$

By (24),

$$\left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y) - 2a\eta(X)\eta(Y) + 2b\mathring{S}(X,Y) = 0.$$

Replacing Y by ξ it follows that

$$-\left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]\eta(X) - a\eta(X) + b\mathring{S}(X,\xi) = 0.$$

By equation (30), we have

$$-\left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]\eta(X) - a\eta(X) = 0.$$

Or,

$$-\left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right) + a\right]\eta(X) = 0.$$

This implies

$$\lambda = \frac{1}{2}\left(p + \frac{2}{2n+1}\right) - a.$$

Thus, the soliton is expanding or steady or shrinking according as $p < 2a\mu^2 - \frac{2}{2n+1}$ or $p = 2a - \frac{2}{2n+1}$ or $p > 2a\mu^2 - \frac{2}{2n+1}$. □

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A Note on Laplacian Coefficients of the Characteristic Polynomial of L-Matrix of a Marked Graph

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Abstract: With this article in mind, we have found the characteristic polynomial of a Laplacian L-matrix of a graph with signs. Using the trace of the Laplacian L-matrix and the number of vertices of the marked graph, the coefficients of the characteristic polynomial have been found. Also we have shown that the same characteristic polynomial coefficients can be obtained using Laplacian eigenvalues of L-matrix. Further, we have obtained an upper bound for the largest eigenvalue of a signed graph.

Key Words: Signed graphs, neutrosophic singed graph, marked graphs, balance, switching.

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§1. Introduction

For all phrases and notes of graph theory, readers should refer to [6]. Here we consider simple and non-loop graphs including the restricted number of vertices and edges.

A *signed graph* $\Gamma = (G(V, E), \triangle)$ is a graph that describes each edge as an edge with positive or negative signs, where G is the graph without sign and $\triangle : E \longrightarrow \{+, -\}$ is a function. Generally, a *neutrosophic singed graph* G^N is such a graph with a bijective $\triangle^N : E \longrightarrow \{+, -\}$ for $e \in E$ neutrosophically, i.e., there is a partition on E with T of positive labels, I of negative labels and F of fail set such that $T \cup I \cup F = E$, where the labels on edges in F are undetermined, which may be all positive, negative or random completely. Particularly, if $F = \emptyset$, a neutrosophic singed graph is just a signed graph.

Usually, people and their relations are represented by a signed graph. One of the main applications of a signed graph is to study the relationships among people as explained in [3].

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Based on the nature of the relationship we can term them as positive or negative. If two people are friendly then their relationship is positive whereas if two people detest each other, there is a negative association between them. These concepts are discussed in [13] and [5].

The applications of signed graphs have been proposed and presented in [7], [8] and [19]. Harary [9] invented the phrase *balanced signed graph* to describe how a balanced signed graph has an even number of negative edges in each cycle.

The positive cycle in a signed graph is the cycle in which the product of signs of the edges is positive. This is referred to as balanced signed graph. Unbalanced signed graph is one which is not balanced (see Harary [9]). A simple *balanced signed graph* can be found using the graph algorithm created by Harary et.al.[11].

A *marking* of a graph Γ is a function $\mu : V(G) \to \{+,-\}$. A *marking* of Γ is a function $\mu : V(G) \to \{+,-\}$; A signed graph Γ together with a marking μ is denoted by Γ_μ. Given a signed graph Γ one can easily define a marking μ of Γ as for any vertex $v \in V(\Gamma)$,

$$\mu(v) = \prod_{uv \in E(\Gamma)} \triangle(uv)$$

and the marking μ of Γ is called *canonical marking* of Γ.

R. Abelson and Rosenberg in [16] were the first to present the *switching signed graph* as a tool to study social behavior. In [19] T.Zaslavasky has clearly explained the relevance of *switching signed graphs* mathematically. *Switching signed graph*, denoted as $\Gamma_\mu(\Gamma)$ and referred to as Γ_μ is a function that employs the marking $\flat$ to alter the mark of each edge of Γ.

Signed graphs $\Gamma_1 = (G_1, \triangle)$ and $\Gamma_2 = (G_2, \triangle')$ are isomorphic if their underline graphs G_1 and G_2 are also *isomorphic*. Here G_1 and G_2 are not signed graphs. Therefore, Γ_1 and Γ_2 are *switching equivalent* which is represented as $\Gamma_1 \sim \Gamma_2$. Specifically $\Gamma_\mu(\Gamma_1) \sim \Gamma_2$ for any marking μ and G_1 and G_2 remain unaltered. Furthermore, two signed graphs Γ_1, Γ_2 are said to cycle *isomorphic* if the cycles of two signed graphs have the same sign.

The following proposition is the characterization of switching signed graph, given by T.Zaslavasky [18].

Proposition 1.1 ([18]) *Two signed graphs Γ_1 and Γ_2 with the same underline graphs are switching equivalent if and only if they are cycle isomorphic.*

In a signed graph, the degree of each vertex can be calculated by $d = d^+ + d^-$ so that the degree of vertex in a signed graph Γ and their underline graphs is the same and in the adjacent matrix of a signed graph, if two vertices are adjacent then the entry a_{ij} is 1 along with the sign of the edge, otherwise the entry is zero. Furthermore, in a Laplacian matrix, if two vertices v_i and v_j are adjacent then the entries a_{ij} are 1 with the opposite sign of the corresponding adjacent edge v_iv_j, otherwise a_{ij} is zero and the diagonal entries a_{ii} being the degree of the vertex.

We know that $L(\Gamma,+)$ and $L(\Gamma,-)$ are the Laplacian matrices of the signed graphs $(\Gamma,+)$ and $(\Gamma,-)$ whose edges are all positive and negative respectively. Also $L(\Gamma,+)$ is the signless Laplacian matrix of Γ which is the sum of the diagonal matrix and the adjacent matrix.

Prof. E. Sampathkumar and M. A Sriraj in [21] have introduced a new matrix $A_{\mathtt{L}}(G)$ called L-matrix of a vertex labelled graph G=(V,E), whose elements are defined as follows:

$$a_{ij} = \begin{cases} 2, \text{ if } v_i \text{ and } v_j & \text{are adjacent with } \mu(v_i) = \mu(v_j) \\ 1, \text{ if } v_i \text{ and } v_j & \text{are adjacent with } \mu(v_i) \neq \mu(v_j) \\ -1, \text{ if } v_i \text{ and } v_j & \text{are non adjacent with } \mu(v_i) = \mu(v_j) \\ 0, \text{ otherwise.} \end{cases}$$

With the motivation of $A_{\mathtt{L}}(G)$ (or $A_{\mathtt{L}}(\Gamma)$) we have Laplacian L-matrix of a signed graph $\mathtt{L}(\Gamma)$ which is defined as follows:

$$b_{ij} = \begin{cases} -2, \text{ if } v_i \text{ and } v_j & \text{are adjacent with } \mu(v_i) = \mu(v_j) \\ -1, \text{ if } v_i \text{ and } v_j & \text{are adjacent with } \mu(v_i) \neq \mu(v_j) \\ 1, \text{ if } v_i \text{ and } v_j & \text{are non adjacent with } \mu(v_i) = \mu(v_j) \\ d(v_i), \text{ if i = j} \\ 0, \text{ otherwise.} \end{cases}$$

Also, $\mathtt{L}(\Gamma) = D(\Gamma) - A_{\mathtt{L}}(\Gamma)$, where $D(\Gamma)$ is the diagonal matrix and $A_{\mathtt{L}}(\Gamma)$ is the Adjacent L-matrix of a signed graph Γ.

Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots \geq \lambda_n$ be the eigenvalues of the Laplacian L-matrix of signed graph $\Gamma = (G, \triangle)$, having n vertices. In 2003 Yaoping Hou. et. al.[20] have established new bounds for eigenvalues of a signed graph as stated in the following theorem.

Theorem 1.2 ([20]) *Let $\Gamma = (G, \triangle)$ be a signed graph with n vertices. Then*

$$\lambda_1 \leq 2(n-1)$$

with equality holds if and only if, Γ is switching equivalent to a complete graph with all edges being negative.

§2. Laplacian Coefficients of the Characteristic Polynomial of L-Matrix of a Marked Graph

The coefficients of a characteristic polynomial are useful in the study of chemical properties of molecules. In [14], Ivailo M. Mladenov et. al. have given very elegant algorithm to find coefficients of a characteristic polynomial using trace of the Laplacian matrix. In [2], Carla Silva Oliveria et. al. have found second and third Laplacian coefficients of a characteristic polynomial. Francesco Belardo et. al. [12] have found Laplacian coefficients of a marked signed graph as stated in the following theorem.

Theorem 2.1 ([12]) *The Laplacian characteristic polynomial of any signed graph* Γ *is*

$$\psi(\Gamma, x) = x^n + q_1 x^{n-1} + \cdots + q_{n-1}x + q_n,$$

where

$$q_p = (-1)^p \sum_{H \in H_p} w(H)$$

and the set of signed TU-sub graphs of Γ *with* p *edges is* H_p.

Using the above theorem and by the motivation of Faddeev LeVerrier algorithm and [14], [1] we present a new algorithm to find signed Laplacian coefficients of a characteristic polynomial using the trace of a Laplacian $\mathtt{L}$-matrix of Γ, whose underline graph is complete and $\mu(v_i) = \mu(v_j)$.

Proposition 2.2 *If* $\mathtt{L}$ *is Laplacian* $\mathtt{L}$*-matrix of a complete signed graph* Γ *and* $\mu(v_i) = \mu(v_j)$ *then,*

$$tr(\mathtt{L}^\ell) = (n-1)(n+1)^\ell + (-1)^\ell (n-1)^\ell.$$

Proof Here, the proof is by induction.

$$\begin{aligned} tr(\mathtt{L}) &= \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n \\ &= (n+1) + (n+1) + \ldots + (-1)(n-1) \\ &= (n-1)(n+1) + (-1)(n-1). \end{aligned}$$

We get

$$tr(\mathtt{L}^2) = (n-1)(n+1)^2 + (-1)^2(n-1)^2,$$

$$tr(\mathtt{L}^3) = (n-1)(n+1)^3 + (-1)^3(n-1)^3.$$

Similarly, for an integer k,

$$tr(\mathtt{L}^k) = (n-1)(n+1)^k + (-1)^k(n-1)^k,$$

$$\begin{aligned} tr(\mathtt{L}^{k+1}) &= \lambda_1^{k+1} + \lambda_2^{k+1} + \lambda_3^{k+1} + \cdots + \lambda_n^{k+1} \\ &= (n+1)^{k+1} + (n+1)^{k+1} + ... + (n+1)^{k+1} + (-1)^{k+1}(n-1)^{k+1} \\ &= (n-1)(n+1)^{k+1} + (-1)^{k+1}(n-1)^{k+1}. \end{aligned}$$

Hence, by induction,

$$tr(\mathtt{L}^\ell) = (n-1)(n+1)^\ell + (-1)^\ell (n-1)^\ell.$$

This completes the proof. $\square$

For an example, we put $\ell = 1,2,3,4$ in the above expression, then

$$\begin{aligned}
tr(\mathtt{L}) &= n(n-1),\\
tr(\mathtt{L}^2) &= n^3+2n^2-3n,\\
tr(\mathtt{L}^3) &= n^4+n^3+3n^2-5n,\\
tr(\mathtt{L}^4) &= n^5+4n^4-2n^3+4n^2-7n.
\end{aligned}$$

The coefficients q_1, q_2, q_3 and q_4 of characteristic equation of a Laplacian $\mathtt{L}$-matrix of a signed graph Γ are calculated as follows:

$$\begin{aligned}
q_1 &= -tr(\mathtt{L}) = -n(n-1),\\
q_2 &= -\frac{1}{2}tr\left\{Z_1\mathtt{L}\right\} \quad \text{where } \mathrm{Z}_1 = \mathtt{L}+\mathrm{q}_1\mathrm{I})\\
&= -\frac{1}{2}tr\left\{(\mathtt{L}^2)+(-n^2+n)\mathtt{L}\right\}\\
&= -\frac{1}{2}\left\{tr(\mathtt{L}^2)-n^2tr(\mathtt{L})+ntr(\mathtt{L})\right\}\\
&= -\frac{1}{2}\left\{(n^3+2n^2-3n)-n^2(n^2-n)+n(n^2-n)\right\}\\
&= -\frac{1}{2}\left\{-n^4+3n^3+n^2-3n\right\},\\
q_3 &= -\frac{1}{3}tr\left\{Z_2\mathtt{L}\right\} \quad \text{where } \mathrm{Z}_2 = \mathrm{Z}_1\mathtt{L}+\mathrm{q}_2\mathrm{I})\\
&= -\frac{1}{3}tr\left\{(\mathtt{L}+q_1I)\mathtt{L}^2+q_2L\right\}\\
&= -\frac{1}{3}tr\left\{\mathtt{L}^3+q_1\mathtt{L}^2+q_2\mathtt{L}\right\}\\
&= -\frac{1}{3}tr\left\{\mathtt{L}^3+(-n^2+n)\mathtt{L}^2+(\frac{-1}{2}(-n^4+3n^3+n^2-3n))\mathtt{L}\right\}\\
&= -\frac{1}{3}\left\{tr(\mathtt{L}^3)+(-n^2+n)tr(\mathtt{L}^2)-(\frac{1}{2}(-n^4+3n^3+n^2-3n)tr(\mathtt{L})\right\}\\
&= -\frac{1}{3}\left\{(n^4+n^3+3n^2-5n)+(-n^2+n)(n^3+2n^2-3n)\right.\\
&\quad \left.-\frac{1}{2}(-n^4+3n^3+n^2-3n)(n(n-1)\right\}\\
&= -\frac{1}{6}(n^6-6n^5+2n^4+16n^3-3n^2-10n),\\
q_4 &= -\frac{1}{4}tr(Z_3\mathtt{L}) \quad (\text{where } \mathrm{Z}_3 = \mathrm{Z}_2\mathtt{L}+\mathrm{q}_3\mathrm{I})\\
&= -\frac{1}{4}tr\left((Z_1\mathtt{L}+q_2)\mathtt{L}^2+q_3\mathtt{L}\right)\\
&= -\frac{1}{4}tr\left((\mathtt{L}+q_1)\mathtt{L}^3+q_2\mathtt{L}^2+q_3\mathtt{L}\right)\\
&= -\frac{1}{4}tr(\mathtt{L}^4+q_1\mathtt{L}^3+q_2\mathtt{L}^2+q_3\mathtt{L})\\
&= -\frac{1}{4}\left(tr(\mathtt{L}^4)+q_1tr(\mathtt{L}^3)+q_2tr(\mathtt{L}^2)+q_3tr(\mathtt{L})\right)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{4}\left\{(n^5+4n^4-2n^3+4n^2-7n)+q_1(n^4+n^3+3n^2-5n)\right.\\
&\quad \left.+\,q_2(n^3+2n^2-3n)+q_3(-n(n-1))\right\}\\
&= -\frac{1}{24}\{-n^8+10n^7-17n^6-38n^5+61n^4+70n^3-43n^2-42n\}.
\end{aligned}$$

Theorem 2.3 *If the Laplacian characteristic equation of* $\mathtt{L}$*-matrix of* Γ *is*

$$\psi(\Gamma,t)=y^n+t_1y^{n-1}+\cdots+t_{n-1}y+t_n$$

with $\mu(v_i)=\mu(v_j)$ *and* $t_0=1$*, then,*

$$t_\hbar=\frac{-1}{\hbar}\sum_{j=0}^{\hbar-1}t_jtr(L^{\hbar-j}),$$

where, $t_\hbar$ *are the coefficients of the characteristic polynomial and* $\hbar\neq 0$.

Proof Here,

$$\begin{aligned}
t_1 &= -tr(\mathtt{L}),\\
t_2 &= -\frac{1}{2}tr(t_1\mathtt{L})=-\frac{1}{2}tr((\mathtt{L}+t_1)\mathtt{L})\\
&= -\frac{1}{2}\{tr(\mathtt{L}^2)+t_1tr(\mathtt{L})\}=-\frac{1}{2}\{t_0tr(\mathtt{L}^2)+t_1tr(\mathtt{L})\}.
\end{aligned}$$

Similarly, for an integer k,

$$t_k=-\frac{1}{k}\sum_{j=0}^{k-1}t_j\ tr(\mathtt{L}^{k-j})$$

and we have,

$$\begin{aligned}
t_{k+1} &= -\frac{1}{k+1}tr(t_k\mathtt{L})=-\frac{1}{k+1}tr((t_{k-1}L+t_k)\mathtt{L})\\
&= -\frac{1}{k+1}tr((\mathtt{L}t_{k-2}+t_{k-1})\mathtt{L}^2+t_k\mathtt{L})\\
&= -\frac{1}{k+1}(tr(\mathtt{L}^3t_{k-2})+t_{k-1}tr(\mathtt{L}^2)+t_ktr(\mathtt{L}))\\
&= -\frac{1}{k+1}(tr(\mathtt{L}^4t_{k-3})+tr(\mathtt{L}^3)t_{k-1}+tr(\mathtt{L}^2)t_k+tr(\mathtt{L})),\\
\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
t_{k+1} &= \frac{-1}{k+1}\{t_0tr(\mathtt{L}^{k+1})+t_1tr(\mathtt{L}^k)+t_2tr(\mathtt{L}^{k-1})+\cdots+t_ktr(\mathtt{L})\},
\end{aligned}$$

i.e.,

$$t_{k+1}=-\frac{1}{k+1}\sum_{j=0}^{k}t_j\ tr(\mathtt{L}^{k+1-j}).$$

Hence by induction,

$$t_\hbar = -\frac{1}{\hbar}\sum_{j=0}^{\hbar-1} t_j tr(\mathtt{L}^{\hbar-j}),$$

where $\hbar \neq 0$. □

Corollary 2.4 *For any signed graph Γ, let $\mu(v_i) = \mu(v_j)$ and $t_0 = 1$. Then, the Laplacian characteristic polynomial of $\mathtt{L}-$ matrix of Γ is*

$$\psi(\Gamma, y) = y^n + t_1 y^{n-1} + \cdots + t_{n-1} y + t_n$$

with $t_0 = 1$ and

$$t_\imath = -\frac{1}{\imath}\sum_{j=0}^{\imath-1} t_j((n-1)(n+1)^{\imath-j} + (-1)^{\imath-j}(n-1)^{\imath-j})$$

where $\imath \neq 0$.

Proof By Proposition 2.2, we have

$$tr(\mathtt{L}^{\imath-j}) = (n-1)(n+1)^{\imath-j} + (-1)^{\imath-j}(n-1)^{\imath-j}$$

. By Theorem 2.3, we get

$$t_\imath = -\frac{1}{\imath}\sum_{j=0}^{\imath-1} t_j((n-1)(n+1)^{\imath-j} + (-1)^{\imath-j}(n-1)^{\imath-j})$$

where $\imath \neq 0$. □

Corollary 2.5 *For any signed graph Γ and $\Gamma \sim (K_n, -)$, the Laplacian characteristic polynomial of Γ is $\psi(\Gamma, y) = y^n + t_1 y^{n-1} + \cdots + t_{n-1} y + t_n$ with $t_0 = 1$ and $\lambda_1, \lambda_n, \lambda_{n-1}$ are the Laplacian eigenvalues of $\mathtt{L}$-matrix of signed graph Γ then,*

$$t_\varsigma = -\frac{1}{\varsigma}\sum_{r=0}^{\varsigma-1} t_r \lambda_n(\lambda_1^{\varsigma-r} + (-1)^{\varsigma-r}\lambda_n^{\varsigma-r-1}),$$

where $\varsigma \neq 0$.

Proof Since $\lambda_1 = (n+1)$, $\lambda_n = -(n-1)$and hence by Corollary 2.4,

$$t_\varsigma = -\frac{1}{\varsigma}\sum_{r=0}^{\varsigma-1} t_r \lambda_n(\lambda_1^{\varsigma-r} + (-1)^{\varsigma-r-1}\lambda_n^{\varsigma-r})$$

where $\varsigma \neq 0$. □

With the motivation of Theorem 1.2, we will find the following upper bound.

Proposition 2.6 *For any graph with sign Γ, let λ_1 be the maximum Laplacian eigenvalue of*

L*-matrix of* Γ*. Then,*

$$\lambda_1 \leq \sqrt{n^3 - 2n^2 + 5n}.$$

Proof By the Cauchy-Schwartz inequality,

$$\begin{aligned}(\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n)^2 &\leq n \ (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \cdots + \lambda_n^2),\\ (\lambda_1 + (n-2)(n+1) + 2)^2 &\leq n((n-1)(n+1)^2 + 2^2).\end{aligned}$$

This leads to

$$\lambda_1^2 \leq n^3 - 2n^2 + 5n$$

and the proof is completes. $\square$

§3. Conclusion

In this paper we have found the coefficients of the characteristic polynomial of a Laplacian L-matrix of a signed graph Γ in two ways:

1) By using the number of vertices of Γ as shown in the following table;
2) By using the Laplacian eigenvalues of $\mathtt{L}-matrix$ of signed graph Γ.

Also, we have found an upper bound for the largest eigenvalue of a Laplacian L-matrix.

Number of Vertices	**Coefficients of Characteristic Polynomial**
3	t_1=-6, t_2=0, t_3=32
4	t_1=-12, t_2=30, t_3=100, t_4=-375
5	t_1=-20, t_2=-120, t_3=0, t_4=-2160, t_5=5184
6	t_1=-30, t_2=315, t_3=-980, t_4=-5145, t_5=43218, t_6=-8405,
7	t_1=-42, t_2=672, t_3=-4480, t_4=0, t_5=172032, t_6=-917504, t_7=1572864
8	t_1=-56, t_2=1260, t_3=-13608, t_4=51030, t_5=367416, t_6=-4960116, t_7=21257640, t_8=-33480783
9	t_1=-72, t_2=2160, t_3=-33600, t_4=252000, t_5=0, t_6=-16800000, t_7=144000000, t_8=-540000000, t_9=800000000
10	t_1=-90, t_2=3465, t_3=-72600, t_4=838530, t_5=-3689532, t_6=-33820710, t_7=637761960, t_8=-4384613475, t_9=15005121670, t_{10}=-21221529219

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Pairwise Balanced Designs Arising from Minimum Covering and Maximum Independent Sets of Circulant Graphs

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Abstract: The pairwise balanced designs (PBD's) is a pair (P, B), where P is a finite set of ν points and B is a family of subsets of P, called blocks such that every two distinct points in P appear in exactly one block. Let $\alpha(G) = \alpha$ and $\beta(G) = \beta$ be the vertex covering and independence number of a graph $G = (V, E)$ with the minimum and maximum cardinality of such sets are denoted by α-sets and β-sets of G, respectively (or, simply (α, β)-sets). In this paper, we obtain the total number of (α, β)-sets in different jump sizes of some circulant graphs apart from strongly regular graphs which are the blocks of PBD.

Key Words: Covering number, independence number, circulant graph, pairwise balanced designs, Smarandachely pairwise balanced designs.

AMS(2010): 05C51, 05E30, 05C69.

§1. Introduction

In this paper, we are focusing on nonempty, finite, simple, undirected graphs with notations $p = |V|$ and $q = |E|$ for the number of vertices and edges of a graph $G = (V, E)$, respectively. In general, we use to $\langle X \rangle$ denote the sub graph induced by the set of vertices X. We refer to [12] for unspecified terms of the paper.

1.1. (α, β)-Sets of a Graph

A vertex of graph G is said to cover the edges incident with it, and a vertex cover of G is a set of vertices covering all the edge of G. The smallest cardinality of a vertex cover is called the vertex covering number $\alpha(G)$ or α of G. Further, a subset S of the vertex set $V(G)$ is said to be an independent set if the induced sub graph $\langle S \rangle$ is a trivial graph. The largest number of vertices in such a set is called the vertex independence number $\beta(G)$ or β of a graph G.

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The minimum and maximum cardinality of vertex covering and independence sets are denoted by α-sets and β-sets of a graph G, respectively. The maximum number of α-sets and β-sets of G is denoted by $\tau_\alpha(G)$ and $\tau_\beta(G)$, respectively. For more details on similar concepts, we refer to [6, 11, 12, 13, 16].

1.2. Strongly Regular Graph

A strongly regular graph with parameters (p, l, ω, μ) is a finite graph on p vertices, without loops or multiple edges, regular of degree l (with $0 < l < p-1$, so that there are both edges and no edges), and such that any two distinct vertices have ω common neighbors when they are adjacent, and μ common neighbors when they are nonadjacent. For the related concepts of the strongly regular graph, we refer to [2, 9].

1.3. Pairwise Balanced Designs (PBD's)

The combinatorial design theory is a study of collection of subsets with certain intersection properties. Based on the group and the type of underlying association scheme, there are other subgroups that can be made.

The following conditions satisfy m classes of association scheme on ν vertices (elements or objects) are

(i) If the associates are symmetric, then any two vertices are m^{th} associates, where $1 \leq k \leq m$;

(ii) Each vertex x contains n_k k^{th} associates, the number n_k being independent of vertex x;

(iii) If two vertices x and y are k^{th} associates, then the number of vertices which are a^{th} associates of x and b^{th} associates of y is p^k_{ab} and is independent of the k^{th} associates x and y. Hence $p^k_{ab} = p^k_{ba}$.

The pairwise balanced designs (PBD's) are a specific type of experimental design that offer several advantages over other block designs in certain situations and is defined as follows:

The PBD is a pair $(P, \mathcal{B})$ such that $\mathcal{B}$ is a set of subsets (called blocks) of P, each of cardinality at least two such that every unordered pair of points (elements of P) is contained in a unique block in $\mathcal{B}$. If ν is a positive integer and K is a set of positive integers, each of which is greater than or equal to 2, then we say that $(P, \mathcal{B})$ is a (ν, K)-PBD if $|X| = \nu$, and $|\mathcal{B}| \in K$ for every $B \in \mathcal{B}$. Furthermore, a Smarandachely PBD is contrary to the pairwise balanced designs, which asks for every unordered pair of points in P containing in 2 blocks in $\mathcal{B}$ at least.

Generally, these PBD's are highly efficient in terms of the number of treatments they can accommodate with a limited number of experimental units. They allow for a large number of treatments to be compared while minimizing the number of experimental units required. This efficiency is particularly useful when resources, such as time, money, or subjects, are limited. These designs provide increased precision in estimating treatment effects compared to other block designs. By carefully selecting which treatments are paired together in each block, these designs allow for a more precise estimation of treatment effects by reducing the variation caused by extraneous factors or confounding variables. PBD's ensure that each treatment appears with every other treatment in a block a balanced number of times. This balance helps to control the influence of confounding factors or extraneous variables that may affect the response variable. By balancing the treatment combinations, PBD's can help to separate the true treatment effects from the effects of other variables. For more details on combinatorial design theory with some graph parameters, we refer to [5, 7, 8, 9, 14, 17].

§2. Main Results

For a given positive integer p, let s_1, s_2, $\cdots$, s_t be a sequence of integers where $0 < s_1 < s_2 < \cdots < s_t < \frac{p+1}{2}$. The circulant graph $C_p(S)$ where $S = s_1, s_2, \cdots, s_t$ is the graph on p vertices labeled as v_1, v_2, $\cdots$, v_p with vertex v_i adjacent to each vertex $v_{i\pm s_j (mod\ p)}$ and the values s_t are called jump sizes.

The circulant graphs have been investigated in the fields outside of graph theory. For example, in geometers, circulant graphs are known as star polygons [1]. They have been used to solve problems in group theory (particularly the families of Cayley graphs), as shown in [3] as well as number theory and analysis. They are also, used as models for interconnection networks in telecommunication, VLSI designs, parallel, and distributed computing. For applications and mathematical properties of circulant graphs, see [4], [10] and [15].

2.1 Circulant Graph $C_p(1)$

The jump size of circulant graph is one, known as cycle C_p with $p \geq 3$ vertices. That is, $C_p(1) \cong C_p$, $p \geq 3$. The circulant graph $C_4(1)$ is the only strongly regular graphs.

Proposition 2.1 ([11]) *For any Circulant graph $C_p(1)$; $p \geq 3$ vertices,*

$$\alpha(C_p(1)) = \left\lceil \frac{p}{2} \right\rceil \textit{ and } \beta(C_p(1)) = \left\lfloor \frac{p}{2} \right\rfloor.$$

Theorem 2.1 *The collection of all (α, β)-sets of a circulant graph $C_p(1)$, $p = 2n$, $n \geq 2$ vertices form a PBD with parameters $\nu = p$, $b = 4$, $g_1 = \lceil \frac{p}{2} \rceil$, $g_2 = \lfloor \frac{p}{2} \rfloor$, $r = 2$ and*

$$\lambda_m = \begin{cases} 2, & m \equiv 0 (mod\, 2), \\ 0, & \textit{otherwise.} \end{cases}$$

Proof Let $C_p(1)$ be a circulant graph with $p = 2n$, $n \geq 2$ vertices given by $v_1, v_2, \cdots, v_p$. By Proposition 2.1, we have $\alpha(C_p(1)) = \left\lceil \frac{p}{2} \right\rceil$ and $\beta(C_p(1)) = \left\lfloor \frac{p}{2} \right\rfloor$. Further, $C_p(1)$ with $p = 2n$, $n \geq 2$ have two blocks of (α, β)-set, it implies $b = \tau_\alpha(C_p(1)) + \tau_\beta(C_p(1)) = 4$. Also, we have $g_1 = \alpha(C_p(1)) = \lceil \frac{p}{2} \rceil$ and $g_2 = \beta(C_p(1)) = \lfloor \frac{p}{2} \rfloor$, where g_1 and g_1 are the number of elements contained exactly in their respective blocks. By virtue of the above facts, we have $r = 2$. To obtain the m-associates for the elements, where $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$. The two distinct elements are first associates, if they have jump size 1 and they are k^{th}-associates ($2 \le k \le \lfloor \frac{p}{2} \rfloor$), if they have k jump sizes. Hence the parameters of first kind are given by $\nu = p$, $b = 4$, $g_1 = \lceil \frac{p}{2} \rceil$, $g_2 = \left\lfloor \frac{p}{2} \right\rfloor$, $r = 2$ and

$$\lambda_m = \begin{cases} 2, & m \equiv 0 (mod\, 2), \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the result follows. □

Theorem 2.2 *The collection of all (α, β)-sets of a circulant graph $C_p(1)$; $p = 2n+1$, $n \ge 1$ vertices form a PBD with parameters $\nu = p$, $b = 2p$, $g_1 = \left\lceil \frac{p}{2} \right\rceil$, $g_2 = \left\lfloor \frac{p}{2} \right\rfloor$, $r = p$ and*

$$\lambda_m = \begin{cases} m, & m \equiv 0 (mod\, 2), \\ p - \lambda_{m-1} - \lambda_{m+1}, & \textit{otherwise.} \end{cases}$$

Proof For a given circulant graph $C_p(1)$; $p = 2n+1$, $n \ge 2$ vertices labeled as $v_1, v_2, \cdots, v_p$. By Proposition 2.1, we have $\alpha(C_p(1)) = \left\lceil \frac{p}{2} \right\rceil$ and $\beta(C_p(1)) = \left\lfloor \frac{p}{2} \right\rfloor$. Further, $C_p(1)$; $p = 2n+1$, $n \ge 1$ have p blocks of α-set, it implies $\tau_\alpha(C_p(1)) = \tau_\beta(C_p(1)) = p$. Therefore $b = 2p$. Also, we have $g_1 = \alpha(C_p(1)) = \left\lceil \frac{p}{2} \right\rceil$ and $g_2 = \beta(C_p(1)) = \left\lfloor \frac{p}{2} \right\rfloor$, where g_1 and g_2 are the number of elements contained exactly in their respective blocks. From the above facts, we have $r_\alpha = \left\lceil \frac{p}{2} \right\rceil$ and $r_\beta = \left\lfloor \frac{p}{2} \right\rfloor$, therefore $r = p$. To obtain the m-associates for the elements, where $1 \le m \le \lfloor \frac{p}{2} \rfloor$. The two distinct elements are first associates, if they have jump size 1 and otherwise they are k^{th}-associates ($2 \le k \le \lfloor \frac{p}{2} \rfloor$). Hence the parameters of first kind are given by $\nu = p$, $b = 2p$, $g_1 = \lceil \frac{p}{2} \rceil$, $g_2 = \lfloor \frac{p}{2} \rfloor$, $r = p$, and

$$\lambda_m = \begin{cases} m, & m \equiv 0 (mod\, 2), \\ p - \lambda_{m-1} - \lambda_{m+1}, & \text{otherwise.} \end{cases}$$

Thus, the result follows. □

2.2 Circulant Graph with Odd Jump Sizes

The circulant graph of odd jump size $(1, 3, \cdots, \lfloor \frac{p}{2} \rfloor)$ with $p \ge 2$ is known as a complete bipartite graph K_{p_1,p_2} for $p_1 = p_2$, that is $C_p\left(1, 3, \ldots, \left\lfloor \frac{p}{2} \right\rfloor\right) \cong K_{p_1,p_2}$. Further, all the sequence of an odd jump size from 1 to $\lfloor \frac{p}{2} \rfloor$ are strongly regular graphs. Apart from this, the circulant graphs are not strongly regular graph if the sequence of odd jump sizes were taken randomly excluding jump size 1.

Proposition 2.2 ([11]) *For any circulant graph $C_p(1, 3, \cdots, \lfloor \frac{p}{2} \rfloor)$; $p = 4n-2$ or $4n$, $n \ge 2$ vertices,*

$$\alpha(C_p(1, 3, \cdots, \lfloor \frac{p}{2} \rfloor)) = \beta(C_p(1, 3, \cdots, \lfloor \frac{p}{2} \rfloor)) = \frac{p}{2}.$$

Theorem 2.3 *The collection of all (α,β)-sets of a Circulant graph $C_p(1,3,\cdots,\lfloor\frac{p}{2}\rfloor)$; $p=4n-2$ or $4n$, $n\geq 2$ vertices form a PBD with parameters $\nu=p$, $b=4$, $g_1=g_2=\frac{p}{2}$, $r=2$ and*

$$\lambda_m=\begin{cases}2, & m\equiv 0(mod\,2),\\ 0, & otherwise.\end{cases}$$

Proof For a given circulant graph $C_p(1,3,\cdots,\lfloor\frac{p}{2}\rfloor)$; $\nu_\alpha=\nu_\beta=p=4n-2$ or $4n$, $n\geq 2$ vertices labeled as $v_1,v_2,\cdots,v_p$. By Proposition 2.2, we have $\alpha(C_p(1,3,\cdots,\lfloor\frac{p}{2}\rfloor)=\beta(C_p(1,3,\cdots,\lfloor\frac{p}{2}\rfloor)=\dfrac{p}{2}$. Further, $C_p(1,3,\cdots,\lfloor\frac{p}{2}\rfloor)$; $p=4n-2$ or $4n$, $n\geq 2$ have two blocks of (α,β)-sets, it implies $b=4$. Also, we have $g_1=g_2=\dfrac{p}{2}$, where g_1 and g_2 are the number of elements contained exactly in their respective blocks. From the above facts, we have $r=2$. To obtain the m-associates for the elements, where $1\leq m\leq\lfloor\frac{p}{2}\rfloor$. Two distinct elements are odd associates if they have odd jump size and they are even associates $(2\leq k\leq\lfloor\frac{p}{2}\rfloor)$. Hence the parameters of first kind are given by $\nu=p$, $b=4$, $g_1=g_2=\dfrac{p}{2}$, $r=2$ and

$$\lambda_m=\begin{cases}2, & m\equiv 0(mod\,2),\\ 0, & \text{otherwise.}\end{cases}$$

This completes the proof. □

2.3 Circulant Graph with Even Jump Sizes

The jump size of circulant graph is $2,4,\cdots,\lfloor\frac{p}{2}\rfloor$ is a $C_p(2,4,\cdots,\lfloor\frac{p}{2}\rfloor)$ with $p\geq 4$ vertices. The circulant graphs $C_5(2)$, $C_6(2)$, $C_8(2,4)$, $C_{10}(2,4)$, $C_{12}(2,4,6)$ are some examples of strongly regular graphs.

Proposition 2.3 ([11]) *For any circulant graph $C_p(2,4,\cdots,\lfloor\frac{p}{2}\rfloor)$ with $p\geq 4$ vertices,*

$$\alpha(C_p(2,4,\cdots,\lfloor\frac{p}{2}\rfloor))=\begin{cases}p-3, & p=4n+3,\\ p-2, & otherwise.\end{cases}$$

and

$$\beta(C_p(2,4,\cdots,\lfloor\frac{p}{2}\rfloor))=\begin{cases}3, & p=4n+3,\\ 2, & otherwise.\end{cases}$$

Proof Since the circulant graph $C_p(2,4,\cdots,\lfloor\frac{p}{2}\rfloor)$ is a $(2n-1)$-regular for $p=4n$ or $2n$-regular for $p=4n+1$ or $p=4n+2$ or $p=4n+3$, $n\geq 1$ vertices, the result follows. □

Theorem 2.4 *The collection of all (α,β)-sets of a Circulant graph $C_p(2,4,\cdots,\lfloor\frac{p}{2}\rfloor)$; $p\geq 4$ vertices form a PBD with parameters:*

(*i*) *For $\nu=p=4n$ or $\nu=p=4n+1, n\geq 1$, $b=p(n+1)$, $g_1=p-2$, $g_2=2$, $r=\frac{3p-4}{2}$,*

$\lambda_m(1) = p-4$, and

$$\lambda_m(2) = \begin{cases} 0, & m \equiv 0 (mod\, 2), \\ 1, & otherwise. \end{cases}$$

(ii) For $\nu = p = 4n+2; n \geq 1$, $b = \frac{p^2}{2}$, $g_1 = p-2$, $g_2 = 2$, $r = \frac{3p-4}{2}$, $\lambda_m(1) = p-4$, and

$$\lambda_m(2) = \begin{cases} 0, & m \equiv 0 (mod\, 2), \\ 1, & otherwise, \end{cases}$$

where $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$.

(iii) For $\nu = p = 4n+3; n \geq 1$, $b = 2p$, $g_1 = p-2$, $g_2 = \left\lceil \frac{p}{2n+1} \right\rceil$, $r = \frac{2(n+1)(p-1)}{2n+1}$ and due to the variations in the values of $\lambda_m(1)$ there is no good relation between $\lambda_m(2)$.

Proof For a given circulant graph $C_p(2, 4, \cdots, \lfloor \frac{p}{2} \rfloor)$, $\nu = p = 4n$ or $4n+1$ or $4n+2$ or $4n+3; n \geq 1$, vertices labeled as $v_1, v_2, \cdots, v_p$. By Proposition 2.3, we have the following cases.

Case 1. The circulant graph $C_p(2, 4, \cdots, \lfloor \frac{p}{2} \rfloor)$ with $\nu = p = 4n$ or $4n+1$, $n \geq 1$ have p blocks of α-set and np blocks of β-sets. This implies that $\tau_\alpha(C_p(2, 4, \cdots, \lfloor \frac{p}{2} \rfloor)) = p$ and $\tau_\beta(C_p(2, 4, \cdots, \lfloor \frac{p}{2} \rfloor)) = np$. Therefore $b = p(n+1)$. By Proposition 2.3, we have $g_1 = \alpha(C_p(2, 4, \ldots, \lfloor \frac{p}{2} \rfloor)) = p-2$ and $g_2 = \beta(C_p(2, 4, \ldots, \lfloor \frac{p}{2} \rfloor)) = 2$, where g_1 and g_2 are the number of elements contained exactly in their respective blocks. From the above facts, we have $r = \frac{3p-4}{2}$. To obtain the m-associates for the elements, where $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$. The two distinct elements odd associates, if they have odd jump size and they are even associates $(2 \leq k \leq \lfloor \frac{p}{2} \rfloor)$. Hence the parameters of first kind are given by $\nu = p$, $b = p(n+1)$, $g_1 = p-2$, $g_2 = 2$, $r = \dfrac{3p-4}{2}$, $\lambda_m(1) = p-4$ and

$$\lambda_m(2) = \begin{cases} 0, & m \equiv 0 (mod\, 2), \\ 1, & \text{otherwise}, \end{cases}$$

where $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$.

Case 2. The circulant graph $C_p(2, 4, \cdots, \lfloor \frac{p}{2} \rfloor)$ with $\nu = p = 4n+2$; $n \geq 1$ have p blocks of α-set and $\frac{p^2}{4}$ blocks of β-sets, this implies $\tau_\alpha(C_p(2, 4, \cdots, \lfloor \frac{p}{2} \rfloor)) = \frac{p^2}{4}$ and $\tau_\beta(C_p(2, 4, \cdots, \lfloor \frac{p}{2} \rfloor)) = \frac{p^2}{4}$. Therefore $b = \frac{p^2}{2}$. By Proposition 2.3, we have $g_1 = \alpha(C_p(2, 4, \ldots, \lfloor \frac{p}{2} \rfloor)) = p-2$ and $g_2 = \beta(C_p(2, 4, \ldots, \lfloor \frac{p}{2} \rfloor)) = 2$, where g_1 and g_2 are the number of elements contained exactly in their respective blocks. From the above facts, we have $r = \frac{3p-4}{2}$. To obtain the m-associates for the elements, where $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$.

The two distinct elements are odd associates, if they have odd jump size and they are even associates $(2 \leq k \leq \lfloor \frac{p}{2} \rfloor)$. Hence, the parameters of first kind are given by $\nu = p$, $b = \frac{p^2}{2}$,

$g_1 = p-2$, $g_2 = 2$, $r = \dfrac{3p-4}{2}$, $\lambda_m(1) = p-4$ and

$$\lambda_m(2) = \begin{cases} 0, & m \equiv 0(mod\,2), \\ 1, & \text{otherwise.} \end{cases}$$

where $1 \le m \le \lfloor \frac{p}{2} \rfloor$.

Case 3. The circulant graph $C_p(2,4,\cdots,\lfloor\frac{p}{2}\rfloor)$ with $p = 4n+3$, $n \ge 1$ have p blocks of α-set and p blocks of β-sets. This implies that $\tau_\alpha(C_p(2,4,\cdots,\lfloor\frac{p}{2}\rfloor)) = \tau_\beta(C_p(2,4,\cdots,\lfloor\frac{p}{2}\rfloor)) = p$. Therefore $b = 2p$. By Proposition 2.3, we have $g_1 = \alpha(C_p(2,4,\ldots,\lfloor\frac{p}{2}\rfloor)) = p-2$ and $g_2 = \beta(C_p(2,4,\cdots,\lfloor\frac{p}{2}\rfloor)) = \left\lceil \dfrac{p}{2n+1} \right\rceil$, where g_1 and g_2 are the number of elements contained exactly in a block. From the above facts, we have $r = \dfrac{2(n+1)(p-1)}{2n+1}$. To obtain the m-associates for the elements, where $1 \le m \le \lfloor\frac{p}{2}\rfloor$. The two distinct elements odd associates, if they have odd jump size and they are even associates $(2 \le k \le \lfloor\frac{p}{2}\rfloor)$.

Hence, the parameters of first kind are given by $\nu = p$, $b = 2p$, $g_1 = p-2$, $g_2 = \left\lceil \dfrac{p}{2n+1} \right\rceil$, $r = \dfrac{2(n+1)(p-1)}{2n+1}$ and

$$\lambda_m(1) = \begin{cases} 1, & \text{for } m = 1, \\ 0, & \text{for } 2 \le m \le \lfloor\frac{p-1}{2}\rfloor, \\ 2, & \text{for } m = \lfloor\frac{p}{2}\rfloor. \end{cases}$$

Thus the result follows. □

2.4 **Circulant Graph** $C_p(1,2,\cdots,\lfloor\frac{p}{2}\rfloor)$

The jump size of circulant graph is $(1,2,\cdots,\lfloor\frac{p}{2}\rfloor)$, known as complete graph K_p with $p \ge 3$ that is, $C_p\left(1,2,\cdots,\lfloor\frac{p}{2}\rfloor\right) \cong K_p$. Further, the complete graph K_p is strongly regular for all $p \ge 3$. The status of the trivial singleton graph K_1 is unclear. Since the parameter μ is not well defined on K_2. It is difficult to analyse and conclude whether K_2 is a strongly regular graph or not.

Proposition 2.4 ([11]) *For any circulant graph $C_p(1,2,\cdots,\lfloor\frac{p}{2}\rfloor)$ with $p \ge 3$ vertices,*

$$\alpha\left(C_p\left(1,2,\ldots,\left\lfloor\frac{p}{2}\right\rfloor\right)\right) = p-1 \text{ and } \beta\left(C_p\left(1,2,\ldots,\left\lfloor\frac{p}{2}\right\rfloor\right)\right) = 1.$$

Theorem 2.5 *The collection of all (α,β)-sets of a circulant graph $C_p(1,2,\cdots,\lfloor\frac{p}{2}\rfloor)$, $p \ge 3$ vertices form a PBD with parameters $\nu = p$, $b = 2p$, $g_1 = p-1$, g_2, $r = p$ and $\lambda_m = p-1$.*

Proof For a given circulant graph $C_p(1,2,\cdots,\lfloor\frac{p}{2}\rfloor)$; $\nu = p$ vertices labeled as $v_1, v_2, \cdots, v_p$. By Proposition 2.4, we have $\alpha(C_p(1,2,\cdots,\lfloor\frac{p}{2}\rfloor) = p-1$, and $\beta(C_p(1,2,\cdots,\lfloor\frac{p}{2}\rfloor) = \frac{p}{2}$. Further, $C_p(1,2,\cdots,\lfloor\frac{p}{2}\rfloor)$; have p blocks of (α,β)-sets, it implies $b = 2p$. Also, we have $g_1 = p-1$,

$g_2 = 1$, where g_1 and g_2 is the number of elements contained exactly in a block. From the above facts, we have $r = p$. To obtain the m-associates for the elements, where $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$. Two distinct elements are odd associates, if they have odd jump size and they are even associates $(2 \leq k \leq \lfloor \frac{p}{2} \rfloor)$. Hence the parameters of first kind are given by $\nu = p$, $b = 2p$, $g_1 = p - 1$, $g_2 = 1$, $r = p$, $\lambda_m = p - 1$, $1 \leq m \leq \lfloor \frac{p}{2} \rfloor)$. Thus, the result follows. $\square$

§3. Conclusion

The construction and analysis of PBD's involve the combinatorial mathematics and statistical techniques, which have applications in various fields, including agriculture, biology, medicine, and social sciences. They are used in situations where it is important to compare treatments or conditions in a systematic and balanced way, while minimizing the number of required comparisons. This allows researchers to obtain reliable and statistically valid results with a reduced number of experimental runs or observations. Generally, the PBD's obtained from the families of strongly regular graphs. Interestingly, we determine the total number of (α, β)-sets with its association schemes in different jump sizes of some circulant graphs.

Finally, we pose the following open problem.

Problem 3.1 *Obtain the PBD's associated with (α, β)-sets of an integral circulant graph, a regular circulant graph or a Cayley graph.*

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Edge C_k Symmetric n-Sigraphs

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Abstract: An n-tuple $(a_1, a_2, \cdots, a_n)$ is *symmetric* if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, \cdots, a_n), a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. A *symmetric n-sigraph (symmetric n-marked graph)* is an ordered pair $S_n = (G, \sigma)$ $(S_n = (G, \mu))$, where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \to H_n$ $(\mu : V \to H_n)$ is a function. In this paper, we introduced a new notion edge C_k symmetric n-sigraph of a symmetric n-sigraph and its properties are obtained. Also, we obtained the structural characterization of edge C_k symmetric n-signed graphs.

Key Words: Symmetric n-sigraph, Smarandachely symmetric n-marked graph, symmetric n-marked graph, Smarandachely symmetric n-marked graph, balance, switching, edge C_k symmetric n-sigraph, complementation.

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§1. Introduction

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [1]. We consider only finite, simple graphs free from self-loops.

Let $n \geq 1$ be an integer. An *n-tuple* $(a_1, a_2, \cdots, a_n)$ is *symmetric* if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, \cdots, a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. Note that H_n is a group under coordinate wise multiplication, and the order of H_n is 2^m, where $m = \lceil \frac{n}{2} \rceil$.

A *symmetric n-sigraph (symmetric n-marked graph)* is an ordered pair $S_n = (G, \sigma)$ $(S_n = (G, \mu))$, where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \to H_n$ $(\mu : V \to H_n)$ is a function. Generally, a *Smarandachely symmetric n-sigraph* (*Smarandachely symmetric n-marked graph)* for a subgraph $H \prec G$ is such a graph that $G - E(H)$ is *symmetric n-sigraph (symmetric n-marked graph)*. For example, let H be an edge $e \in E(G)$, a path $P_s \succ G$

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for an integer $s \geq 2$ or a claw $K_{1,3} \prec G$. Certainly, if $H = \emptyset$, a Smarandachely symmetric n-sigraph (or Smarandachely symmetric n-sigraph) is nothing else but a symmetric n-sigraph (or symmetric n-marked graph).

In this paper by an *n-tuple/n-sigraph/n-marked graph* we always mean a symmetric n-tuple/symmetric n-sigraph/symmetric n-marked graph.

An n-tuple $(a_1, a_2, \cdots, a_n)$ is an *identity n-tuple* if $a_k = +$ for $1 \leq k \leq n$. Otherwise, it is a *non-identity n-tuple*. In an n-sigraph $S_n = (G, \sigma)$ an edge labelled with the identity n-tuple is called an *identity edge*, otherwise it is a *non-identity edge*. Further, in an n-sigraph $S_n = (G, \sigma)$, for any $A \subseteq E(G)$ the *n-tuple* $\sigma(A)$ is the product of the n-tuples on the edges of A.

In [9], the authors defined two notions of balance in n-sigraph $S_n = (G, \sigma)$ as follows (See also R. Rangarajan and P.S.K.Reddy [5]:

Definition 1.1 *Let $S_n = (G, \sigma)$ be an n-sigraph. Then,*

(i) *S_n is identity balanced (or i-balanced), if product of n-tuples on each cycle of S_n is the identity n-tuple, and*

(ii) *S_n is balanced, if every cycle in S_n contains an even number of non-identity edges.*

Note 1.1 An i-balanced n-sigraph need not be balanced and conversely.

The following characterization of i-balanced n-sigraphs is obtained in [9].

Theorem 1.1 (E. Sampathkumar et al. [9]) *An n-sigraph $S_n = (G, \sigma)$ is i-balanced if, and only if, it is possible to assign n-tuples to its vertices such that the n-tuple of each edge uv is equal to the product of the n-tuples of u and v.*

In [9], the authors also have defined switching and cycle isomorphism of an n-sigraph $S_n = (G, \sigma)$ (See also [2, 6C8, 11C20, 22]) as follows:

Let $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$, be two n-sigraphs. Then S_n and S'_n are said to be *isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that if uv is an edge in S_n with label $(a_1, a_2, \cdots, a_n)$ then $\phi(u)\phi(v)$ is an edge in S'_n with label $(a_1, a_2, \cdots, a_n)$.

Given an n-marking μ of an n-sigraph $S_n = (G, \sigma)$, *switching* S_n with respect to μ is the operation of changing the n-tuple of every edge uv of S_n by $\mu(u)\sigma(uv)\mu(v)$. The n-sigraph obtained in this way is denoted by $\mathcal{S}_\mu(S_n)$ and is called the *μ-switched n-sigraph* or just *switched n-sigraph*. Further, an n-sigraph S_n *switches* to n-sigraph S'_n (or that they are *switching equivalent* to each other), written as $S_n \sim S'_n$, whenever there exists an n-marking of S_n such that $\mathcal{S}_\mu(S_n) \cong S'_n$.

Two n-sigraphs $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ are said to be *cycle isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that the n-tuple $\sigma(C)$ of every cycle C in S_n equals to the n-tuple $\sigma(\phi(C))$ in S'_n.

We make use of the following known result (see [9]).

Theorem 1.2 (E. Sampathkumar et al. [9]) *Given a graph G, any two n-sigraphs with G as underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

Let $S_n = (G, \sigma)$ be an n-sigraph. Consider the n-marking μ on vertices of S defined as follows: each vertex $v \in V$, $\mu(v)$ is the product of the n-tuples on the edges incident at v. *Complement* of S is an n-sigraph $\overline{S_n} = (\overline{G}, \sigma')$, where for any edge $e = uv \in \overline{G}$, $\sigma'(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S_n}$ as defined here is an i-balanced n-sigraph due to Theorem 1.1.

§2. Edge C_k Symmetric n-Sigraph of an n-Sigraph

The *edge C_k graph* $E_k(G)$ of a graph G is defined in [4] as follows:

The edge C_k graph of a graph $G = (V, E)$ is a graph $E_k(G) = (V', E')$, with vertex set $V' = E(G)$ such that two vertices e and f are adjacent if, and only if, the corresponding edges in G either incident or opposite edges of some cycle C_k. In this paper, we extend the notion of $E_k(G)$ to realm of symmetric n-sigraphs: Given an n-sigraph $S_n = (G, \sigma)$ its *edge C_k n-sigraph* $E_k(S_n) = (E_k(G), \sigma')$ is that n-sigraph whose underlying graph is $E_k(G)$, the edge C_k graph of G, where for any edge e_1e_2 in $E_k(S_n)$, $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$. When $k = 3$, the definition coincides with triangular line n-sigraph of a graph [2], and when $k = 4$, the definition coincides with the edge E_4 n-sigraph of an n-sigraph [12].

Hence, we shall call a given n-sigraph an edge C_k n-sigraph if there exists an n-sigraph S'_n such that $S_n \cong E_k(S'_n)$. In the following subsection, we shall present a characterization of edge C_k n-sigraphs.

The following result indicates the limitations of the notion of edge C_k n-sigraphs as introduced above, since the entire class of i-unbalanced n-sigraphs is forbidden to be edge C_k n-sigraphs.

Theorem 2.1 *For any n-sigraph $S_n = (G, \sigma)$, its edge C_k n-sigraph $E_k(S_n)$ is i-balanced.*

Proof Since the n-tuple of any edge uv in $E_k(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n, by Theorem 1, $E_k(S_n)$ is i-balanced. □

When $k = 3$ and $k = 4$, we can deduce the following results.

Corollary 2.1 (Lokesha et al. [2]) *For any n-sigraph $S_n = (G, \sigma)$, its triangular line n-sigraph $\mathcal{T}(S_n)$ is i-balanced.*

Corollary 2.2 (P.S.K.Reddy et al. [12]) *For any n-sigraph $S_n = (G, \sigma)$, its edge C_4 n-sigraph $E_4(S_n)$ is i-balanced.*

For any positive integer i, the i^{th} iterated edge C_k n-sigraph, $E_k^i(S_n)$ of S_n is defined as follows:

$$E_k^0(S_n) = S_n,\ E_k^i(S_n) = E_k(E_k^{i-1}(S_n)).$$

Corollary 2.3 *For any n-sigraph $S_n = (G, \sigma)$ and any positive integer m, $E_k^m(S_n)$ is i-balanced.*

In [21], the authors obtained the characterizations for the edge C_k graph of a graph G is connected, complete, bipartite etc. The authors have also proved that the edge C_k graph has no forbidden subgraph characterization. The dynamical behavior such as convergence, periodicity, mortality and touching number of $E_k(G)$ are also discussed.

Recall that, the edge C_k graph coincides with the line graph for any acyclic graph. As a case, for a connected graph G, $E_k(G) = G$ if, and only if $G = C_n, n \neq k$ ([4]).

We now characterize n-sigraphs that are switching equivalent to their the edge C_k n-sigraphs.

Theorem 2.2 *For any n-sigraph $S_n = (G, \sigma)$, $S_n \sim E_k(S_n)$ if and only if $G \cong C_n$, where $n \geq 5$ and S_n is i-balanced.*

Proof Suppose $S_n \sim E_k(S_n)$. This implies, $G \cong E_k(G)$ and hence G is isomorphic to C_n, where $n \geq 5$, Theorem 3 implies that $E_k(S_n)$ is i-balanced and hence if S_n is i-unbalanced and its $E_k(S_n)$ being i-balanced can not be switching equivalent to S_n in accordance with Theorem 1.2. Therefore, S_n must be i-balanced.

Conversely, suppose that S_n is an i-balanced n-sigraph and its undrelying G is isomorphic to C_n, where $n \geq 5$. Then, since $E_k(S_n)$ is i-balanced as per Theorem 3 and since $G \cong E_k(G)$, the result follows from Theorem 1.2 again. □

In [21], we obtained the following result.

Theorem 2.3 (P.S.K.Reddy et al. [21]) *For a graph $G = (V, E)$, $E_k(G) \cong L(G)$ if, and only if G is C_k-free.*

In view of the above result, we have the following characterization.

Theorem 2.4 *For any n-sigraph $S_n = (G, \sigma)$, $E_k(S_n) \cong L(S_n)$ if, and only if G is C_k-free.*

For any $m \in H_n$, the *m-complement* of $a = (a_1, a_2, \cdots, a_n)$ is $a^m = am$. For any $M \subseteq H_n$, and $m \in H_n$, the *m-complement* of M is $M^m = \{a^m : a \in M\}$.

For any $m \in H_n$, the *m-complement* of an n-sigraph $S_n = (G, \sigma)$, written (S_n^m), is the same graph but with each edge label $a = (a_1, a_2, \cdots, a_n)$ replaced by a^m.

For an n-sigraph $S_n = (G, \sigma)$, the $R(S_n)$ is i-balanced. We now examine, the condition under which m-complement of $E_k(S_n)$ is i-balanced, where for any $m \in H_n$.

Theorem 2.5 *Let $S_n = (G, \sigma)$ be an n-sigraph. Then, for any $m \in H_n$, if $E_k(G)$ is bipartite then $(E_k(S_n))^m$ is i-balanced.*

Proof Since, by Theorem 2.1, $E_k(S_n)$ is i-balanced, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in $E_k(S_n)$ whose k^{th} co-ordinate are $-$ is even. Also, since $E_k(G)$ is bipartite, all cycles have even length; thus, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in $E_k(S_n)$ whose k^{th} co-ordinate are $+$ is also even. This implies that the same thing is true in any m-complement, where for any $m, \in H_n$. Hence $(E_k(S_n))^t$ is i-balanced. □

In [3], the authors proved that for a connected complete multipartite graph G, $E_k(G)$ is complete. The following result follows from the above observation and Theorem 2.1.

Theorem 2.6 *For a connected n-sigraph $S_n = (G, \sigma)$, $E_k(S_n)$ is complete i-balanced signed* *graph if, and only if G is complete multipartite graph.*

In [21], the authors proved that: For a connected graph $G = (V, E)$, $E_k(G)$ is bipartite if, and only if, G is either a path or an even cycle of length $r \neq k$. The following result follows from the above result and Theorem 2.1.

Theorem 2.7 *For a connected n-sigraph $S_n = (G, \sigma)$, $E_k(S_n)$ is bipartite i-balanced signed graph if, and only if G is isomorphic to either path or C_{2n}, where $n \geq 3$.*

§3. Characterization of Edge C_k Signed Graphs

The following result characterize n-sigraphs which are edge C_k n-sigraphs.

Theorem 3.1 *An n-sigraph $S_n = (G, \sigma)$ is an edge C_k n-sigraph if, and only if S_n is i-balanced n-sigraph and its underlying graph G is an edge C_k graph.*

Proof Suppose that S_n is i-balanced and G is an edge C_k graph. Then there exists a graph Γ' such that $E_k(G') \cong G$. Since S_n is i-balanced, by Theorem 1, there exists an n-marking ζ of G such that each edge uv in S_n satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. Now consider the n-sigraph $S'_n = (G', \sigma')$, where for any edge e in G', $\sigma'(e)$ is the marking of the corresponding vertex in G. Then clearly, $E_k(S'_n) \cong S_n$. Hence S_n is an edge C_k n-sigraph.

Conversely, suppose that $S_n = (G, \sigma)$ is an edge C_k n-sigraph. Then there exists an n-sigraph $S'_n = (G', \sigma')$ such that $E_k(S'_n) \cong S_n$. Hence G is the edge C_k graph of G' and by Theorem 3, S_n is i-balanced. □

If we take $k = 3$ and $k = 4$ in $E_k(S_n)$, then we can deduce the triangular line n-sigraph and edge C_4 n-sigraph respectively. In [2,12], the authors obtained structural characterizations of triangular line n-sigraphs and edge C_4 n-sigraphs and clearly Theorem 2.7 is the generalization of above said notions.

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Perfect Roman Domination of Some Cycle Related Graphs

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Abstract: In this paper, we continue the study of perfect Roman dominating functions in graphs. A perfect Roman dominating PRD function on a graph $G = (V, E)$ is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex v with $f(v) = 0$ is adjacent to exactly one vertex neighbor u with $f(u) = 2$. The weight of PRD function is the sum of its function values over all the vertices. The perfect Roman domination number of G denoted by $\gamma_R^P(G)$ is the minimum weight of a PRD function in G. We present the perfect Roman domination number of some cycle related graphs such as helm graphs, sunlet graphs and flower snark graphs. If G is a spider web graph, we show that $\gamma_R^P(G) \leq \frac{2}{3}|G|$.

Key Words: Perfect Roman domination, Smarandachely perfect Roman domination, domination number, web graph.

AMS(2010): 05C69, 05C76.

§1. Introduction and Preliminaries

Let $G = (V, E)$ be an undirected simple graph with vertex set V and edge set E. The graph G has order $n = |V|$. For every vertex $v \in V$, the open neighborhood of v is the set $\{u|uv \in E\}$ denoted by $N(v)$ and the closed neighborhood of v is the set $N(v) \cup \{v\}$ denoted as $N[v]$. The cardinality of $N(v)$ is the degree of vertex v denoted by $d_G(v) = |N(v)|$. A vertex of degree one is called a pendant vertex. We denote the cycle graph with n vertices by C_n. A *wheel* graph W_n is the join of $K_1 + C_n$. The vertex corresponding to K_1 in W_n is known as apex vertex and vertices of the cycle C_n in W_n are rim vertices. And a *helm* graph H_n is the graph obtained from a wheel graph W_n by attaching a pendant vertex to each rim vertex, a sunlet graph denoted by L_n is a graph that contains a cycle $C_{\frac{n}{2}}$ with pendant vertex attached to each vertex of the cycle $C_{\frac{n}{2}}$.

A *dominating set* of G is a set $V \setminus D$ such that every vertex not in D has a neighbor in D. The minimum cardinality of all dominating sets of G is the *domination number*, denoted as $\gamma(G)$. More on dominating sets can be found in [9].

A *Roman dominating function* of a graph G, denoted as a *RD-function* is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u with $f(u) = 0$ is adjacent to at least one vertex v with $f(v) = 2$. The weight of a vertex v is its value, $f(v)$, assigned to it

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under f. The weight, wt_f, of f is the sum, $\sum_{u\in V(G)} = f(u)$, of the weights of the vertices. The *Roman domination number*, denoted $\gamma_R(G)$ is the minimum weight of an RD-function in G; i.e. $\gamma_R(G) = min\{wt_f | f$ is an RD $-$ function in $G\}$. In [7], Roman domination was first studied, see the papers [2], [4], [5], [8], [12],[13], [14], [15] for more results on Roman domination number. In [6], upper bounds for Roman domination were considered.

A *perfect Roman dominating function* (or *PRD-function*) on G is a Roman dominating function $f(V) = \{0, 1, 2\}$ on G such that for each vertex u with $f(u) = 0$ there exists exactly one vertex v with $f(v) = 2$, for which $uv \in E(G)$ and contrarily, a *Smarandachely perfect Roman domination* is such a Roman dominating function $f(V) = \{0, 1, 2\}$ on G that each vertex u with $f(u) = 0$ has at least two vertices v with $f(v) = 2$ in its neighborhood $N_G(u)$. Generally, the *perfect Roman domination number* of G, denoted by $\gamma_R^P(G)$ is the minimum weight of a PRD-function on G. A PRD-function f with weight $wt_f(G) = \gamma_R^P(G)$ is called $\gamma_R^P(G)$-function of G. The perfect Roman domination is a variation of the Roman domination. It was introduced and first studied in 2018 by Henning et al. [11] and further studied in [10] for regular graphs. More recent studies on perfect Roman domination number can be found in [1], [16], [17], [3].

In this paper, we continue the study of perfect Roman domination number for some irregular graphs which are cycle related graphs, mainly Helm graphs, Sunlet graphs and spider web graphs. Also, we gave perfect Roman domination number of flower snark graph.

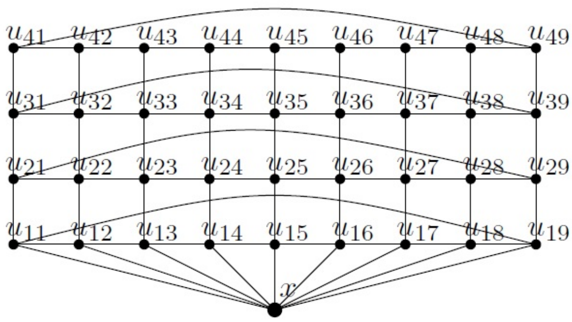

Figure 1. Web graph $W(4, 9)$

§2. Perfect Roman Domination Number of Cycle Related Graphs

In this section, we shall consider the perfect Roman dominating functions of Helm graph, sunlet graph and flower snark graph. We begin with the following definition.

Definition 2.1 (Flower Snark Graph) *For $n \geq 3$, take the union of n copies of $K_{1,3}$. Denote the vertex with degree 3 in the i-th copy of $K_{1,3}$ as x^i, $1 \leq i \leq n$ and the other three vertices in the i-th copy as w^i, y^i, z^i. Next, construct cycle C_n through vertices $w^1, w^2, \cdots, w^n$ and cycle C_{2n} through vertices $y^1, y^2, \cdots, y^n, z^1, z^2, \cdots, z^n$. Let the $i-th$ copy of $K_{1,3}$ be denoted by J^i and its vertices are x^i, w^i, y^i, z^i. Denote flower snark graph by J_n and note that J_n contains n copies of $K_{1,3}$.*

Next, we have the following results on the perfect Roman domination number of Helm graph, sunlet graph and flower snark graph.

Theorem 2.1 *Let H_n be a Helm graph forn ≥ 9. Then,* $\gamma_R^P(H_n) = \frac{n-1}{2} + 2$.

Proof Let $x, z_i, y_i, 1 \leq i \leq \frac{n-1}{2}$ denotes the center vertex, vertices of the cycle contained in H_n and pendant vertices respectively. Define a function $f : V(H_n) \to \{0, 1, 2\}$ as follow: $f(x) = 2$, $f(z_i) = 0$, $f(y_i) = 1$, where $1 \leq i \leq \frac{n-1}{2}$. It+ is clear to see from the labeling that f is a perfect Roman dominating function of H_n since each vertex with label 0 is adjacent to exactly one vertex with label 2. Hence, from the above labeling

$$\gamma_R^P(H_n) = \frac{n-1}{2} + 2.$$

Assume that $\gamma_R^P(H_n) < \frac{n-1}{2} + 2$. Then we split the problem into the following two cases.

Case 1. $f(x) < 2$.

In this case, $\sum_{i=1}^{\frac{n-1}{2}} f(z_i) > 2$ which implies that

$$wt_f(H_n) > \frac{n-1}{2} + 2,$$

which is a contradiction.

Case 2. $\sum_{i=1}^{\frac{n-1}{2}} f(y_i) < \frac{n-1}{2}$.

In this case, $f(y_t) = 0$ for some $1 \leq t \leq \frac{n-1}{2}$. If $f(y_t) = 0$, this implies that $1 \leq f(z_t) \leq 2$, which will give the condition in Case 1, that is

$$wt_f(H_n) > \frac{n-1}{2} + 2$$

a contradiction. Hence $\gamma_R^P(H_n) = \frac{n-1}{2} + 2$. □

Theorem 2.2 *Let L_n be a sunlet graph for $n \geq 6$. Then,*

$$\gamma_R^P(L_n) = \begin{cases} \frac{2}{3}n, & \text{if } n \equiv 0 \bmod 3 \\ \frac{2}{3}(n+2), & \text{if } n \equiv 1 \bmod 3 \\ \frac{2}{3}(n+1), & \text{if } n \equiv 2 \bmod 3. \end{cases}$$

Proof Notice that a sunlet graph L_n consist of cycle $C_t, t = \frac{n}{2}$ with pendant vertex attached to each vertex of the cycle C_t. Let $x_1, x_2, \cdots, x_t$ be the vertices of the cycle in L_n and $y_1, y_2, \cdots, y_t$ be the pendant vertices in L_n such that $x_i y_i \in E(L_n)$ for $1 \leq i \leq t$. Next, we describe the construction of perfect Roman domination of L_n. For any $t = 3q + r$, where $q \geq 1$ and $0 \leq r \leq 2$, partition $V(C_t) = \{x_1, x_2, \cdots, x_t\}$ into q sets of cardinality 3. Define a function

$f : V(L_n) \to \{0, 1, 2\}$ as follows: Assign labeling 2 to a single vertex and 0 to the remaining vertices in each q set. such that vertex with label 2 in q_i is not adjacent to any vertices in q_{i+1}. Furthermore, assign 1 to r vertices. Assign 0 to q pendant vertices y_i such that for every $x_iy_i \in E(L_n)$, $f(x_i) = 2$. Lastly, assign label 1 to the remaining pendant vertices. From the above labeling, the function f is a perfect Roman dominating function of L_n since each vertex with label 0 is adjacent to exactly one vertex with label 2. Thus we have

$$\begin{aligned} wt_f(L_n) &= \sum_{i=1}^{t} f(x_i) + \sum_{i=1}^{t} f(y_i) \\ &= 2q + r + t - q = 2q + r + 3q + r - q = 4q + 2r. \end{aligned} \tag{1}$$

Now, we consider the problem in the following three cases.

Case 1. $n \equiv 0 \mod 3$ i.e. $r = 0$ and $t = 3q$.

By equation (1),

$$wt_f(L_n) = 4q = \frac{4}{3}.t = \frac{2}{3}n \quad \text{because of} \quad t = \frac{n}{2}.$$

Case 2. $n \equiv 1 \mod 3$ i.e. $r = 2$ and $t = 3q + 2$.

By equation (1),

$$\begin{aligned} wt_f(L_n) &= 4q + 4 = \frac{4}{3} \cdot t + 4 \\ &= \frac{2}{3}n + \frac{4}{3} \quad (\text{since } t = \frac{n}{2}) \\ &= \frac{2}{3}(n + 2). \end{aligned}$$

Case 3. $n \equiv 2 \mod 3$, i.e. $r = 1$ and $t = 3q + 1$.

By equation (1),

$$\begin{aligned} wt_f(L_n) &= 4q + 2 = \frac{4}{3} \cdot t + 2 \\ &= \frac{2}{3}n + \frac{2}{3} \quad (\text{since } t = \frac{n}{2}) \\ &= \frac{2}{3}(n + 1). \end{aligned}$$

Hence, we get the result. □

Theorem 2.3 *Let J_n be a flower snark graph for $n \geq 3$. Then, $\gamma_R^P(J_n) = 2n$.*

Proof Define a function $f : V(J_n) \to \{0, 1, 2\}$ as follows:

$$f(x^i) = 2, f(w^i) = f(y^i) = f(z^i) = 0, \ 1 \leq i \leq n.$$

The function f gives perfect Roman dominating function since each vertex with label 0 is adjacent to only one vertex with label 2. Thus we have $wt_f(J_n) = 2n$.

Assume that $wt_f(J_n) < 2n$, then we have that $f(x^t) < 2$ for some $t < n$. If $f(x^t) < 2$, then $f(w^t \neq 0$, $f(y^t) \neq 0$, $f(z^t) \neq 0$. This implies that the copy J^t will have weight greater than 2. Assume that the statement is true for J_{n-1}, that is, $\gamma_R^P(J_{n-1}) = 2(n-1)$. Then we have

$$\begin{aligned} wt_f(J_n) &= wt_f(J^i) + wt_f(J^t), \quad i = 1, 2, \cdots, n-1 \\ &= 2(n-1) + wt_f(J^t) > 2n. \end{aligned}$$

since $wt_f(J^t) > 2$, a contradiction. Hence, $\gamma_R^P(J_n) = 2n$. □

§3. Perfect Roman Domination Number of Spider Web Graph

The following result present the upper bound for the perfect Roman domination number of a spider web graph. We begin with the following definition.

Definition 3.1 (Web graph) *Let $p, q \geq 5$, the spider web graph $W(p,q)$ is constructed from p cycles of length q and a vertex x, as shown in Figure* 1. *Let $\{u_1, u_2, \cdots, u_q\}$ be the vertices in each cycle and let x denote the center vertex. The spider web graph has p rows and q columns, where all the vertices on the row $p-th$ row are adjacent to a common vertex x. Denote the vertex in row i and column j by $u_{ij}, 1 \leq i \leq p$ and $1 \leq j \leq q$. The vertex set of*

$$V(W(p,q)) = \{x, u_{11}, u_{12}, \cdots, u_{1q}, u_{21}, u_{22}, \cdots, u_{2q}, u_{31}, u_{32}, \cdots, u_{3q}, \cdots, u_{p1}, u_{p2}, \cdots, u_{pq}\}.$$

To define the edge set of this graph, let

$$\begin{aligned} A_i &= \{u_{i1} - u_{i2}, u_{i2} - u_{i3}, \cdots, u_{i(q-1)} - u_{iq}, u_{iq} - u_{i1}\} \\ B_j &= \{x - u_{pj}, u_{pj} - u_{p-1j}, \cdots, u_{(2)j} - u_{1j}\}. \end{aligned}$$

Then, $E(W(p,q)) = \bigcup_{i,j}(A_i \cup B_j)$. A web graph $W(p,q)$ has $pq+1$ vertices.

The next result gives the upper bound of the perfect Roman domination number of spider web graph.

Theorem 3.1 *Let $p, q \geq 5$. If $G = W(p,q)$ then, $\gamma_R^P(G) \leq \frac{2}{3}|G|$.*

Proof Note that the web graph $W(p,q)$ contains p rows, q columns and a common vertex, i.e. $W(p,q)$ contains $pq+1$ vertices. The problem will be split into the following 3 cases.

Case 1. $q \equiv 0 \; mod \; 3$ or $p \equiv 0 \; mod \; 3$.

If $q = 3k$ for some k, for $i = 1, 2, 3, \cdots, p-1$, label each vertex in the column $2 + 3t$ for $t \in \{0, 1, \cdots, k\}$ and the common vertex x with 2. When $i = p$, label the vertices on the column $2 + 3t$ for $t \in \{0, 1, \cdots, k\}$ with 1 and the remaining vertices with 0, such as those shown in Figure 2.

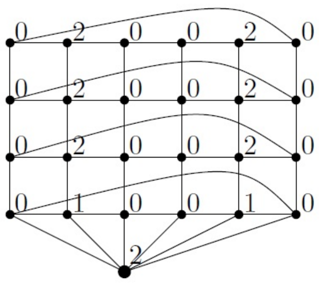

Figure 2. Web graph $W(4,6)$

The labelling holds for all p. It is easy to see that the labeling in Figure 2 gives a perfect Roman domination since each vertex label with 0 is adjacent to exactly one vertex with the label 2.

Let assume that $q = 3k$ and let f be the perfect Roman domination functing on G. If $j = 2 + 3t$, $t \in \{0, 1, \cdots, k-1\}$,

$$\begin{aligned} \sum_{i=p} &= 2(p-1)(\frac{q}{3}) \\ f(x) &= 2. \end{aligned}$$

Thus,

$$\begin{aligned} wt_f(G) &= 2(p-1)(\frac{q}{3}) + \frac{q}{3} + 2 = \frac{2pq}{3} + 2 - \frac{q}{3} \\ &\leq \frac{2pq}{3} + \frac{2}{3} \qquad \text{(since } q > 5) \\ &= \frac{2}{3}(pq+1) = \frac{2}{3}|G| \end{aligned}$$

We need the following functions for the remaining cases.

Define a function $f : V(G) \to \{0, 1, 2\}$ as follows

$$f(u_{i,j}) = \begin{cases} 2, & \text{if } i \equiv 0 \text{mod } 3, i \neq p \text{ and } j \equiv 3 \text{mod } 6 \\ 2, & \text{if } i \equiv 1 \text{mod } 3, i \neq p \text{ and } j \equiv 1 \text{mod } 6 \\ 2, & \text{if } i \equiv 2 \text{mod } 3, i \neq p \text{ and } j \equiv 5 \text{mod } 6 \\ 1, & \text{if } i \equiv 0 \text{mod } 3, i \neq p \text{ and } j \equiv 0 \text{mod } 6 \\ 1, & \text{if } i \equiv 1 \text{mod } 3, i \neq p \text{ and } j \equiv 4 \text{mod } 6 \\ 1, & \text{if } i \equiv 2 \text{mod } 3, i \neq p \text{ and } j \equiv 2 \text{mod } 6 \\ 0, & \text{otherwise.} \end{cases}$$

and $f(x) = 2$. The function f reoccur at every six columns and at every three rows. The labeling above gives a perfect Roman dominating function f since each vertex u_{ij} with label zero is adjacent to only one vertex with label 2.

Case 2. $q \equiv 1 \ mod \ 3$.

Next, we divide the proof into the following three subcases.

Subcase 2.1 $p \equiv 0 \ \ mod \ \ 3$ and $q \equiv 1 \text{mod } 6$.

In this subcase, define the function $f^* : V(G) \to \{0, 1, 2\}$ as follows:

$$f^*(u_{i,j}) = \begin{cases} 1, & \text{if } i = 1, \text{ and } j \equiv 3 \text{mod } 6 \\ 1, & \text{if } i = p-1, \text{ and } j \not\equiv 1 \text{mod } 6 \\ f(u_{ij}), & \text{otherwise.} \end{cases}$$

. Then, the function f^* is a PRD on G, see Figure 3.

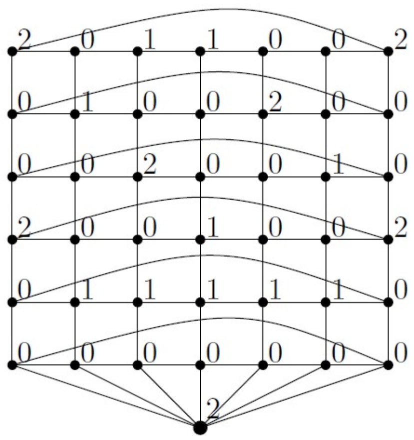

Figure 3. Web graph $W(p, q)$, $p \equiv 0 \text{mod } 6$ and $q \equiv 1 \text{mod } 6$

Notice that

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-3)}{3} + 2 \text{ if } j \equiv 1 \text{mod } 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-3)}{3} + 1 \text{ if } j \equiv 2 \text{mod } 6,$$
$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-3)}{3} + 2 \text{ if } j \equiv 3 \text{mod } 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-3)}{3} + 2 \text{if } j \equiv 4 \text{mod } 6,$$
$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-3)}{3} + 1 \text{ if } j \equiv 5 \text{mod } 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-3)}{3} + 1 \text{ if} j \equiv 6 \text{mod } 6.$$

Then, we have

$$\begin{aligned} wt_f(G) &= \left(9\left(\frac{p-3}{3}\right) + 9\right)\left(\frac{q-1}{6}\right) + 2\left(\frac{p-3}{3}\right) + 2 + 2 \\ &= \frac{1}{2}pq + \frac{1}{6}p + 2 \leq \frac{1}{2}pq + \frac{1}{6}pq + \frac{2}{3} \\ &= \frac{2}{3}pq + \frac{2}{3} = \frac{2}{3}(pq+1) = \frac{2}{3}|G| \end{aligned}$$

Subcase 2.2 $p = 3y + 1$, for some integer y.

In this subcase, if $q \equiv 1 \text{mod } 6$ define a function $f^* : V(G) \to \{0, 1, 2\}$ as follows:

$$f^*(u_{i,j}) = \begin{cases} 1, & \text{if } i = 1, \text{ and } j \equiv 3 \text{mod } 6 \\ 1, & \text{if } i = p-2 \text{ and } j \equiv 3 \text{mod } 6 \\ 1, & \text{if } i = p-1, \text{ and } j \not\equiv 5 \text{mod } 6 \\ 1, & \text{if } i = p-2, \text{ and } j = 3 \text{mod } 6 \\ f(u_{ij}), & \text{otherwise.} \end{cases}$$

The function f^* is a PRD on G, see Figure 4. Notice that

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-4)}{3} + 3 \text{ if } j \equiv 1 \text{mod } 6, \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-4)}{3} + 2 \text{ if } j \equiv 2 \text{mod } 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-4)}{3} + 3 \text{ if } j \equiv 3 \text{mod } 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-4)}{3} + 2 \text{ if } j \equiv 4 \text{mod } 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-4)}{3} + 2 \text{ if } j \equiv 5 \text{mod } 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-4)}{3} + 1 \text{ if } j \equiv 6 \text{mod } 6.$$

Then, we have

$$\begin{aligned} wt_f(G) &= \left(9\left(\frac{p-4}{3}\right) + 13\right)\left(\frac{q-1}{6}\right) + 2\left(\frac{p-4}{3}\right) + 3 + 2 \\ &= \frac{1}{2}pq + \frac{1}{6}p + \frac{1}{6}q + \frac{13}{6} \le \frac{1}{2}pq + \frac{1}{6}pq + \frac{2}{3} \\ &= \frac{2}{3}pq + \frac{2}{3} = \frac{2}{3}(pq+1) = \frac{2}{3}|G| \end{aligned}$$

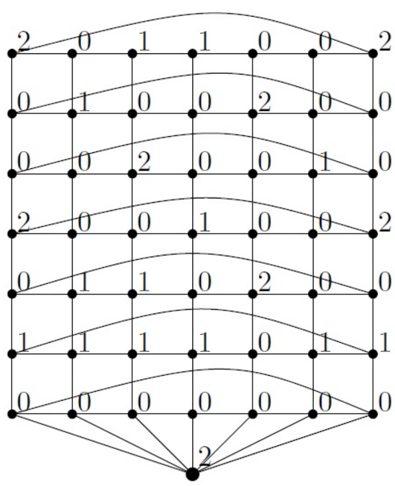

Figure 4. Web graph $W(p, q)$, $p \equiv 1 \text{mod } 6$ and $q \equiv 1 \text{mod } 6$

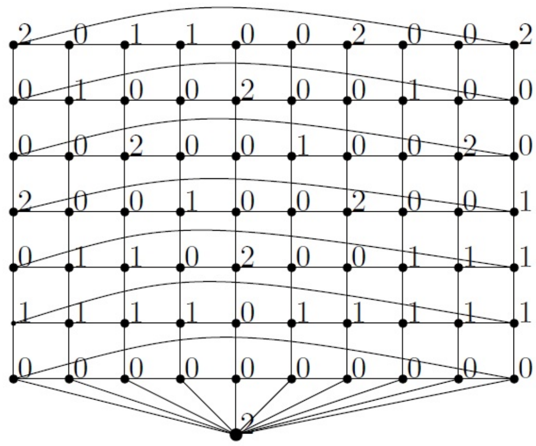

Figure 5. Web graph $W(p,q)$, $p \equiv 1 \bmod 6$ and $q \equiv 4 \bmod 6$

If $q \equiv 4 \bmod 6$, define a function $f^* : V(G) \to \{0,1,2\}$ such that

$$f^*(u_{i,j}) = \begin{cases} 1, & \text{if } i = p-1, \text{ and } j \not\equiv 5 \bmod 6 \\ 2, & \text{if } i = 1 \text{ and } j = q \\ 1, & \text{if } i = 1, \text{ and } j \equiv 3 \bmod 6 \text{ and } j \neq q-1 \\ 1, & \text{if } i = p-2, \text{ and } j \equiv 3 \bmod 6 \\ 1, & \text{if } i \equiv 2 \bmod 3, i \neq 2, j = q \\ f(u_{ij}), & \text{otherwise.} \end{cases}$$

Therefore, f^* is a PRD function on G, see Figure 5. Notice that

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-4)}{3} + 3 \text{ if } j \equiv 1 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-4)}{3} + 2 \text{ if } j \equiv 2 \bmod 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-4)}{3} + 3 \text{ if } j \equiv 3 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-4)}{3} + 2 \text{ if } j \equiv 4 \bmod 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-4)}{3} + 2 \text{ if } j \equiv 5 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-4)}{3} + 1 \text{ if } j \equiv 6 \bmod 6.$$

Now, we have

$$\begin{aligned} wt_f(G) &= \left(9\left(\frac{p-4}{3}\right) + 13\right)\left(\frac{q-4}{6}\right) + 7\left(\frac{p-4}{3}\right) + 10 + 2 \\ &= \frac{1}{2}pq + \frac{1}{3}p + \frac{1}{6}q + 2 \leq \frac{1}{2}pq + \frac{1}{6}pq + \frac{2}{3}, \quad (p, q \geq 7) \\ &= \frac{2}{3}pq + \frac{2}{3} = \frac{2}{3}(pq+1) = \frac{2}{3}|G|. \end{aligned}$$

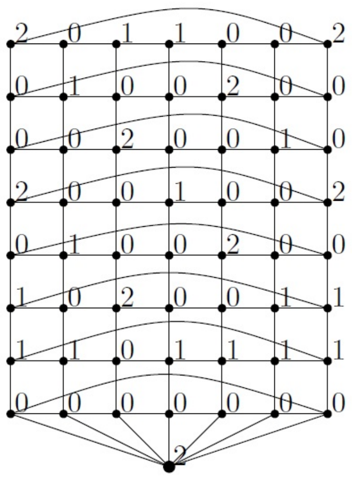

Figure 6. Web graph $W(p,q)$ with function f^*, $q = 1\text{mod } 6$ and $p = 3y+2$

Subcase 2.3 $p = 3y+2$ for some integer y.

Let $q \equiv 1\text{mod } 6$. Define the function $f^* : V(G) \to \{0,1,2\}$ as follows:

$$f^*(u_{i,j}) = \begin{cases} 1, & \text{if } i=1, \text{ and } j \equiv 3\text{mod } 6 \\ 1, & \text{if } i=p-2 \text{ and } j \equiv 1\text{mod } 6 \\ 1, & \text{if } i=p-1, \text{ and } j \not\equiv 3\text{mod } 6 \\ f(u_{ij}), & \text{otherwise.} \end{cases}$$

Therefore, f^* is a PRD function on G, see Figure 6. Notice that

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-5)}{3} + 4 \text{ if } j \equiv 1\text{mod } 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-5)}{3} + 2 \text{ if } j \equiv 2\text{mod } 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-5)}{3} + 3 \text{ if } j \equiv 3\text{mod } 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-5)}{3} + 2 \text{ if } j \equiv 4\text{mod } 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-5)}{3} + 3 \text{ if } j \equiv 5\text{mod } 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-5)}{3} + 2 \text{ if } j \equiv 6\text{mod } 6.$$

Now, we have

$$\begin{aligned} wt_f(G) &= \left(9\left(\frac{p-5}{3}\right) + 16\right)\left(\frac{q-1}{6}\right) + 2\left(\frac{p-5}{3}\right) + 4 + 2 \\ &= \frac{1}{2}pq + \frac{1}{6}p + \frac{1}{6}q + \frac{15}{6} \le \frac{1}{2}pq + \frac{1}{6}pq + \frac{2}{3}, \quad (p \ge 8, q \ge 7) \\ &= \frac{2}{3}pq + \frac{2}{3} = \frac{2}{3}(pq+1) = \frac{2}{3}|G| \end{aligned}$$

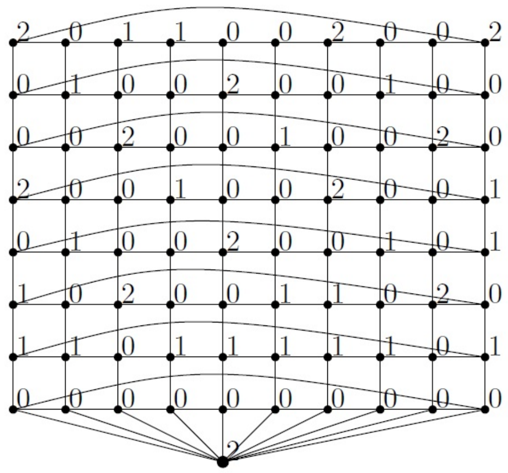

Figure 7. Web graph $W(p,q)$ with function f^*, $q = 4\text{mod } 6$ and $p = 3y+2$

If $q \equiv 4\text{mod } 6$, define a function $f^* : V(G) \to \{0,1,2\}$ as follows:

$$f^*(u_{i,j}) = \begin{cases} 1, & \text{if } i = 1, \text{ and } j \equiv 3\text{mod } 6, j \neq q-1 \\ 1, & \text{if } i = p-2 \text{ and } j \equiv 1\text{mod } 6 \\ 1, & \text{if } i = p-1, \text{ and } j \not\equiv 3\text{mod } 6 \\ 0, & \text{if } i = p-1, \text{ and } j \equiv 3\text{mod } 6 \\ 2, & \text{if } i = 1, \text{ and } j = q \\ 1, & \text{if } i \equiv 2\text{mod } 3, j = q, \\ f(u_{ij}), & \text{otherwise.} \end{cases}$$

Then, f^* is a PRD function on G, see Figure 7. Notice that

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-5)}{3} + 3 \text{ if } j \equiv 1\text{mod } 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-5)}{3} + 2 \text{ if } j \equiv 2\text{mod } 6,$$
$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-5)}{3} + 3 \text{ if } j \equiv 3\text{mod } 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-5)}{3} + 2 \text{ if } j \equiv 4\text{mod } 6,$$
$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-5)}{3} + 3 \text{ if } j \equiv 5\text{mod } 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-5)}{3} + 2 \text{ if } j \equiv 6\text{mod } 6.$$

Now, we have

$$\begin{aligned} wt_f(G) &= \left(9\left(\frac{p-5}{3}\right) + 16\right)\left(\frac{q-4}{6}\right) + 7\left(\frac{p-5}{3}\right) + 11 + 2 \\ &= \frac{1}{2}pq + \frac{1}{3}p + \frac{1}{6}q + \frac{2}{3} \leq \frac{1}{2}pq + \frac{1}{6}pq + \frac{2}{3}, \quad (p,q \geq 8) \\ &= \frac{2}{3}pq + \frac{2}{3} = \frac{2}{3}(pq+1) = \frac{2}{3}|G| \end{aligned}$$

Case 3. $q \equiv 2 \ mod \ 3$.

Subcase 3.1 If $p = 3y$, assume that $q \equiv 2 \text{mod } 6$. Then define a function $f^* : V(G) \to \{0, 1, 2\}$ as follows:

$$f^*(u_{i,j}) = \begin{cases} 1, & \text{if } i = 1, \text{ and } j \equiv 3 \text{mod } 6 \\ 1, & \text{if } i \equiv 1 \text{mod } 3, \text{ and } j = r \\ 1, & \text{if } i \equiv 0 \text{mod } 3, \text{ and } j = r \\ 1, & \text{if } i = p-1, \text{ and } j \not\equiv 1 \text{mod } 6 \\ f(u_{ij}), & \text{otherwise.} \end{cases}$$

Then, f^* is a PRD function on G, see Figure 8.

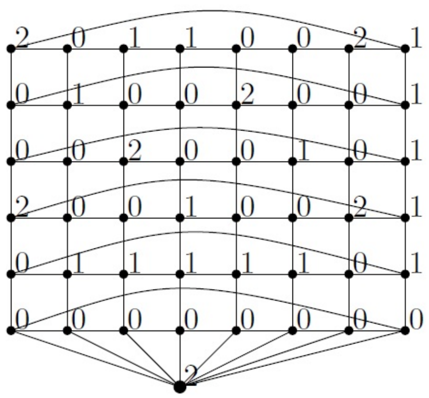

Figure 8. Web graph $W(p, q)$ with function f^*, $q \equiv 2 \text{mod } 6$ and $p = 3y$

Notice that

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-3)}{3} + 2 \text{ if } j \equiv 1 \text{mod } 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-3)}{3} + 1 \text{ if } j \equiv 2 \text{mod } 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-3)}{3} + 2 \text{ if } j \equiv 3 \text{mod } 6; \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-3)}{3} + 2 \text{ if } j \equiv 4 \text{mod } 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-3)}{3} + 1 \text{ if } j \equiv 5 \text{mod } 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-3)}{3} + 2 \text{ if } j \equiv 6 \text{mod } 6.$$

Thus,

$$\begin{aligned} wt_f(G) &= \left(9\left(\frac{p-3}{3}\right) + 9\right)\left(\frac{q-2}{6}\right) + 3\left(\frac{p-3}{3}\right) + 4 + 2 \\ &= \frac{1}{2}pq + 5 \leq \frac{1}{2}pq + \frac{1}{6}pq + \frac{2}{3}, \quad (p \geq 6, q \geq 8) \\ &= \frac{2}{3}pq + \frac{2}{3} = \frac{2}{3}(pq+1) = \frac{2}{3}|G| \end{aligned}$$

Subcase 3.2 If $p = 3y + 1$, assume that $q \equiv 2 \bmod 6$. Then define a function $f^* : V(G) \to \{0, 1, 2\}$ as follows:

$$f^*(u_{i,j}) = \begin{cases} 1, & \text{if } i = 1, \text{ and } j \equiv 3 \bmod 6 \\ 1, & \text{if } i \equiv 1 \bmod 3, \text{ and } j = r \\ 1, & \text{if } i \equiv 0 \bmod 3, \text{ and } j = r \\ 1, & \text{if } i = p - 2, \text{ and } j \equiv 3 \bmod 6 \\ 1, & \text{if } i = p - 1, \text{ and } j \not\equiv 5 \bmod 6 \\ f(u_{ij}), & \text{otherwise.} \end{cases}$$

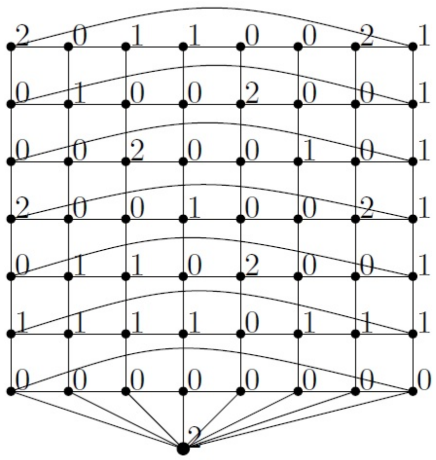

Figure 9. Web graph $W(p, q)$ with function f^*, $q \equiv 2 \bmod 6$ and $p = 3y + 1$

Then, f^* is a PRD function on G, see Figure 9. Notice that

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-4)}{3} + 3 \text{ if } j \equiv 1 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-4)}{3} + 2 \text{ if } j \equiv 2 \bmod 6,$$
$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-4)}{3} + 3 \text{ if } j \equiv 3 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-4)}{3} + 2 \text{ if } j \equiv 4 \bmod 6,$$
$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-4)}{3} + 2 \text{ if } j \equiv 5 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-4)}{3} + 1 \text{ if } j \equiv 6 \bmod 6.$$

Thus,

$$\begin{aligned} wt_f(G) &= \left(9\left(\frac{p-4}{3}\right) + 13\right)\left(\frac{q-2}{6}\right) + 4\left(\frac{p-4}{3}\right) + 7 + 2 \\ &= \frac{1}{2}pq + \frac{1}{6}p + \frac{1}{6}q + \frac{10}{3} \le \frac{1}{2}pq + \frac{1}{6}pq + \frac{2}{3}, \quad (p \ge 7, q \ge 8) \\ &= \frac{2}{3}pq + \frac{2}{3} = \frac{2}{3}(pq + 1) = \frac{2}{3}|G| \end{aligned}$$

Assume that $q \equiv 5 \bmod 6$ and $p = 3y + 1$. Define a function $f^* : V(G) \to \{0, 1, 2\}$ as follows:

$$f^*(u_{i,j}) = \begin{cases} 1, & \text{if } i = 1, \text{ and } j \equiv 3 \bmod 6 \\ 1, & \text{if } i \equiv 2 \bmod 3, \text{ and } j = q-1, q \\ 1, & \text{if } i \equiv 0 \bmod 3 \text{ and } j = q \\ 1, & \text{if } i = p-1, \text{ and } j \not\equiv 5 \bmod 6 \text{ and } j \neq q \\ 1, & \text{if } i = p-2, \text{ and } j \equiv 3 \bmod 6 \\ 1, & \text{if } i = p-1, j = q \\ f(u_{ij}), & \text{otherwise.} \end{cases}$$

Now, f^* is a PRD function on G, see Figure 10.

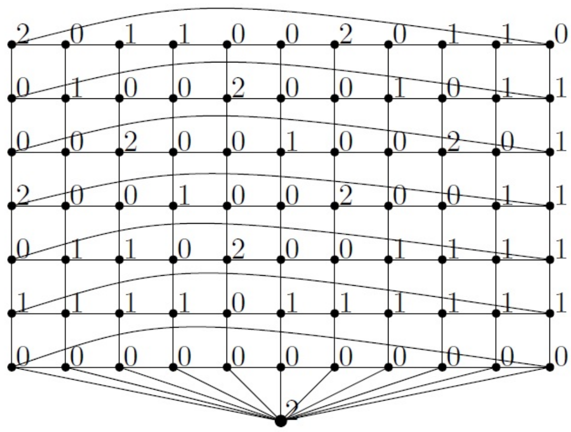

Figure 10. Web graph $W(p, q)$ with function f^*, $q \equiv 5 \bmod 6$ and $p = 3y + 1$

Notice that

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-4)}{3} + 3 \text{ if } j \equiv 1 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-4)}{3} + 2 \text{ if } j \equiv 2 \bmod 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-4)}{3} + 3 \text{ if } j \equiv 3 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-4)}{3} + 2 \text{ if } j \equiv 4 \bmod 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-4)}{3} + 2 \text{ if } j \equiv 5 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-4)}{3} + 1 \text{ if } j \equiv 6 \bmod 6.$$

Thus

$$\begin{aligned} wt_f(G) &= \left(9\left(\frac{p-4}{3}\right) + 16\right)\left(\frac{q-5}{6}\right) + 9\left(\frac{p-4}{3}\right) + 13 + 2 \\ &= \frac{1}{2}pq + \frac{1}{2}p + \frac{2}{3}q - \frac{1}{3} \leq \frac{1}{2}pq + \frac{1}{6}pq + \frac{2}{3}, \quad (p \geq 7, q \geq 11) \\ &= \frac{2}{3}pq + \frac{2}{3} = \frac{2}{3}(pq+1) = \frac{2}{3}|G| \end{aligned}$$

Subcase 3.3 $q = 3k + 2$ and $p = 3y + 2$ for some positive integer y.

In this case, assume that $q \equiv 2 \bmod 6$. Define a function $f^* : V(G) \to \{0, 1, 2\}$ as follows:

$$f^*(u_{i,j}) = \begin{cases} 1, & \text{if } i = 1, \text{ and } j \equiv 3 \bmod 6 \\ 1, & \text{if } i \equiv 1 \bmod 3, \text{ and } j = q \\ 1, & \text{if } i \equiv 0 \bmod 3 \text{ and } j = q \\ 1, & \text{if } i = p-2, \text{ and } j \equiv 1 \bmod 6 \\ 1, & \text{if } i = p-1, j \not\equiv 3 \bmod 6 \\ f(u_{ij}), & \text{otherwise.} \end{cases}$$

Now, f^* is a PRD function on G, see Figure 11.

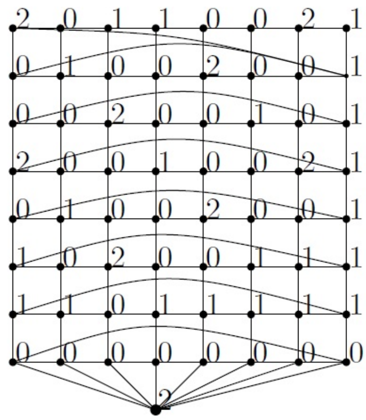

Figure 11. Web graph $W(p, q)$ with function f^*, $q \equiv 2 \bmod 6$ and $p = 3y + 2$

Notice that

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-5)}{3} + 4 \text{ if } j \equiv 1 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-5)}{3} + 2 \text{ if } j \equiv 2 \bmod 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-5)}{3} + 3 \text{ if } j \equiv 3 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-5)}{3} + 2 \text{ if } j \equiv 4 \bmod 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-5)}{3} + 2 \text{ if } j \equiv 5 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-5)}{3} + 2 \text{ if } j \equiv 6 \bmod 6.$$

Thus,

$$\begin{aligned} wt_f(G) &= \left(9\left(\frac{p-5}{3}\right) + 16\right)\left(\frac{q-2}{6}\right) + 5\left(\frac{p-5}{3}\right) + 8 + 2 \\ &= \frac{1}{2}pq + \frac{2}{3}p + \frac{1}{6}q + \frac{4}{3} \leq \frac{1}{2}pq + \frac{1}{6}pq + \frac{2}{3}, \quad (p, q \geq 8) \\ &= \frac{2}{3}pq + \frac{2}{3} = \frac{2}{3}(pq+1) = \frac{2}{3}|G| \end{aligned}$$

Assume that $q \equiv 5 \bmod 6$ and $p = 3y + 2$. We define a function $f^* : V(G) \to \{0, 1, 2\}$ as follows:

$$f^*(u_{i,j}) = \begin{cases} 1, & \text{if } i = 1, \text{ and } j \equiv 3 \bmod 6 \\ 1, & \text{if } i \equiv 2 \bmod 3, \text{ and } j = q-1, q \\ 1, & \text{if } i \equiv 0 \bmod 3 \text{ and } j = q \\ 1, & \text{if } i = p-2, \text{ and } j \equiv 1 \bmod 6 \\ 1, & \text{if } i = p-1, j \not\equiv 3 \bmod 6 \\ f(u_{ij}), & \text{otherwise.} \end{cases}$$

Therefore, f^* is a PRD function on G, see Figure 12.

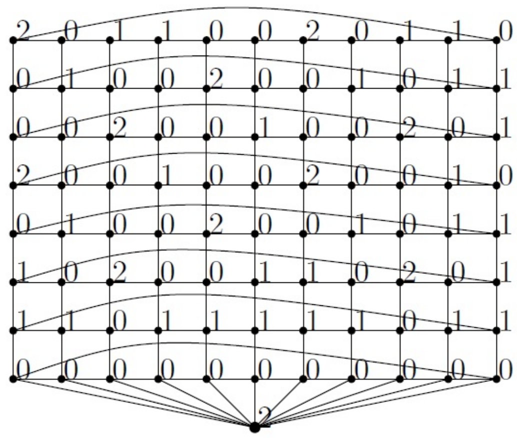

Figure 12. Web graph $W(p, q)$ with function f^*, $q \equiv 5 \bmod 6$ and $p = 3y + 2$

Notice that

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-5)}{3} + 4 \text{ if } j \equiv 1 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-5)}{3} + 2 \text{ if } j \equiv 2 \bmod 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-5)}{3} + 3 \text{ if } j \equiv 3 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-5)}{3} + 2 \text{ if } j \equiv 4 \bmod 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-5)}{3} + 3 \text{ if } j \equiv 5 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-5)}{3} + 2 \text{ if } j \equiv 6 \bmod 6.$$

Thus,

$$\begin{aligned} wt_f(G) &= \left(9\left(\frac{p-5}{3}\right) + 16\right)\left(\frac{q-5}{6}\right) + 9\left(\frac{p-5}{3}\right) + 15 + 2 \\ &= \frac{1}{2}pq + \frac{1}{3}p + \frac{1}{6}q + \frac{7}{6} \leq \frac{1}{2}pq + \frac{1}{6}pq + \frac{2}{3}, \quad (p \geq 8, q \geq 11) \\ &= \frac{2}{3}pq + \frac{2}{3} = \frac{2}{3}(pq+1) = \frac{2}{3}|G| \end{aligned}$$

This completes the proof. $\square$

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On Equitable Associate Symmetric n-Sigraphs

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Abstract: In this paper we introduced a new notion equitable associate symmetric n-sigraph of a symmetric n-sigraph and its properties are obtained. Further, we discuss structural characterization of equitable associate symmetric n-sigraphs.

Key Words: Symmetric n-sigraphs, Smarandachely symmetric n-marked graph, symmetric n-marked graphs, Smarandachely symmetric n-marked graph, balance, switching, equitable associate n-sigraphs, Smarandachely equitable dominating set, complementation.

AMS(2010): 05C22.

§1. Introduction

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [2]. We consider only finite, simple graphs free from self-loops.

Let $n \geq 1$ be an integer. An *n-tuple* $(a_1, a_2, \cdots, a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, \cdots, a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. Note that H_n is a group under coordinate wise multiplication, and the order of H_n is 2^m, where $m = \lceil \frac{n}{2} \rceil$.

A *symmetric n-sigraph (symmetric n-marked graph)* is an ordered pair $S_n = (G, \sigma)$ ($S_n = (G, \mu)$), where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \to H_n$ ($\mu : V \to H_n$) is a function. Generally, a *Smarandachely symmetric n-sigraph* (*Smarandachely symmetric n-marked graph)* for a subgraph H is such a graph that $G - E(H)$ is *symmetric n-sigraph (symmetric n-marked graph)*. For example, let H be a path $P_2 \succ G$ or a cycle $C_3 \prec G$. Certainly, if $H = \emptyset$, a Smarandachely symmetric n-sigraph (or Smarandachely symmetric n-sigraph) is nothing else but a symmetric n-sigraph (or symmetric n-marked graph).

In this paper by an *n-tuple/n-sigraph/n-marked graph* we always mean a symmetric n-tuple/symmetric n-sigraph/symmetric n-marked graph.

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An n-tuple $(a_1, a_2, \cdots, a_n)$ is the *identity n-tuple*, if $a_k = +$, for $1 \leq k \leq n$, otherwise it is a *non-identity n-tuple*. In an n-sigraph $S_n = (G, \sigma)$ an edge labelled with the identity n-tuple is called an *identity edge*, otherwise it is a *non-identity edge*. Further, in an n-sigraph $S_n = (G, \sigma)$, for any $A \subseteq E(G)$ the *n-tuple* $\sigma(A)$ is the product of the n-tuples on the edges of A.

In [2], the authors defined two notions of balance in n-sigraph $S_n = (G, \sigma)$ as follows (See also R. Rangarajan and P.S.K.Reddy [4]):

Definition 1.1 *Let $S_n = (G, \sigma)$ be an n-sigraph. Then,*

(i) *S_n is* identity balanced *(or* i-balanced*), if product of n-tuples on each cycle of S_n is the identity n-tuple, and*

(ii) *S_n is* balanced*, if every cycle in S_n contains an even number of non-identity edges.*

Observation 1.2 An i-balanced n-sigraph need not be balanced and conversely.

The following characterization of i-balanced n-sigraphs is obtained in [8].

Theorem 1.3 (E. Sampathkumar et al. [8]) *An n-sigraph $S_n = (G, \sigma)$ is i-balanced if, and only if, it is possible to assign n-tuples to its vertices such that the n-tuple of each edge uv is equal to the product of the n-tuples of u and v.*

Let $S_n = (G, \sigma)$ be an n-sigraph. Consider the n-marking μ on vertices of S_n defined as follows: each vertex $v \in V$, $\mu(v)$ is the n-tuple which is the product of the n-tuples on the edges incident with v. *Complement* of S_n is an n-sigraph $\overline{S_n} = (\overline{G}, \sigma^c)$, where for any edge $e = uv \in \overline{G}$, $\sigma^c(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S_n}$ as defined here is an i-balanced n-sigraph due to Proposition 1 in [10].

In [8], the authors also have defined switching and cycle isomorphism of an n-sigraph $S_n = (G, \sigma)$ as follows (See also [3, 5C7, 10C20]):

Let $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$, be two n-sigraphs. Then S_n and S'_n are said to be *isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that if uv is an edge in S_n with label $(a_1, a_2, \cdots, a_n)$ then $\phi(u)\phi(v)$ is an edge in S'_n with label $(a_1, a_2, \cdots, a_n)$.

Given an n-marking μ of an n-sigraph $S_n = (G, \sigma)$, *switching* S_n with respect to μ is the operation of changing the n-tuple of every edge uv of S_n by $\mu(u)\sigma(uv)\mu(v)$. The n-sigraph obtained in this way is denoted by $\mathcal{S}_\mu(S_n)$ and is called the *μ-switched n-sigraph* or just *switched n-sigraph*. Further, an n-sigraph S_n *switches* to n-sigraph S'_n (or that they are *switching equivalent* to each other), written as $S_n \sim S'_n$, whenever there exists an n-marking of S_n such that $\mathcal{S}_\mu(S_n) \cong S'_n$.

Two n-sigraphs $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ are said to be *cycle isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that the n-tuple $\sigma(C)$ of every cycle C in S_n equals to the n-tuple $\sigma(\phi(C))$ in S'_n.

We make use of the following known result (see [8]).

Theorem 1.4 (E. Sampathkumar et al. [8]) *Given a graph G, any two n-sigraphs with G as underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

Let $S_n = (G, \sigma)$ be an n-sigraph. Consider the n-marking μ on vertices of S defined as follows: each vertex $v \in V$, $\mu(v)$ is the product of the n-tuples on the edges incident at v. *Complement* of S is an n-sigraph $\overline{S_n} = (\overline{G}, \sigma')$, where for any edge $e = uv \in \overline{G}$, $\sigma'(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S_n}$ as defined here is an i-balanced n-sigraph due to Theorem 1.3.

§2. Equitable Associate n-Sigraph of an n-Sigraph

A subset D of $V(\Gamma)$ is called an *equitable dominating set* of a graph Γ, if for every $v \in V - D$ there exists a vertex $v \in D$ such that $uv \in E(\Gamma)$ and $|d(u)-d(v)| \leq 1$. The minimum cardinality of such a dominating set is denoted by γ_e and is called equitable domination number of Γ. A vertex $u \in V$ is said to be *degree equitable* with a vertex $v \in V$ if $|deg(u) - deg(v)| \leq 1$ (see [21]) and to be *Smarandachely degree equitable* if $|deg(u) - deg(v)| \geq 2$.

Generally, a subset D of V is called an *equitable dominating set* if for every $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$ and a *Smarandachely equitable dominating set* if for every $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \geq 2$. Further, a vertex $u \in V$ is said to be *degree equitable* with a vertex $v \in V$ if $|deg(u) - deg(v)| \leq 1$ and *Smarandachely degree equitable* if $|deg(u) - deg(v)| \geq 1$.

In [1], Dharmalingam introduced a new class of intersection graphs in the field of domination theory. The equitable associate graphs is denoted by $\mathcal{E}(G)$ is the graph which has the same vertex set as G with two vertices u and v are adjacent if and only if u and v are adjacent and degree equitable in G.

Motivated by the existing definition of complement of an n-sigraph, we extend the notion of equitable associate graphs to n-sigraphs as follows:

The *equitable associate n-sigraph* $\mathcal{E}(S_n)$ of an n-sigraph $S_n = (G, \sigma)$ is an n-sigraph whose underlying graph is $\mathcal{E}(G)$ and the n-tuple of any edge uv is $\mathcal{E}(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n. Further, an n-sigraph $S_n = (G, \sigma)$ is called equitable associate n-sigraph, if $S_n \cong \mathcal{E}_t(S'_n)$ for some n-sigraph S'_n. The following result indicates the limitations of the notion $\mathcal{E}(S_n)$ as introduced above, since the entire class of i-unbalanced n-sigraphs is forbidden to be equitable associate n-sigraphs.

Theorem 2.1 *For any n-sigraph $S_n = (G, \sigma)$, its equitable associate n-sigraph $\mathcal{E}(S_n)$ is i-balanced.*

Proof Since the n-tuple of any edge uv in $\mathcal{E}(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n, by Theorem 1.1, $\mathcal{E}(S_n)$ is i-balanced. □

For any positive integer k, the k^{th} iterated equitable associate n-sigraph $\mathcal{E}(S_n)$ of S_n is defined as follows:

$$(\mathcal{E})^0(S_n) = S_n, \quad (\mathcal{E})^k(S_n) = \mathcal{E}((\mathcal{E})^{k-1}(S_n)).$$

Corollary 2.2 *For any n-sigraph $S_n = (G, \sigma)$ and any positive integer k, $(\mathcal{E})^k(S_n)$ is i-balanced.*

The following result characterize n-sigraphs which are equitable associate n-sigraphs.

Theorem 2.3 *An n-sigraph $S_n = (G, \sigma)$ is an equitable associate n-sigraph if, and only if, S_n is i-balanced n-sigraph and its underlying graph G is an equitable associate graph.*

Proof Suppose that S_n is i-balanced and G is a $\mathcal{E}(G)$. Then there exists a graph H such that $\mathcal{E}(H) \cong G$. Since S_n is i-balanced, by Theorem 1.3, there exists an n-marking μ of G such that each edge uv in S_n satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the n-sigraph $S'_n = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the n-marking of the corresponding vertex in G. Then clearly, $\mathcal{E}(S'_n) \cong S_n$. Hence S_n is an equitable associate n-sigraph.

Conversely, suppose that $S_n = (G, \sigma)$ is an equitable associate n-sigraph. Then there exists an n-sigraph $S'_n = (H, \sigma')$ such that $\mathcal{E}(S'_n) \cong S_n$. Hence G is the $\mathcal{E}(G)$ of H and by Theorem 2.1, S_n is i-balanced. $\square$

In [1], the author characterized graphs for which $\overline{\mathcal{E}(G)} \cong \mathcal{E}(\overline{G})$.

Theorem 2.4 (K. M. Dharmalingam [1]) *For any graph $G = (V, E)$, $\overline{\mathcal{E}(G)} \cong \mathcal{E}(\overline{G})$ if and only if every edge of G is equitable.*

We now characterize n-sigraphs whose complementary equitable associate n-sigraphs and equitable associate n-sigraphs are switching equivalent.

Theorem 2.5 *For any n-sigraph $S_n = (G, \sigma)$, $\overline{\mathcal{E}(S_n)} \sim \mathcal{E}(\overline{S_n})$ if and only if every edge of G is equitable.*

Proof Suppose $\overline{\mathcal{E}(S_n)} \sim \mathcal{E}(\overline{S_n})$. This implies, $\overline{\mathcal{E}(G)} \cong \mathcal{E}(\overline{G})$ and hence by Theorem 2.4, every edge of G is equitable.

Conversely, suppose that every edge of G is equitable. Then $\overline{\mathcal{E}(G)} \cong \mathcal{E}(\overline{G})$ by Theorem 2.4. Now, if S_n is an n-sigraph with each edge of G is equitable, by the definition of complementary n-sigraph and Theorem 2.1, $\overline{\mathcal{E}(S_n)}$ and $\mathcal{E}(\overline{S_n})$ are i-balanced and hence, the result follows from Theorem 1.4. $\square$

Theorem 2.6 *For any two n-sigraphs S_n and S'_n with the same underlying graph, their equitable associate n-sigraphs are switching equivalent.*

Proof Suppose $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ be two n-sigraphs with $G \cong G'$. By Theorem 2.1, $\mathcal{E}(S_n)$ and $\mathcal{E}(S'_n)$ are i-balanced and hence, the result follows from Theorem 1.4. $\square$

For any $m \in H_n$, the *m-complement* of $a = (a_1, a_2, \cdots, a_n)$ is: $a^m = am$. For any $M \subseteq H_n$, and $m \in H_n$, the *m-complement* of M is $M^m = \{a^m : a \in M\}$.

For any $m \in H_n$, the *m-complement* of an n-sigraph $S_n = (G, \sigma)$, written (S_n^m), is the same graph but with each edge label $a = (a_1, a_2, \cdots, a_n)$ replaced by a^m.

For an n-sigraph $S_n = (G, \sigma)$, the $\mathcal{E}(S_n)$ is i-balanced. We now examine, the condition under which m-complement of $\mathcal{E}(S_n)$ is i-balanced, where for any $m \in H_n$.

Theorem 2.7 *Let $S_n = (G, \sigma)$ be an n-sigraph. Then, for any $m \in H_n$, if $\mathcal{E}(G)$ is bipartite then $(\mathcal{E}(S_n))^m$ is i-balanced.*

Proof Since, by Theorem 2.1, $\mathcal{E}(S_n)$ is i-balanced, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in $\mathcal{E}(S_n)$ whose k^{th} co-ordinate are $-$ is even. Also, since $\mathcal{E}(G)$ is bipartite, all cycles have even length; thus, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in $\mathcal{E}(S_n)$ whose k^{th} co-ordinate are $+$ is also even. This implies that the same thing is true in any m-complement, where for any $m, \in H_n$. Hence $(\mathcal{E}(S_n))^t$ is i-balanced. □

Notice that Theorem 2.6 provides an easy solutions to other n-sigraph switching equivalence relations, which are given in the following results.

Corollary 2.8 *For any two n-sigraphs S_n and S'_n with the same underlying graph, $\mathcal{E}(S_n)$ and $\mathcal{E}((S'_n)^m)$ are switching equivalent.*

Corollary 2.9 *For any two n-sigraphs S_n and S'_n with the same underlying graph, $\mathcal{E}((S_n)^m)$ and $\mathcal{E}(S'_n)$ are switching equivalent.*

Corollary 2.10 *For any two n-sigraphs S_n and S'_n with the same underlying graph, $\mathcal{E}((S_n)^m)$ and $\mathcal{E}((S'_n)^m)$ are switching equivalent.*

Corollary 2.11 *For any two n-sigraphs $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ with the $G \cong G'$ and G, G' are bipartite, $(\mathcal{E}(S_n))^m$ and $\mathcal{E}(S'_n)$ are switching equivalent.*

Corollary 2.12 *For any two n-sigraphs $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ with the $G \cong G'$ and G, G' are bipartite, $\mathcal{E}(S_n)$ and $(\mathcal{E}(S'_n))^m$ are switching equivalent.*

Corollary 2.13 *For any two n-sigraphs $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ with the $G \cong G'$ and G, G' are bipartite, $(\mathcal{E}(S_1))^m$ and $(\mathcal{E}(S_2))^m$ are switching equivalent.*

Corollary 2.14 *For any n-sigraph $S_n = (G, \sigma)$, $S_n \sim \mathcal{E}((S_n)^m)$ if and only if G is K_n and S_n is i-balanced.*

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PD-Divisor Labeling of Graphs

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Abstract: Let $G = (V(G), E(G))$ be a simple, finite and undirected graph of order n. Given a bijection $f : V(G) \to \{1, 2, \cdots, |V(G)|\}$, we associate two integers $P = f(u)f(v)$ and $D = |f(u) - f(v)|$ with every edge uv in $E(G)$. The labeling f induces on edge labeling $f' : E(G) \to \{0, 1\}$ such that for any edge uv in $E(G)$, $f'(uv) = 1$ if $D \mid P$ and $f'(uv) = 0$ if $D \nmid P$. Let $e_{f'}(i)$ be the number of edges labeled with $i \in \{0, 1\}$. We say f is an PD-divisor labeling if $f'(uv) = 1$ for all $uv \in E(G)$. Moreover, G is PD-divisor if it admits an PD-divisor labeling. We say f is an PD-divisor cordial labeling if $|e_{f'}(0) - e_{f'}(1)| \leq 1$. Moreover, G is PD-divisor cordial if it admits an PD-divisor cordial labeling. In this paper, we define PD-divisibility and PD-divisor pair of numbers and establish some of its properties. Also, we are dealing in PD-divisor labeling of some standard graphs.

Key Words: Divisor cordial labeling; PD-divisor labeling; PD-divisor graph, Smarandachely SD-divisor cordial labeling, Smarandachely SD-divisor cordial graph.

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§1. Introduction

Let $G = (V(G), E(G))$ (or $G = (V, E)$) be a simple, finite and undirected graph of order $|V(G)| = n$ and size $|E(G)| = m$. All notations not defined in this paper can be found in [4].

Definition 1.1 ([2]) *Let a and b be two integers. If a divides b means that there is a positive integer k such that $b = ka$. It is denoted by $a \mid b$. If a does not divide b , then we denote $a \nmid b$.*

Definition 1.2 ([1]) *Let $G = (V, E)$ be a graph. A mapping $f : V(G) \to \{0, 1\}$ is called binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f. For an edge $e = uv$, the induced edge labeling $f' : E(G) \to \{0, 1\}$ is given by $f'(e) = |f(u) - f(v)|$. Let $v_f(0), v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and $e_{f'}(0), e_{f'}(1)$ be the number of edges having labels 0 and 1 respectively under f'. This labeling is called cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_{f'}(0) - e_{f'}(1)| \leq 1$. A graph G is cordial if it admits cordial labeling.*

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Definition 1.3 ([9]) *A bijection $f : V \to \{1,2,\cdots,n\}$ induces an edge labeling $f' : E \to \{0,1\}$ such that for any edge uv in G, $f'(uv) = 1$ if gcd$(f(u), f(v)) = 1$, and $f'(uv) = 0$ otherwise. We say that f is a prime cordial labeling if $|e_{f'}(0) - e_{f'}(1)| \leq 1$. Moreover, G is prime cordial if it admits a prime cordial labeling.*

Definition 1.4 ([10]) *Let $G = (V, E)$ be a simple graph and $f : V \to \{1,2,\cdots,n\}$ be a bijection. For each edge uv, assign the label 1 if either $f(u) \mid f(v)$ or $f(v) \mid f(u)$ and the label 0 otherwise. We say that f is a divisor cordial labeling if $|e_{f'}(0) - e_{f'}(1)| \leq 1$. Moreover, G is divisor cordial if it admits a divisor cordial labeling.*

Given a bijection $f : V \to \{1,2,\cdots,n\}$, we associate two integers $S = f(u) + f(v)$ and $D = |f(u) - f(v)|$ with every edge uv in E.

Definition 1.5 ([7]) *A bijection $f : V \to \{1,2,\cdots,n\}$ induces an edge labeling $f' : E \to \{0,1\}$ such that for any edge uv in G, $f'(uv) = 1$ if gcd$(S, D) = 1$, and $f'(uv) = 0$ otherwise. We say f is an SD-prime labeling if $f'(uv) = 1$ for all $uv \in E$. Moreover, G is SD-prime if it admits an SD-prime labeling.*

Definition 1.6 ([6]) *A bijection $f : V \to \{1,2,\cdots,n\}$ induces an edge labeling $f' : E \to \{0,1\}$ such that for any edge uv in G, $f'(uv) = 1$ if gcd$(S, D) = 1$, and $f'(uv) = 0$ otherwise. The labeling f is called an SD-prime cordial labeling if $|e_{f'}(0) - e_{f'}(1)| \leq 1$. We say that G is SD-prime cordial if it admits an SD-prime cordial labeling.*

Definition 1.7 ([5]) *Let $G = (V(G), E(G))$ be a simple graph and a bijection $f : V(G) \to \{1,2,3,\cdots,|V(G)|\}$ induces an edge labeling $f' : E(G) \to \{0,1\}$ such that for any edge uv in $E(G)$, $f'(uv) = 1$ if $D \mid S$ and $f'(uv) = 0$ if $D \nmid S$. We say f is an SD-divisor labeling if $f'(uv) = 1$ for all $uv \in E(G)$. Moreover, G is SD-divisor if it admits an SD-divisor labeling.*

Definition 1.8 ([5]) *Let $G = (V(G), E(G))$ be a simple graph and a bijection $f : V(G) \to \{1,2,3,\cdots,|V(G)|\}$ induces an edge labeling $f' : E(G) \to \{0,1\}$ such that for any edge uv in $E(G)$, $f'(uv) = 1$ if $D \mid S$ and $f'(uv) = 0$ if $D \nmid S$. The labeling f is called an SD-divisor cordial labeling if $|e_{f'}(0) - e_{f'}(1)| \leq 1$. We say that G is SD-divisor cordial if it admits an SD-divisor cordial labeling.*

Generally, the labeling f in Definition 1.8 is said to be *Smarandachely SD-divisor cordial* if $|e_{f'}(0) - e_{f'}(1)| \geq 2$ and G is said to be a Smarandachely SD-divisor cordial graph. In [5], we introduced two new types of labeling called SD-divisor and SD-divisor cordial labeling. Also, we proved some graphs are SD-divisor. Motivated by the concepts of SD-divisor and SD-divisor cordial labeling, we introduce two new types of labeling called PD-divisor and PD-divisor cordial labeling. In this paper, we define PD-divisibility and PD-divisor pair of numbers and establish some of its properties. Also, we are dealing in PD-divisor labeling of some standard graphs.

§2. PD-Divisibility and its Properties

First, we define PD-divisibility of two positive integers.

Definition 2.1 *Let a and b be the two distinct positive integers, we say that a PD-divides b if $|a-b| \mid ab$. It is denoted by $a \mid_{PD} b$. If a does not PD-divide b, then it is denoted by $a \nmid_{PD} b$.*

Example 2.2 $4 \mid_{PD} 6$.

Example 2.3 $2 \nmid_{PD} 8$.

Notice that

1. From the Examples 2.2 and 2.3, divisibility and PD-divisibility are different concepts.
2. By Definition 2.1, PD-divisibility is not reflexive.
3. From Definition 2.1, $a \mid_{PD} b \Rightarrow |a-b| \mid ab$

$$\Rightarrow |b-a| \mid ba$$
$$\Rightarrow b \mid_{PD} a.$$

Thus, PD-divisibility is symmetric.

4. PD-divisibility is not transitive.

Example 2.4 $1 \mid_{PD} 2$ and $2 \mid_{PD} 6$ but $1 \nmid_{PD} 6$.

PD-divisibility is not an equivalence relation.

Observation 2.5 Its known that if k and $k+1$ are two consecutive integers, then $k \nmid k+1$ for $k \geq 2$.

Proposition 2.6 1 *PD-divides only to the integer* 2.

Proof Let $a = 1$ and $b > 1$ be the any positive integer. If $1 \mid_{PD} b$, then $(b-1) \mid b$. This means that two consecutive integers divide. This is possible only if $b = 2$. $\square$

Proposition 2.7 2 *PD-divides only to the integers* $1, 3, 4$ *and* 6.

Proof Let $a = 2$ and b be the any positive integer. If $2 \mid_{PD} b$, then $|b-2| \mid 2b$. This is possible only if $b = 1, 3, 4$ and 6. $\square$

Proposition 2.8 3 *PD-divides only to the integers* $2, 4, 6$ *and* 12.

Proof Let $a = 3$ and b be the any positive integer. If $3 \mid_{PD} b$, then $|b-3| \mid 3b$. This is possible only if $b = 2, 4, 6$ and 12. $\square$

Observation 2.9 Let $a \geq 2$ be the any positive integer. Then $a-1, a+1, 2a$ and $a(a+1)$ are PD-divisible by a.

Observation 2.10 Let $a \geq 4$ be the any positive even integer. Then $a-2$, $a-1$, $a+1$, $a+2$, $a+4$, $2a$, $3a$ and $a(a+1)$ are PD-divisible by a.

Proposition 2.11 *Let a and b be the two consecutive odd integers, then $a \nmid_{PD} b$.*

Proof Let $a = 2k+1$ and $b = 2k+3$ for $k \geq 0$. Then $|a-b| = |2k+1-2k-3| = 2$ and $ab = (2k+1)(2k+3)$.

Clearly, $2 \nmid (2k+1)(2k+3)$. Then $a \nmid_{PD} b$. $\square$

Proposition 2.12 *Let a and b be the two consecutive even integers, then $a \mid_{PD} b$.*

Proof Let $a = 2k+2$ and $b = 2k+4$ for $k \geq 0$. Then $|a-b| = |2k+2-2k-4| = 2$ and $ab = (2k+2)(2k+4)$.

Clearly, $2 \mid (2k+2)(2k+4)$. Then $a \mid_{PD} b$. □

§3. PD-Divisor Pair

Definition 3.1 *Let a and b be the two distinct positive integers. If $a \mid_{PD} b$, then we say that (a, b) is called PD-divisor pair.*

Example 3.2 For $k \geq 1$, $(k, k+1)$ is PD-divisor pair.

Notice that if $l \geq 1$ is any positive integer, then $(lk, l(k+1))$ is PD-divisor pair. We know the following results.

Proposition 3.3 *If the pair (a, b) is PD-divisor, then (ka, kb) is PD-divisor pair for $k \geq 1$.*

Proof Let a and b the PD-divisor pair. Without loss of generality, we take $a > b$.

Then,
$$\begin{aligned} a \mid_{PD} b &\Rightarrow (a-b) \mid ab \\ &\Rightarrow k(a-b) \mid kab \text{ for } k \geq 1 \\ &\Rightarrow (ka-kb) \mid kab \\ &\Rightarrow (ka-kb) \mid k^2ab \Rightarrow ka \mid_{PD} kb. \end{aligned}$$
□

Proposition 3.4 *Let $k \geq 3$ be an odd integer. Then $(k+1, k-1)$ is PD-divisor pair.*

Proof Let $a = k+1$ and $b = k-1$ for all odd integer $k \geq 3$. Then $|a-b| = |k+1-k+1| = 2$ and $ab = (k+1)(k-1) = k^2-1$.

Clearly $2 \mid k^2-1$. Thus, $(k+1, k-1)$ is PD-divisor pair for all odd integer $k \geq 3$. □

Proposition 3.5 *Let $k \geq 2$ be an even integer. Then $(k+1, k-1)$ is not PD-divisor pair.*

Proof Let $a = k+1$ and $b = k-1$ for all even integer $k \geq 2$. Then, $|a-b| = |k+1-k+1| = 2$ and $ab = (k+1)(k-1) = k^2-1$.

Clearly, $2 \nmid k^2-1$. Thus, $(k+1, k-1)$ is not PD-divisor pair for all even integer $k \geq 2$.□

Proposition 3.6 *Let $k \geq 0$. Then $(2^k, 2^{k+1})$ is PD-divisor pair.*

Proof Let $a = 2^k$ and $b = 2^{k+1}$ for $k \geq 0$. Then $|a-b| = |2^k - 2^{k+1}| = 2^k$ and $ab = (2^k)(2^{k+1})$.

Clearly, $|a-b| \mid ab$. Thus, $(2^k, 2^{k+1})$ is PD-divisor pair for $k \geq 0$. □

Proposition 3.7 *Let $k \geq 0$. Then $(3^k, 3^{k+1})$ is not PD-divisor pair.*

Proof Let $a = 3^k$ and $b = 3^{k+1}$ for $k \geq 0$. Then, $|a-b| = |3^k - 3^{k+1}| = 2 \cdot 3^k$ and $ab = (3^k)(3^{k+1})$.

Clearly, $|a-b| \nmid ab$. Thus, $(3^k, 3^{k+1})$ is not PD-divisor pair for $k \geq 0$. □

Proposition 3.8 *Let $l \geq 3$ and $k \geq 0$. Then (l^k, l^{k+1}) is not PD-divisor pair.*

Proof Let $a = l^k$ and $b = l^{k+1}$ for $l \geq 3$, $k \geq 0$. Then, $|a-b| = |l^k - l^{k+1}| = l^k(l-1)$ and $ab = (l^k)(l^{k+1})$.

Clearly, $l - 1 \nmid l^{k+1}$ for $l \geq 3$ and $k \geq 0$. Thus, (l^k, l^{k+1}) is not a PD-divisor pair for $l \geq 3$ and $k \geq 0$. □

Definition 3.9 *Let S be a set of any distinct positive integers. Then S is said to be PD-divisor set if every pair of integers in S is PD-divisor.*

we always use notation $[n] = \{1, 2, \cdots, n\}$ in this paper.

Example 3.10 [2] is PD-divisor set.

§4. PD-Divisor Labeling of Graphs

Now, we introduce two new types of labeling called PD-divisor and PD-divisor cordial labeling. Given a bijection $f : V \to \{1, 2, \cdots, n\}$, we associate two integers $P = f(u)f(v)$ and $D = |f(u) - f(v)|$ with every edge uv in E.

Definition 4.1 *Let $G = (V(G), E(G))$ be a simple graph and a bijection $f : V(G) \to \{1, 2, 3, \cdots, |V(G)|\}$ induces an edge labeling $f' : E(G) \to \{0, 1\}$ such that for any edge uv in $E(G)$, $f'(uv) = 1$ if $D \mid P$ and $f'(uv) = 0$ if $D \nmid P$. We say f is an PD-divisor labeling if $f'(uv) = 1$ for all $uv \in E(G)$. Moreover, G is PD-divisor if it admits an PD-divisor labeling.*

Example 4.2 Consider the following graph G.

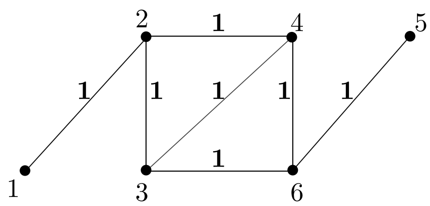

Figure 1

We see that $e_{f'}(1) = 7$. Hence G is PD-divisor.

Definition 4.2 *Let $G = (V(G), E(G))$ be a simple graph and a bijection $f : V(G) \to \{1, 2, 3, ..., |V(G)|\}$ induces an edge labeling $f' : E(G) \to \{0, 1\}$ such that for any edge uv in $E(G)$, $f'(uv) = 1$ if $D \mid P$ and $f'(uv) = 0$ if $D \nmid P$. The labeling f is called an PD-divisor cordial labeling if $|e_{f'}(0) - e_{f'}(1)| \leq 1$. We say that G is PD-divisor cordial if it admits an PD-divisor cordial labeling.*

Example 4.3 Consider the the labeling of G in Figure 3.

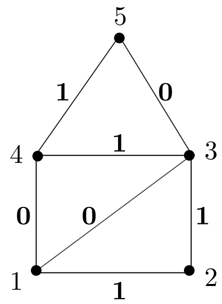

Figure 2

We see that $e_{f'}(0) = 3$ and $e_{f'}(1) = 4$. Thus $|e_{f'}(0) - e_{f'}(1)| \leq 1$ and hence G is PD-divisor cordial.

Now, we prove path and some path related graphs are PD-divisor. Also, we prove some standard graphs such as star, cycle, complete, complete bipartite and wheel graphs are not PD-divisor.

Theorem 4.5 *A path P_n is PD-divisor.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of path P_n. Let $V(P_n) = \{v_i : 1 \leq i \leq n\}$ and $E(P_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$. Therefore, P_n is of order n and size $n-1$. Define $f : V(P_n) \to \{1, 2, 3, \cdots, n\}$ to be

$$f(v_i) = i, \quad 1 \leq i \leq n.$$

From the above labeling pattern we get, $e_{f'}(1) = n - 1$. Hence, P_n is PD-divisor. $\square$

Example 4.6 Consider the labeling of P_8 in Figure 3.

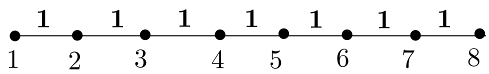

Figure 3

Here $e_{f'}(1) = 7$. Hence, P_8 is PD-divisor.

Theorem 4.7 *A comb $P_n \odot K_1$ is PD-divisor.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of path P_n. Let $V(P_n \odot K_1) = \{v_i, u_i : 1 \leq i \leq n\}$ and $E(P_n \odot K_1) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \bigcup \{v_i u_i : 1 \leq i \leq n\}$. Therefore, $P_n \odot K_1$ is of order $2n$ and size $2n - 1$. Define $f : V(P_n \odot K_1) \to \{1, 2, 3, \cdots, 2n\}$ to be $f(v_i) = 2i$, $1 \leq i \leq n$, $f(u_i) = 2i - 1$, $1 \leq i \leq n$.

From the above labeling pattern we get, $e_{f'}(1) = 2n - 1$. So $P_n \odot K_1$ is PD-divisor. $\square$

Example 4.8 Consider the labeling of $P_6 \odot K_1$ in Figure 4.

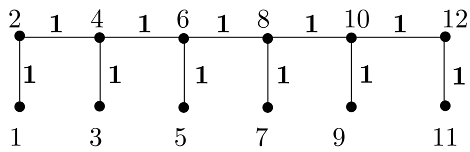

Figure 4

Here, $e_{f'}(1) = 11$. Hence, $P_6 \odot K_1$ is PD-divisor.

Theorem 4.9 *A double comb $P_n \odot 2K_1$ is PD-divisor.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of path P_n. Let $V(P_n \odot 2K_1) = \{v_i, u_i, w_i : 1 \leq i \leq n\}$ and $E(P_n \odot 2K_1) = \{v_iv_{i+1} : 1 \leq i \leq n-1\} \bigcup \{v_iu_i : 1 \leq i \leq n\} \bigcup \{v_iw_i : 1 \leq i \leq n\}$. Therefore, $P_n \odot 2K_1$ is of order $3n$ and size $3n-1$. Define $f : V(P_n \odot 2K_1) \to \{1, 2, 3, \cdots, 3n\}$ to be

$$
\begin{aligned}
f(v_{2i-1}) &= 6i-4, \quad 1 \leq i \leq \lceil \frac{n}{2} \rceil, \\
f(v_{2i}) &= 6i-2, \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, \\
f(u_i) &= 3i, \quad 1 \leq i \leq n, \\
f(w_{2i-1}) &= 6i-5, \quad 1 \leq i \leq \lceil \frac{n}{2} \rceil, \\
f(w_{2i}) &= 6i-1, \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor.
\end{aligned}
$$

From the above labeling pattern we get, $e_{f'}(1) = 3n-1$. Hence, $P_n \odot 2K_1$ is PD-divisor.□

Example 4.10 Consider the labeling of $P_8 \odot 2K_1$ in Figure 5.

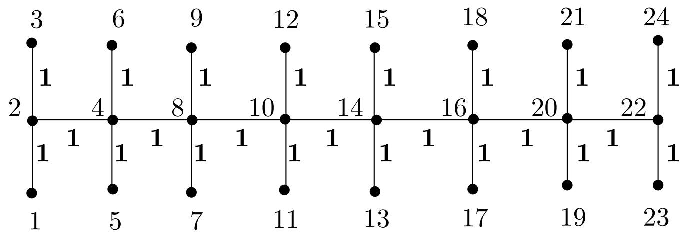

Figure 5

Here $e_{f'}(1) = 23$. Hence, $P_8 \odot 2K_1$ is PD-divisor.

Theorem 4.11 *A crown $C_n \odot K_1$ is PD-divisor.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of cycle C_n. Let $V(C_n \odot K_1) = \{v_i, u_i : 1 \leq i \leq n\}$ and $E(C_n \odot K_1) = \{v_iv_{i+1} : 1 \leq i \leq n-1\} \bigcup \{v_nv_1, v_iu_i : 1 \leq i \leq n\}$. Therefore, $C_n \odot K_1$ is of order $2n$ and size $2n$. Define $f : V(C_n \odot K_1) \to \{1, 2, 3, \cdots, 2n\}$ to be

$$\begin{aligned}
f(v_i) &= 4i-2, \quad 1 \le i \le \left\lceil \frac{n}{2} \right\rceil, \\
f(v_{n+1-i}) &= 4i, \qquad\ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor, \\
f(u_i) &= 4i-3, \quad 1 \le i \le \left\lceil \frac{n}{2} \right\rceil, \\
f(u_{n+1-i}) &= 4i-1, \quad 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor.
\end{aligned}$$

From the above labeling pattern we get, $e_{f'}(1) = 2n$. Hence, $C_n \odot K_1$ is PD-divisor. □

Example 4.12 Consider the labeling of $C_{11} \odot K_1$ in Figure 6.

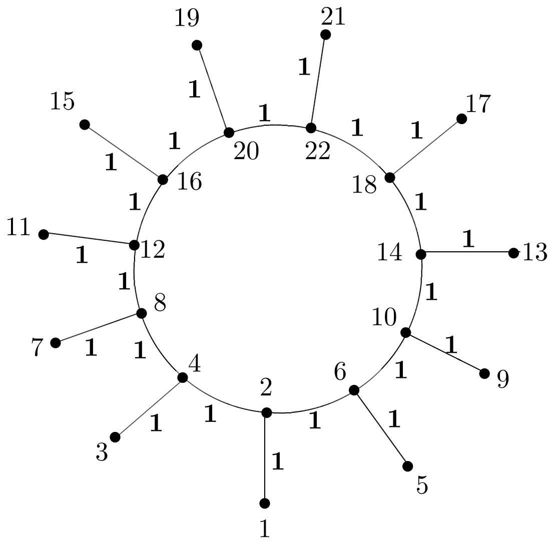

Figure 6

Here, $e_{f'}(1) = 22$. Hence, $C_{11} \odot K_1$ is PD-divisor.

Next, we will investigate whether the star graph $K_{1,n}$ is PD-divisor or not. Clearly, $K_{1,1}$ and $K_{1,2}$ are PD-divisor, and also $K_{1,3}$ is PD-divisor in the following labeling.

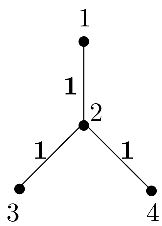

Figure 7

Next, we will prove that $K_{1,n}$ is not PD-divisor for $n \ge 4$.

Theorem 4.13 *For $n \ge 4$, the star graph $K_{1,n}$ is not PD-divisor.*

Proof Consider the set $\{1, 2, \cdots, n+1\}$, $n \geq 4$. Let v be the central vertex of $K_{1,n}(n \geq 4)$.

If we label 1 to v and other numbers to the end vertices of $K_{1,n}$, then it follows from Proposition 2.6, 1 does not PD-divide $3, 4, 5, \cdots, n+1$.

If we label 2 to v and other numbers to the end vertices of $K_{1,n}$, then it follows from Proposition 2.7, 2 does not PD-divide $5, 7, 8, \cdots, n+1$.

Suppose, we label $n \geq 3$ to v. Since any one of the end vertex has the label 1, then it follows from Proposition 2.6, 1 does not PD-divide to the label of v.

Thus, $K_{1,n}$ is not PD-divisor for $n \geq 4$. □

Theorem 4.14 *If $\delta(G) \geq 2$, then G is not PD-divisor.*

Proof Suppose G is PD-divisor. Let v be the vertex of degree $\delta(G) \geq 2$, which is labeled with 1. Then, any one of the δ adjacent vertices of v must have the labels other than 2, say w.

From Proposition 2.6, it follows that 1 does not PD-divide the label of w. This is contradiction to G is PD-divisor. □

Remark 4.15 *If $\delta(G) = 1$, then it is not necessary that G is PD-divisor from Theorem* 4.13 *its follows.*

Corollary 4.16 *For $n \geq 3$, the cycle graph C_n is not PD-divisor.*

Proof Since $\delta(C_n) \geq 2$ for $n \geq 3$, the result follows from Theorem 4.14. □

Corollary 4.17 *For $n \geq 3$, the complete graph K_n is not PD-divisor.*

Proof Since $\delta(K_n) \geq 2$ for $n \geq 3$, the result follows from Theorem 4.14. □

Corollary 4.18 *For $m, n \geq 2$, the complete bipartite graph $K_{m,n}$ is not PD-divisor.*

Proof Since $\delta(K_{m,n}) \geq 2$ for $m, n \geq 2$, the result follows from Theorem 4.14. □

Corollary 4.19 *The wheel graph $W_{n+1}(n \geq 2)$ is not PD-divisor.*

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A QSPR Analysis for Square Root Stress-Sum Index

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Abstract: A QSPR analysis is carried for square root stress-sum index of molecular graphs and physical properties of lower alkanes and linear regression models are presented for boiling points, molar volumes, molar refractions, heats of vaporization and critical temperatures.

Key Words: Topological index, square root stress-sum index, lower alkanes.

AMS(2010): 05C09, 05C92.

§1. **Introduction**

Let $G = (V, E)$ be a graph (simple, finite, connected and undirected). A shortest path between two vertices u and v in G is called a geodesic between u and v. The molecular graph of a chemical compound is a simple connected graph considering atoms of chemical compounds as vertices and the chemical bonds between them as edges. For a glossary of graph theory terms, turn to the Harary's textbook [2]. As and when necessary, the non-standard notions will be provided in this document.

The topological indices are graph invariants (theoretical molecular descriptors) that play an important role in chemistry (see [3, 6, 7]). Many important degree or distance based topological indices can be found in literature for graphs having numerous applications in Chemistry [6] like Zagreb index, Wiener index, Harary index etc.

The quantitative structure-property relationship (QSPR) converts the quantitative physical characteristics of chemical compounds into numerical data, allowing researchers to investigate the relationships between these characteristics and the chemical compounds' structures while simultaneously creating regression models.

Shimbel [8] introduced the idea of stress of a vertex in a network (graph) as a centrality

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measure in 1953. K. Bhargava, N. N. Dattatreya, and R. Rajendra [1] have investigated the concepts of stress number of a graph and stress regular graphs.

The stress of a vertex v in a graph G, denoted by $\mathrm{str}_G(v)$ or briefly by $\mathrm{str}(v)$, is the number of geodesics passing through it. The square root stress-sum index $\mathcal{SRS}(G)$ of a simple graph G is defined (see [5]) by

$$\mathcal{SRS}(G) = \sum_{uv \in E(G)} \sqrt{\mathrm{str}(u) + \mathrm{str}(v)}. \tag{1}$$

In this paper, we present best linear regression models for boiling points, molar volumes, molar refractions, heats of vaporization and critical temperatures of low alkanes through a QSPR analysis for physical properties of lower alkanes with square root stress-sum index of molecular graph.

§2. A QSPR Analysis

We carry a QSPR analysis for the physical properties - boiling points, molar volumes, molar refractions, heats of vaporization, critical temperatures, critical pressures and surface tensions of lower alkanes with square root stress-sum index of molecular graphs.

Table 1 gives the square root stress-sum index $\mathcal{SRS}(G)$ of molecular graphs and the experimental values for the physical properties - Boiling points (bp) $^\circ C$, molar volumes (mv) cm^3, molar refractions (mr) cm^3, heats of vaporization (hv) kJ, critical temperatures (ct) $^\circ C$, critical pressures (cp) atm, and surface tensions(st) $dyne\ cm^{-1}$ of considered alkanes. The values given in the columns 3 to 9 in the Table 1 are taken from Needham et al. [3] (the same values can be found in [7]).

Table 1. First Stress Index, boiling points, molar volumes, molar refractions, heats of vaporization, critical temperatures, critical pressures and surface tensions of low alkanes

Alkane	$\mathcal{SRS}(G)$	$\frac{bp}{^\circ C}$	$\frac{mv}{cm^3}$	$\frac{mr}{cm^3}$	$\frac{hv}{kJ}$	$\frac{ct}{^\circ C}$	$\frac{cp}{atm}$	$\frac{st}{dyne\ cm^{-1}}$
Pentane	8.756	36.1	115.2	25.27	26.4	196.6	33.3	16
2-Methylbutane	9.033	27.9	116.4	25.29	24.6	187.8	32.9	15
2,2-Dimethylpropane	9.798	9.5	122.1	25.72	21.8	160.6	31.6	
Hexane	13.789	68.7	130.7	29.91	31.6	234.7	29.9	18.42
2-Methylpentane	14.059	60.3	131.9	29.95	29.9	224.9	30	17.38
3-Methylpentane	13.757	63.3	129.7	29.8	30.3	231.2	30.8	18.12
2,2-Dimethylbutane	14.606	49.7	132.7	29.93	27.7	216.2	30.7	16.3
2,3-Dimethylbutane	14.325	58	130.2	29.81	29.1	227.1	31	17.37
Heptane	19.929	98.4	146.5	34.55	36.6	267	27	20.26
2-Methylhexane	20.207	90.1	147.7	34.59	34.8	257.9	27.2	19.29
3-Methylhexane	19.753	91.9	145.8	34.46	35.1	262.4	28.1	19.79
3-Ethylhexane	25.828	93.5	143.5	34.28	35.2	267.6	28.6	20.44
2,2-Dimethylpentane	20.706	79.2	148.7	34.62	32.4	247.7	28.4	18.02
2,3-Dimethylpentane	20.025	89.8	144.2	34.32	34.2	264.6	29.2	19.96
2,4-Dimethylpentane	20.485	80.5	148.9	34.62	32.9	247.1	27.4	18.15
3,3-Dimethylpentane	16.563	86.1	144.5	34.33	33	263	30	19.59

2,3,3-Trimethylbutane	20.975	80.9	145.2	34.37	32	258.3	29.8	18.76
Octane	27.179	125.7	162.6	39.19	41.5	296.2	24.64	21.76
2-Methylheptane	27.468	117.6	163.7	39.23	39.7	288	24.8	20.6
3-Methylheptane	26.902	118.9	161.8	39.1	39.8	292	25.6	21.17
4-Methylheptane	26.772	117.7	162.1	39.12	39.7	290	25.6	21
3-Ethylhexane	32.034	118.5	160.1	38.94	39.4	292	25.74	21.51
2,2-Dimethylhexane	27.955	106.8	164.3	39.25	37.3	279	25.6	19.6
2,3-Dimethylhexane	27.055	115.6	160.4	38.98	38.8	293	26.6	20.99
2,4-Dimethylhexane	27.191	109.4	163.1	39.13	37.8	282	25.8	20.05
2,5-Dimethylhexane	27.757	109.1	164.7	39.26	37.9	279	25	19.73
3,3-Dimethylhexane	27.137	112	160.9	39.01	37.9	290.8	27.2	20.63
3,4-Dimethylhexane	26.618	117.7	158.8	38.85	39	298	27.4	21.62
3-Ethyl-2-methylpentane	26.109	115.7	158.8	38.84	38.5	295	27.4	21.52
3-Ethyl-3-methylpentane	26.288	118.3	157	38.72	38	305	28.9	21.99
2,2,3-Trimethylpentane	26.607	109.8	159.5	38.92	36.9	294	28.2	20.67
2,2,4-Trimethylpentane	28.244	99.2	165.1	39.26	36.1	271.2	25.5	18.77
2,3,3-Trimethylpentane	29.101	114.8	157.3	38.76	37.2	303	29	21.56
2,3,4-Trimethylpentane	27.338	113.5	158.9	38.87	37.6	295	27.6	21.14
Nonane	35.537	150.8	178.7	43.84	46.4	322	22.74	22.92
2-Methyloctane	35.839	143.3	179.8	43.88	44.7	315	23.6	21.88
3-Methyloctane	35.181	144.2	178	43.73	44.8	318	23.7	22.34
4-Methyloctane	34.963	142.5	178.2	43.77	44.8	318.3	23.06	22.34
3-Ethylheptane	33.801	143	176.4	43.64	44.8	318	23.98	22.81
4-Ethylheptane	32.034	141.2	175.7	43.49	44.8	318.3	23.98	22.81
2,2-Dimethylheptane	36.327	132.7	180.5	43.91	42.3	302	22.8	20.8
2,3-Dimethylheptane	35.259	140.5	176.7	43.63	43.8	315	23.79	22.34
2,4-Dimethylheptane	35.265	133.5	179.1	43.74	42.9	306	22.7	21.3
2,5-Dimethylheptane	35.483	136	179.4	43.85	42.9	307.8	22.7	21.3
2,6-Dimethylheptane	36.141	135.2	180.9	43.93	42.8	306	23.7	20.83
3,3-Dimethylheptane	35.303	137.3	176.9	43.69	42.7	314	24.19	22.01
3,4-Dimethylheptane	34.599	140.6	175.3	43.55	43.8	322.7	24.77	22.8
3,5-Dimethylheptane	34.825	136	177.4	43.64	43	312.3	23.59	21.77
4,4-Dimethylheptane	35.052	135.2	176.9	43.6	42.7	317.8	24.18	22.01
3-Ethyl-2-methylhexane	33.729	138	175.4	43.66	43.8	322.7	24.77	22.8
4-Ethyl-2-methylhexane	34.102	133.8	177.4	43.65	43	330.3	25.56	21.77
3-Ethyl-3-methylhexane	33.963	140.6	173.1	43.27	43	327.2	25.66	23.22
3-Ethyl-4-methylhexane	33.434	140.46	172.8	43.37	44	312.3	23.59	23.27
2,2,3-Trimethylhexane	35.742	133.6	175.9	43.62	41.9	318.1	25.07	21.86
2,2,4-Trimethylhexane	35.971	126.5	179.2	43.76	40.6	301	23.39	20.51
2,2,5-Trimethylhexane	36.629	124.1	181.3	43.94	40.2	296.6	22.41	20.04
2,3,3-Trimethylhexane	35.344	137.7	173.8	43.43	42.2	326.1	25.56	22.41
2,3,4-Trimethylhexane	34.895	139	173.5	43.39	42.9	324.2	25.46	22.8
2,3,5-Trimethylpentane	35.560	131.3	177.7	43.65	41.4	309.4	23.49	21.27
2,4,4-Trimethylhexane	35.605	130.6	177.2	43.66	40.8	309.1	23.79	21.17

3,3,4-Trimethylhexane	34.935	140.5	172.1	43.34	42.3	330.6	26.45	23.27
3,3-Diethylpentane	32.854	146.2	170.2	43.11	43.4	342.8	26.94	23.75
2,2-Dimethyl-3-ethylpentane	34.576	133.8	174.5	43.46	42	338.6	25.96	22.38
2,3-Dimethyl-3-ethylpentane	34.253	142	170.1	42.95	42.6	322.6	26.94	23.87
2,4-Dimethyl-3-ethylpentane	34.021	136.7	173.8	43.4	42.9	324.2	25.46	22.8
2,2,3,3-Tetramethylpentane	36.075	140.3	169.5	43.21	41	334.5	27.04	23.38
2,2,3,4-Tetramethylpentane	34.848	133	173.6	43.44	41	319.6	25.66	21.98
2,2,4,4-Tetramethylpentane	37.118	122.3	178.3	43.87	38.1	301.6	24.58	20.37
2,3,3,4-Tetramethylpentane	35.635	141.6	169.9	43.2	41.8	334.5	26.85	23.31

§3. Regression Models

Using Table 1, a study was carried out with a linear regression model

$$P = A + B \cdot \mathcal{SRS}(G)$$

where P = Physical property and $\mathcal{SRS}(G)$ = square root stress-sum index. The correlation coefficient r, its square r^2, standard error (se), t-value and p-value are computed and tabulated in Table 2 followed by linear regression models.

Table 2. r, r^2, se, t and p for the physical properties (P) and square root stress-sum index

P	r	r^2	se	t	p
bp	0.9493	0.9013	(4.4601) (0.1501)	(1.8548) (24.7396)	(0.06801894400) (2.06038$E-35$)
mv	0.9785	0.9576	(1.6138) (0.0543)	(63.2757) (38.9117)	(1.73757$E-61$) (1.01051$E-47$)
mr	0.9856	0.9715	(0.4026) (0.0135)	(51.9956) (47.7961)	(6.91797$E-56$) (1.68657$E-53$)
hv	0.9322	0.8690	(0.8810) (0.0296)	(23.7143) (21.0820)	(2.63733$E-34$) (2.78818$E-31$)
ct	0.9294	0.8638	(6.3691) (0.2144)	(25.8642) (20.6137)	(1.38438$E-36$) (1.03014$E-30$)
cp	-0.8803	0.7749	(0.5641) (0.0189)	(61.2577) (-15.1892)	(1.46981$E-60$) (2.19803$E-23$)
st	0.7941	0.6306	(0.5619) (0.0185)	(27.3738) (10.4534)	(4.82637$E-37$) (1.78543$E-15$)

The linear regression models for boiling points, molar volumes, molar refractions, heats of vaporization, critical temperatures, critical pressures and surface tensions of low alkanes are as follows:

$$\begin{aligned}
bp &= 8.272946679 + 3.714688178 \cdot \mathcal{SRS}(G), & (2)\\
mv &= 102.1199905 + 2.114125806 \cdot \mathcal{SRS}(G), & (3)\\
mr &= 20.9376134 + 0.647931461 \cdot \mathcal{SRS}(G), & (4)\\
hv &= 20.8936634 + 0.625307734 \cdot \mathcal{SRS}(G), & (5)\\
ct &= 164.7337437 + 4.419939221 \cdot \mathcal{SRS}(G), & (6)\\
cp &= 34.55588274 - 0.288452993 \cdot \mathcal{SRS}(G), & (7)\\
st &= 15.38165627 + 0.193802145 \cdot \mathcal{SRS}(G). & (8)
\end{aligned}$$

The values of r,r^2, se, t and p in Table 2 for the physical properties are good except for critical pressures and surface tensions. As a result the linear regression models (2) – (6) can be used as predictive tools.

§4. Conclusion

Table 2 reveals that the linear regression models (2) – (7) are useful tools in predicting the physical properties - boiling points, molar volumes, molar refractions, heats of vaporization and critical temperatures of low alkanes. It shows that square root stress-sum index can be used as predictive tool in QSPR researches.

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Gallai and Anti-Gallai Symmetric n-Sigraphs

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Abstract: An *n-tuple* $(a_1, a_2, \cdots, a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, \cdots, a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. A *symmetric n-sigraph (symmetric n-marked graph)* is an ordered pair $S_n = (G, \sigma)$ $(S_n = (G, \mu))$, where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \to H_n$ $(\mu : V \to H_n)$ is a function. In this paper, we introduced a new notions Gallai and anti-Gallai symmetric n-sigraph of a symmetric n-sigraph and its properties are obtained. Also we give the relation between Gallai symmetric n-sigraphs and anti-Gallai symmetric n-sigraphs. Further, we discuss structural characterizations of these notions.

Key Words: Symmetric n-sigraph, Smarandachely symmetric n-sigraph, symmetric n-marked graph, Smarandachely symmetric n-marked graph, balance, switching, Gallai symmetric n-sigraphs, Smarandachely Gallai symmetric n-sigraph, anti-Gallai symmetric n-sigraph, Smarandachely anti-Gallai n-sigraph, complementation.

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§1. Introduction

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [3]. We consider only finite, simple graphs free from self-loops.

Let $n \geq 1$ be an integer. An *n-tuple* $(a_1, a_2, \cdots, a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, \cdots, a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. Note that H_n is a group under coordinate wise multiplication, and the order of H_n is 2^m, where $m = \lceil \frac{n}{2} \rceil$.

A *symmetric n-sigraph (symmetric n-marked graph)* is an ordered pair $S_n = (G, \sigma)$ $(S_n = (G, \mu))$, where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \to H_n$ $(\mu : V \to H_n)$ is a function. Generally, a *Smarandachely symmetric n-sigraph* (*Smarandachely*

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symmetric n-marked graph) for a subgraph $H \prec G$ is such a graph that $G - E(H)$ is *symmetric n-sigraph (symmetric n-marked graph)*. For example, let H be a path $P_2 \succ G$ or a claw $K_{1,3} \prec G$. Certainly, if $H = \emptyset$, a Smarandachely symmetric n-sigraph (or Smarandachely symmetric n-sigraph) is nothing else but a symmetric n-sigraph (or symmetric n-marked graph).

In this paper by an *n-tuple/n-sigraph/n-marked graph* we always mean a symmetric n-tuple/symmetric n-sigraph/symmetric n-marked graph.

An n-tuple $(a_1, a_2, \cdots, a_n)$ is the *identity n-tuple*, if $a_k = +$, for $1 \leq k \leq n$, otherwise it is a *non-identity n-tuple*. In an n-sigraph $S_n = (G, \sigma)$ an edge labelled with the identity n-tuple is called an *identity edge*, otherwise it is a *non-identity edge*.

Further, in an n-sigraph $S_n = (G, \sigma)$, for any $A \subseteq E(G)$ the *n-tuple* $\sigma(A)$ is the product of the n-tuples on the edges of A.

In [11], the authors defined two notions of balance in n-sigraph $S_n = (G, \sigma)$ as follows (See also R. Rangarajan and P.S.K.Reddy [7]):

Definition 1.1 *Let $S_n = (G, \sigma)$ be an n-sigraph. Then,*

(i) S_n is identity balanced *(or* i-balanced*), if product of n-tuples on each cycle of S_n is the identity n-tuple, and*

(ii) S_n is balanced*, if every cycle in S_n contains an even number of non-identity edges.*

Notice that an i-balanced n-sigraph need not be balanced and conversely. The following characterization of i-balanced n-sigraphs is obtained in [11].

Proposition 1.1 (E. Sampathkumar et al. [11]) *An n-sigraph $S_n = (G, \sigma)$ is i-balanced if, and only if, it is possible to assign n-tuples to its vertices such that the n-tuple of each edge uv is equal to the product of the n-tuples of u and v.*

Let $S_n = (G, \sigma)$ be an n-sigraph. Consider the n-marking μ on vertices of S_n defined as follows: each vertex $v \in V$, $\mu(v)$ is the n-tuple which is the product of the n-tuples on the edges incident with v. *Complement* of S_n is an n-sigraph $\overline{S_n} = (\overline{G}, \sigma^c)$, where for any edge $e = uv \in \overline{G}$, $\sigma^c(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S_n}$ as defined here is an i-balanced n-sigraph due to Proposition 1 in [11].

In [11], the authors also have defined switching and cycle isomorphism of an n-sigraph $S_n = (G, \sigma)$ as follows: (See also [5, 8C10, 13C23]).

Let $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$, be two n-sigraphs. Then S_n and S'_n are said to be *isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that if uv is an edge in S_n with label $(a_1, a_2, \cdots, a_n)$ then $\phi(u)\phi(v)$ is an edge in S'_n with label $(a_1, a_2, \cdots, a_n)$.

Given an n-marking μ of an n-sigraph $S_n = (G, \sigma)$, *switching* S_n with respect to μ is the operation of changing the n-tuple of every edge uv of S_n by $\mu(u)\sigma(uv)\mu(v)$. The n-sigraph obtained in this way is denoted by $\mathcal{S}_\mu(S_n)$ and is called the *μ-switched n-sigraph* or just *switched n-sigraph*.

Further, an n-sigraph S_n *switches* to n-sigraph S'_n (or that they are *switching equivalent* to each other), written as $S_n \sim S'_n$, whenever there exists an n-marking of S_n such that $\mathcal{S}_\mu(S_n) \cong S'_n$.

Two n-sigraphs $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ are said to be *cycle isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that the n-tuple $\sigma(C)$ of every cycle C in S_n equals to the n-tuple $\sigma(\phi(C))$ in S'_n.

We make use of the following known result (see [11]).

Proposition 1.2 (E. Sampathkumar et al. [11]) *Given a graph G, any two n-sigraphs with G as underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

§2. Gallai n-Sigraphs

The Gallai graph $\mathcal{GL}(G)$ of a graph $G = (V, E)$ is the graph whose vertex-set $V(\mathcal{GL}(G)) = E(G)$ and two distinct vertices e_1 and e_2 are adjacent in $\mathcal{GL}(G)$ if e_1 and e_2 are incident in G, but do not span a triangle in G (see [4]). In fact, this concept was introduced by Gallai [2] in his examination of comparability graphs and this notation was suggested by Sun [24]. The author Sun wasted Gallai graphs $\mathcal{GL}(G)$ to characterize a nice class of perfect graphs. Gallai graphs are also wasted in the polynomial time algorithm to determinate complete bipartite $K_{1,3}$-free perfect graphs by the authors Chvátal and Sbihi [1].

Motivated by the existing definition of complement of an n-sigraph, we extend the notion of Gallai graphs to n-sigraphs as follows:

The *Gallai n-sigraph* $\mathcal{GL}(S_n)$ of an n-sigraph $S_n = (G, \sigma)$ is an n-sigraph whose underlying graph is $\mathcal{GL}(G)$ and the n-tuple of any edge uv in $\mathcal{GL}(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n and similarly, the *Smarandachely Gallai symmetric n-sigraph* on s subgraph $H \prec G$ is the Gallai Smarandachely symmetric n-sigrpah on H. Further, an n-sigraph $S_n = (G, \sigma)$ is called Gallai n-sigraph if $S_n \cong \mathcal{GL}(S'_n)$ for some n-sigraph S'_n. The following result indicates the limitations of the notion $\mathcal{GL}(S_n)$ as introduced above, since the entire class of i-unbalanced n-sigraphs is forbidden to be Gallai n-sigraphs.

Proposition 2.1 *For any n-sigraph $S_n = (G, \sigma)$, its Gallai n-sigraph $\mathcal{GL}(S_n)$ is i-balanced.*

Proof Since the n-tuple of any edge uv in $\mathcal{GL}(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n, by Proposition 1.1, $\mathcal{GL}(S_n)$ is i-balanced. □

For any positive integer k, the k^{th} iterated Gallai n-sigraph $\mathcal{GL}(S_n)$ of S_n is defined as

$$(\mathcal{GL})^0(S_n) = S_n, \quad (\mathcal{GL})^k(S_n) = \mathcal{GL}((\mathcal{GL})^{k-1}(S_n)).$$

Corollary 2.1 *For any n-sigraph $S_n = (G, \sigma)$ and any positive integer k, $(\mathcal{GL})^k(S_n)$ is i-balanced.*

In [4], the author characterize the graphs for which $\mathcal{GL}(G) \cong G$.

Theorem 2.1 *Let $G = (V, E)$ be any graph, Gallai graph $\mathcal{GL}(G)$ is isomorphic to G if, and only if, $G \cong C_n$, where $n \geq 4$.*

In view of the above result, we now characterize the n-sigraphs for which Gallai n-sigraph

$\mathcal{GL}(S_n)$ and S_n are switching equivalent.

Theorem 2.2 *For any n-sigraph $S_n = (G, \sigma)$, the Gallai n-sigraph $\mathcal{GL}(S_n)$ and S_n are switching equivalent if, and only if, S_n is i-balanced n-sigraph and G is isomorphic to C_n, where $n \geq 4$.*

Proof Suppose $S_n \sim \mathcal{GL}(S_n)$. This implies, $G \cong \mathcal{GL}(G)$ and hence G is isomorphic to C_n, where $n \geq 4$. Now, if S_n is any n-sigraph with underlying graph as cycle C_n, where $n \geq 4$, Proposition 2.1 implies that $\mathcal{GL}(S_n)$ is i-balanced and hence if S_n is i-unbalanced and its $\mathcal{GL}(S_n)$ being i-balanced can not be switching equivalent to S_n in accordance with Proposition 1.2. Therefore, S_n must be i-balanced.

Conversely, suppose that S_n is an i-balanced n-sigraph and G is isomorphic to C_n, where $n \geq 4$. Then, since $\mathcal{GL}(S_n)$ is i-balanced as per Proposition 2.1 and since $G \cong \mathcal{GL}(G)$, the result follows from Proposition 1.2 again. □

Proposition 2.2 *For any two S_n and S'_n with the same underlying graph, their Gallai n-sigraphs are switching equivalent.*

Now, we characterize Gallai n-sigraphs. The following result characterize n-sigraphs which are Gallai n-sigraphs.

Theorem 2.3 *An n-sigraph $S_n = (G, \sigma)$ is a Gallai n-sigraph if, and only if, S_n is i-balanced n-sigraph and its underlying graph G is a Gallai graph.*

Proof Suppose that S_n is i-balanced and G is a $\mathcal{GL}(G)$. Then there exists a graph H such that $\mathcal{GL}(H) \cong G$. Since S_n is i-balanced, by Proposition 1.1, there exists an n-marking μ of G such that each edge uv in S_n satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the n-sigraph $S'_n = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the n-marking of the corresponding vertex in G. Then clearly, $\mathcal{GL}(S'_n) \cong S_n$. Hence S_n is a Gallai n-sigraph.

Conversely, suppose that $S_n = (G, \sigma)$ is a Gallai n-sigraph. Then there exists an n-sigraph $S'_n = (H, \sigma')$ such that $\mathcal{GL}(S'_n) \cong S_n$. Hence G is the $\mathcal{GL}(G)$ of H and by Proposition 2.1, S_n is i-balanced. □

§3. Anti-Gallai n-Sigraph of a n-Sigraph

The anti-Gallai graph $\mathcal{AGL}(G)$ of a graph $G = (V, E)$ is the graph whose vertex-set $V(\mathcal{AGL}(G)) = E(G)$; two distinct vertices e_1 and e_2 are adjacent in $\mathcal{AGL}(G)$ if e_1 and e_2 are incident in G and lie on a triangle in G (see [4]). Equivalently, the anti-Gallai graph $\mathcal{AGL}(G)$ is the complement of Gallai graph $\mathcal{GL}(G)$ in the line graph $L(G)$. We can easily observe that the Gallai graphs $\mathcal{GL}(G)$ and anti-Gallai graphs $\mathcal{AGL}(G)$ are the spanning subgraphs of the line graph $L(G)$ (See [4] for details).

Motivated by the existing definition of complement of an n-sigraph, we extend the notion of anti-Gallai graphs to n-sigraphs as follows:

The *anti-Gallai n-sigraph* $\mathcal{AGL}(S_n)$ of an n-sigraph $S_n = (G, \sigma)$ is an n-sigraph whose underlying graph is $\mathcal{AGL}(G)$ and the n-tuple of any edge uv is $\mathcal{AGL}(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n. Similarly, the *Smarandachely anti-Gallai n-sigraph* of a Smarandachely n-sigraph $S_n = (G, \sigma)$ on $H \prec G$ is the anti-Gallai n-sigraph of the Smarandachely n-sigraph on H. Further, an n-sigraph $S_n = (G, \sigma)$ is called anti-Gallai n-sigraph, if $S_n \cong \mathcal{AGL}(S'_n)$ for some n-sigraph S'_n. The following result indicates the limitations of the notion $\mathcal{AGL}(S_n)$ as introduced above, since the entire class of i-unbalanced n-sigraphs is forbidden to be anti-Gallai n-sigraphs.

Proposition 3.1 *For any n-sigraph $S_n = (G, \sigma)$, its anti-Gallai n-sigraph $\mathcal{AGL}(S_n)$ is i-balanced.*

Proof Since the n-tuple of any edge uv in $\mathcal{AGL}(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n, by Proposition 1.1, $\mathcal{AGL}(S_n)$ is i-balanced. □

For any positive integer k, the k^{th} iterated anti-Gallai n-sigraph $\mathcal{AGL}(S_n)$ of S_n is defined to be

$$(\mathcal{AGL})^0(S_n) = S_n, \quad (\mathcal{AGL})^k(S_n) = \mathcal{AGL}((\mathcal{AGL})^{k-1}(S_n)).$$

Corollary 3.1 *For any n-sigraph $S_n = (G, \sigma)$ and any positive integer k, $(\mathcal{AGL})^k(S_n)$ is i-balanced.*

In [4], the author characterize the graphs for which $\mathcal{AGL}(G) \cong G$.

Theorem 3.1 *Let $G = (V, E)$ be any graph, anti-Gallai graph $\mathcal{AGL}(G)$ is isomorphic to G if, and only if $G \cong K_3$.*

In view of the above result, we now characterize the n-sigraphs for which anti-Gallai n-sigraph $\mathcal{AGL}(S)$ and S are switching equivalent.

Theorem 3.2 *For any n-sigraph $S_n = (G, \sigma)$, the anti-Gallai signed graph $\mathcal{AGL}(S_n)$ and S are switching equivalent if, and only if, S_n is i-balanced and G is isomorphic to K_3.*

Proof Suppose $S_n \sim \mathcal{AGL}(S_n)$. This implies, $G \cong \mathcal{AGL}(G)$ and hence G is isomorphic to K_3. Now, if S_n is any n-sigraph with underlying graph as C_3, Proposition 2.1 implies that $\mathcal{AGL}(S_n)$ is i-balanced and hence if S_n is i-unbalanced and its $\mathcal{AGL}(S_n)$ being i-balanced can not be switching equivalent to S_n in accordance with Proposition 1.2. Therefore, S_n must be i-balanced.

Conversely, suppose that S_n is an i-balanced n-sigraph and G is isomorphic to C_3. Then, since $\mathcal{AGL}(S_n)$ is i-balanced as per Proposition 3 and since $G \cong \mathcal{AGL}(G)$, the result follows from Proposition 1.2 again. □

Proposition 3.2 *For any two S_n and S'_n with the same underlying graph, their anti-Gallai n-sigraphs are switching equivalent.*

Now, we characterize Gallai n-sigraphs. The following result characterize n-sigraphs which are Gallai n-sigraphs.

Theorem 3.3 *An n-sigraph $S_n = (G, \sigma)$ is an anti-Gallai n-sigraph if, and only if, S_n is i-balanced n-sigraph and its underlying graph G is an anti-Gallai graph.*

Proof Suppose that S_n is i-balanced and G is a $\mathcal{AGL}(G)$. Then there exists a graph H such that $\mathcal{AGL}(H) \cong G$. Since S_n is i-balanced, by Proposition 1.1, there exists an n-marking μ of G such that each edge uv in S_n satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the n-sigraph $S'_n = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the n-marking of the corresponding vertex in G. Then clearly, $\mathcal{AGL}(S'_n) \cong S_n$. Hence S_n is an anti-Gallai n-sigraph.

Conversely, suppose that $S_n = (G, \sigma)$ is an anti-Gallai n-sigraph. Then there exists an n-sigraph $S'_n = (H, \sigma')$ such that $\mathcal{AGL}(S'_n) \cong S_n$. Hence G is the $\mathcal{AGL}(G)$ of H and by Proposition 2.1, S_n is i-balanced. $\square$

We now characterize n-sigraphs whose Gallai n-sigraphs and anti-Gallai n-sigraphs are switching equivalent. In case of graphs the following result is due to Palathingal and Aparna Lakshmanan [6].

Theorem 3.4 *For any graph $G = (V, E)$, the graphs $\mathcal{GL}(G)$ and $\mathcal{AGL}(G)$ are isomorphic if, and only if, G is $nK_3 \cup nK_{1,3}$.*

Theorem 3.5 *For any n-sigraph $S_n = (G, \sigma)$, $\mathcal{GL}(S_n) \sim \mathcal{AGL}(S_n)$ if, and only if, G is $nK_3 \cup nK_{1,3}$.*

Proof Suppose $\mathcal{GL}(S_n) \sim \mathcal{AGL}(S_n)$. This implies, $\mathcal{GL}(G) \cong \mathcal{AGL}(G)$ and hence by Theorem 3.4, we see that the graph G must be isomorphic to $nK_3 \cup nK_{1,3}$.

Conversely, suppose that G is isomorphic to $nK_3 \cup nK_{1,3}$. Then $\mathcal{GL}(G) \cong \mathcal{AGL}(G)$ by Theorem 3.4. Now, if S_n is an n-sigraph with underlying graph as $nK_3 \cup nK_{1,3}$, by Propositions 2.1 and 3.1, $\mathcal{GL}(S_n)$ and $\mathcal{GL}(S_n)$ are i-balanced. The result follows from Proposition 1.2. $\square$

§4. Complementation

In this section, we investigate the notion of complementation of a graph whose edges have signs (a *sigraph*) in the more general context of graphs with multiple signs on their edges. We look at two kinds of complementation: complementing some or all of the signs, and reversing the order of the signs on each edge.

For any $m \in H_n$, the *m-complement* of $a = (a_1, a_2, \cdots, a_n)$ is: $a^m = am$. For any $M \subseteq H_n$, and $m \in H_n$, the *m-complement* of M is $M^m = \{a^m : a \in M\}$.

For any $m \in H_n$, the *m-complement* of an n-sigraph $S_n = (G, \sigma)$, written (S_n^m), is the same graph but with each edge label $a = (a_1, a_2, \cdots, a_n)$ replaced by a^m.

For an n-sigraph $S_n = (G, \sigma)$, the $\mathcal{GL}(S_n)$ ($\mathcal{AGL}(S_n)$) is i-balanced. We now examine, the condition under which m-complement of $\mathcal{GL}(S_n)$ is i-balanced, where for any $m \in H_n$.

Proposition 4.1 *Let $S_n = (G, \sigma)$ be an n-sigraph. Then, for any $m \in H_n$, if $\mathcal{GL}(G)$ ($\mathcal{AGL}(G)$) is bipartite then $(\mathcal{GL}(S_n))^m$ ($(\mathcal{AGL}(S_n))^m$) is i-balanced.*

Proof Since, by Proposition 2.1 (Proposition 3.1), $\mathcal{GL}(S_n)$ ($\mathcal{AGL}(S_n)$) is i-balanced, for each k,

$1 \leq k \leq n$, the number of n-tuples on any cycle C in $\mathcal{GL}(S_n)$ ($\mathcal{AGL}(S_n)$) whose k^{th} co-ordinate are $-$ is even. Also, since $\mathcal{GL}(G)$ ($\mathcal{AGL}(G)$) is bipartite, all cycles have even length; thus, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in $\mathcal{GL}(S_n)$ ($\mathcal{AGL}(S_n)$) whose k^{th} co-ordinate are $+$ is also even. This implies that the same thing is true in any m-complement, where for any $m, \in H_n$. Hence $(\mathcal{GL}(S_n))^t$ $((\mathcal{AGL}(S_n))^t)$ is i-balanced. $\square$

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Open Neighborhood Coloring of a Generalized Antiprism Graph

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Abstract: An open neighborhood coloring of a graph is a coloring in which vertices adjacent with a common vertex are colored differently. The minimum number of colors used in an open neighborhood coloring of a graph G is called the open neighborhood chromatic number of G. We determine this parameter for a generalization of the antiprism graph in this paper.

Key Words: Coloring, chromatic number, open neighborhood coloring, Smarandachely open neighborhood coloring, antiprism.

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§1. Introduction

A *vertex coloring* or simply, a *coloring* of a graph $G = (V, E)$ is an assignment of colors to the vertices of G. A k-coloring of G is a surjection $c : V \to \{1, 2, \cdots, k\}$. A proper coloring of G is an assignment of colors to the vertices of G so that adjacent vertices are colored differently. A *proper k-coloring* of G is a surjection $c : V \to \{1, 2, \cdots, k\}$ such that $c(u) \neq c(v)$ if u and v are adjacent in G. The minimum k for which there is a proper k-coloring of G is called the *chromatic number* of G denoted $\chi(G)$.

An *open neighborhood coloring* [5] of a graph is a coloring in which vertices adjacent with a common vertex are colored differently. In other words, an open neighborhood coloring of a graph $G(V, E)$ is a coloring $c : V \to Z^+$ such that for each $w \in V$ and every $u, v \in N(w)$, $c(u) \neq c(v)$ and generally, for a subgraph Γ such as $P_2, K_{1,3}$ of G if there is an open neighborhood coloring c on graph $G - \Gamma$, G is said to be a *Smarandachely open neighborhood coloring* on Γ. Clearly, if $\Gamma = \emptyset$, a Smarandachely open neighborhood coloring of G is nothing else but an open neighborhood coloring of G. The minimum number of colors used in an open neighborhood coloring of a graph G is called the *open neighborhood chromatic number* of G, denoted $\chi_{onc}(G)$.

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The concept of open neighborhood coloring was introduced in the year 2013 by Geetha *et* al. [5]. Further, in [6], this parameter has been obtained for the Prism graph which is a particular case of the generalised Petersen graph $GP(n, k)$. In [8, 9], the open neighborhood chromatic number has been obtained for the class of antiprism graphs and some path related graphs such as line graph, total graph and transformation graphs of a path.

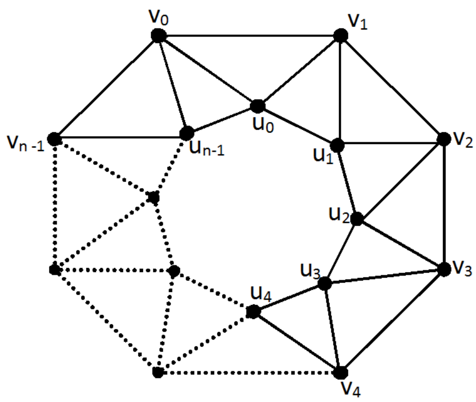

Figure 1. n-Antiprism graph

The graph obtained by replacing the faces of a polyhedron with its edges and vertices is called the *skeleton* [3] of the polyhedron. An *n-antiprism* [2], $n \geq 3$, is a semiregular polyhedron constructed with $2n$-gons and $2n$ triangles. It is made up of two n-gons on top and bottom, separated by a ribbon of $2n$ triangles, with the two n-gons being offset by one ribbon segment. The graph corresponding to the skeleton of an n-antiprism is called the *$n-$antiprism graph*, denoted by Q_n, $n \geq 3$ as shown in Figure 1. As seen from this figure, Q_n has $2n$ vertices and $4n$ edges, and is isomorphic to the circulant graph $\mathrm{Ci}_{2n}(1, 2)$.

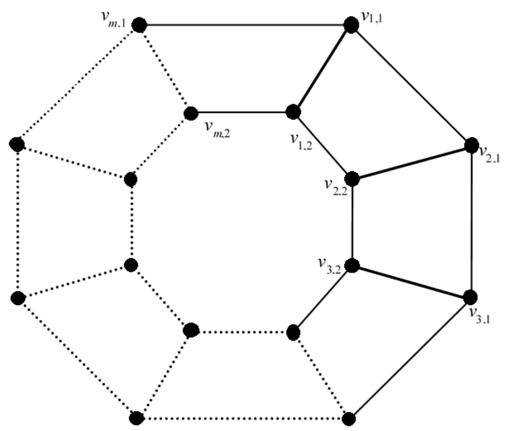

Figure 2. Prism graph

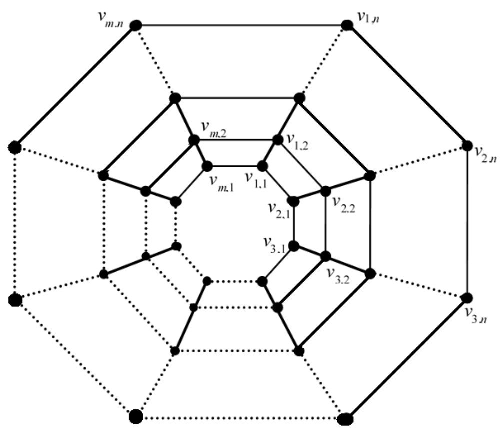

Figure 3. Generalized prism graph

A *prism graph* [7] Y_n is a graph corresponding to the skeleton of an n-prism and has $2n$ vertices and $3n$ edges as shown in Figure 2. A *generalized prism* graph [4], denoted $Y_{m,n} = C_m \times P_n$, is the graph having mn vertices and $m(2n-1)$ edges as shown in Figure 3. Its vertex set is given by $V(Y_{m,n}) = V(C_m \times P_n) = \{v_{i,j} : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$ and its edge set $E(Y_{m,n}) = E(C_m \times P_n) = \{v_{i,j}v_{i+1,j} : 1 \leqslant i \leqslant m-1, 1 \leqslant j \leqslant n\} \bigcup \{v_{m,j}v_{1,j} : 1 \leqslant j \leqslant n\} \bigcup \{v_{i,j}v_{i,j+1} : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n-1\}$.

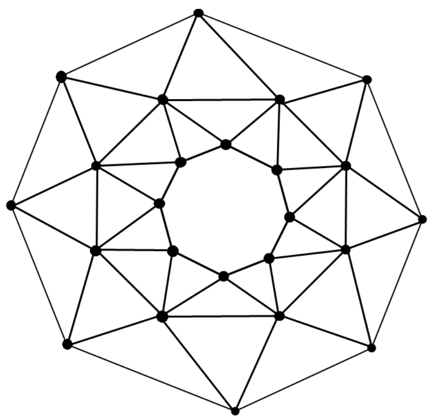

Figure 4. Generalized antiprism graph A_8^3

As introduced in [1], the *generalized antiprism* A_m^n can be obtained by completing the generalized prism $C_m \times P_n$ by edges $\{v_{i,j+1}v_{i+1,j} : 1 \leqslant i \leqslant m-1, 1 \leqslant j \leqslant n-1\} \bigcup \{v_{m,j+1}v_{1,j} : 1 \leqslant j \leqslant n-1\}$ where $V(A_m^n) = V(C_m \times P_n) = \{v_{i,j} : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$ is the vertex set of A_m^n. Thus, $E(A_m^n) = E(C_m \times P_n) \bigcup \{v_{i,j+1}v_{i+1,j} : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n-1\}$ is the edge set of A_m^n, where i is taken modulo m. In particular, if $n = 2$, we obtain the antiprism graph Q_m. For example, the graph A_8^3 is as shown in Figure 4.

§2. Open Neighborhood Coloring of A_m^3

In this section, we obtain the open neighborhood chromatic number of the generalized antiprism graph A_m^n for $m \geqslant 3$ and $n = 3$. Using this result, we determine the open neighborhood chromatic number of the generalized antiprism graph A_m^n for any $m \geqslant 3, n \geqslant 2$ in the following section.

To begin with, we recall some of the important definitions and results obtained by various authors for immediate reference.

Theorem 2.1 ([5]) *If f is an open neighborhood k-coloring of $G(V, E)$ with $\chi_{onc}(G) = k$, then $f(u) \neq f(v)$ holds where u, v are the end vertices of a path of length 2 in G.*

Theorem 2.2 ([5]) *For any graph $G(V, E)$, $\chi_{onc}(G) \geq \Delta(G)$.*

Theorem 2.3 ([5]) *If H is a connected subgraph of G, then $\chi_{onc}(H) \leq \chi_{onc}(G)$.*

Theorem 2.4 ([5]) *Let $G(V, E)$ be a connected graph on $n \geq 3$ vertices. Then $\chi_{onc}(G) = n$ if and only if $N(u) \bigcap N(v) \neq \emptyset$ holds for every pair of vertices $u, v \in V(G)$.*

Theorem 2.5 ([8]) *For an antiprism graph $Q_n, n \geq 3$,*

$$\chi_{onc}(Q_n) = \begin{cases} 5, & \textit{if } n \equiv 0 (mod\ 5), \\ 7, & \textit{if } n = 7, \\ 8, & \textit{if } n = 4, \\ 6, & \textit{otherwise.} \end{cases}$$

Observation 2.6 For $m \geqslant 3$, $Q_m \subseteq A^3_m$ so that $\chi_{onc}(A^3_m) \geqslant \chi_{onc}(Q_m)$.

Definition 2.7 ([6]) *In a graph G, a subset V_1 of $V(G)$ such that no two vertices of V_1 are end vertices of a path of length two in G is called a P_3-independent set of G.*

Lemma 2.8 *For the generalized antiprism graph $A^3_m, m \geq 3$, $\chi_{onc}(A^3_m) \geq 7$.*

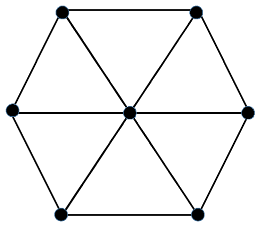

Figure 5. Subgraph H of A^3_m

Proof For each $m \geq 3$, it is easy to observe that A^3_m contains the subgraph H in Figure 5. Further, in H, there is a path of length two between every pair of vertices so that $\chi_{onc}(H) = 7$. Hence, by Theorem 2.3, $\chi_{onc}(A^3_m) \geq 7$. □

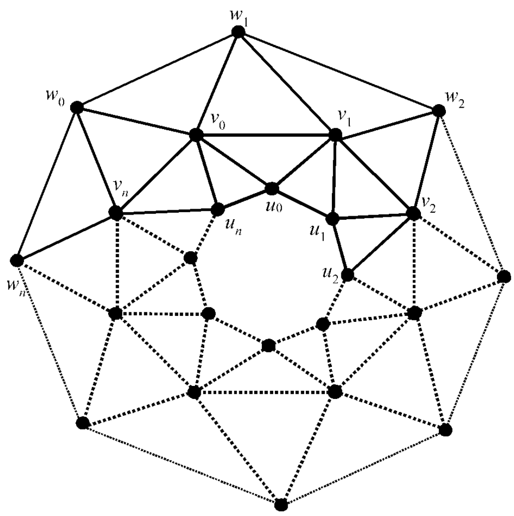

Figure 6. Generalized antiprism graph A^3_m

Observation 2.9 In the generalized antiprism graph A_m^3 as shown in Figure 6,

(i) The only vertices that are connected to a vertex $u_i, 0 \le i \le m-1$ by a path of length two are $u_{i\pm1}, u_{i\pm2}, v_i, v_{i\pm1}, v_{i+2}, w_i, w_{i+1}, w_{i+2}$ where the suffix is under modulo m;

(ii) The only vertices that are connected to a vertex $v_i, 0 \le i \le m-1$ by a path of length two are $u_i, u_{i\pm1}, u_{i-2}, v_{i\pm1}, v_{i\pm2}, w_{i\pm1}, w_i, w_{i+2}$ where the suffix is under modulo m;

(iii) The only vertices that are connected to a vertex $w_i, 0 \le i \le m-1$ by a path of length two are $u_i, u_{i-1}, u_{i-2}, v_{i\pm1}, v_i, v_{i-2}, w_{i\pm1}, w_{i\pm2}$ where the suffix is under modulo m.

Lemma 2.10 *For the generalized antiprism graph* A_m^3,

$$\chi_{onc}(A_m^3) = \begin{cases} 9, & \textit{if } m = 3, 6, \\ 8, & \textit{if } m = 4, 5. \end{cases}$$

Proof We prove the result by considering the following cases.

Case 1. $m = 3$.

It is easy to observe that, in A_3^3, every vertex is connected to every other vertex by a path of length two. Thus, in any open neighborhood coloring of A_3^3, every vertex has to be given a different color so that $\chi_{onc}(A_3^3) = 9$.

Case 2. $m = 4$.

By Observation 2.6 and Theorem 2.5, we have $\chi_{onc}(A_4^3) \geqslant \chi_{onc}(Q_4) = 8$. The reverse inequality can be established from Figure 7. Hence, $\chi_{onc}(A_4^3) = 8$.

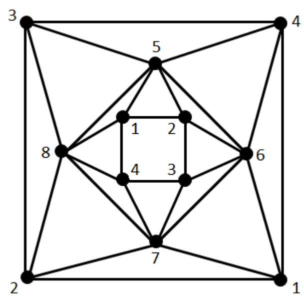

Figure 7. An open neighborhood coloring of the graph A_4^3

Case 3. $m = 5, 6$.

By observation, it is seen that a color can be assigned to not more than two vertices in any open neighborhood coloring of A_5^3 and A_6^3 so that $\chi_{onc}(A_5^3) \geqslant 8$ and $\chi_{onc}(A_6^3) \geqslant 9$. Further, the open neighborhood coloring of A_5^3 and A_6^3 as shown in Figure 8 ensure that $\chi_{onc}(A_5^3) \leqslant 8$ and $\chi_{onc}(A_6^3) \leqslant 9$.

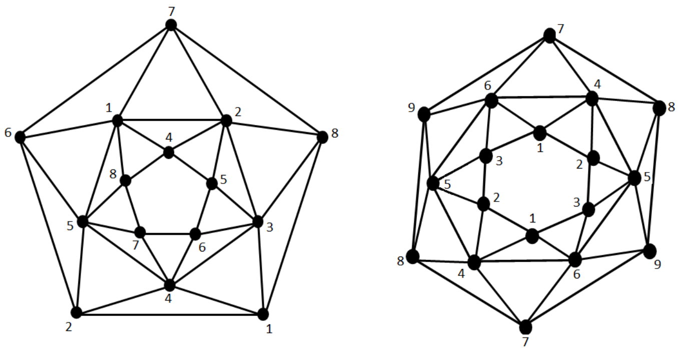

Figure 8. An open neighborhood coloring of the graphs A_5^3 and A_6^3

This completes the proof. $\square$

Lemma 2.11 *In the generalized antiprism graph* $A_m^3, m \geqslant 7$*, for each* l*,* $0 \leq l \leq 6$*, the set* $S_l = \{u_i, v_j, w_k |\ i \equiv l(mod\ 7), j \equiv l + 3(mod\ 7) and\ k \equiv l - 1(mod\ 7)\}$ *is a* P_3- *independent set if and only if* $m \equiv 0(mod\ 7)$*.*

Proof Let $m \equiv 0(mod\ 7)$. It is given that $S_l = \{u_i, v_j, w_k |\ i \equiv l(mod\ 7), j \equiv l + 3(mod\ 7)\ and\ k \equiv l - 1(mod\ 7)\}$. It is easy to observe that each $S_i, 0 \leqslant i \leqslant 6$, is a P_3-independent set of A_m^3.

We prove the converse by the method of contraposition. Suppose that $m \not\equiv 0(mod\ 7)$. Then, we have the following cases.

Case 1. $m \equiv 1(mod\ 7)$. Then, $u_0, u_{m-1} \in S_0$. In such a case, S_0 is not a P_3- independent set as u_0 and u_{m-1} are end vertices of a path of length two.

Case 2. $m \equiv 2(mod\ 7)$. Then, $u_0, u_{m-2} \in S_0$. But u_0 and u_{m-2} are end vertices of a path of length two so that S_0 is not a P_3- independent set.

The other cases follow similarly. $\square$

Theorem 2.12 *For the generalized antiprism graph* $A_m^3, m \geqslant 3$*,* $\chi_{onc}(A_m^3) = 7$ *if and only if* $m \equiv 0(mod\ 7)$*.*

Proof Suppose $m \equiv 0(mod\ 7)$. From Lemma 2.8, we have $\chi_{onc}(A_m^3) \geqslant 7$. Further, by Lemma 2.11, each $S_l = \{u_i, v_j, w_k |\ i \equiv l(mod\ 7), j \equiv l + 3(mod\ 7) and\ k \equiv l - 1(mod\ 7)\}, 0 \leqslant l \leqslant 6$ is a P_3-independent set of A_m^3.

Define a function $c : V(A_m^3) \to \{1, 2, \cdots, 7\}$ as $c(v) = l + 1$ such that $v \in S_l, 0 \leqslant l \leqslant 6$. Clearly, c is an open neighborhood coloring of A_m^3 so that $\chi_{onc}(A_m^3) \leqslant 7$. Thus, $\chi_{onc}(A_m^3) = 7$ if $m \equiv 0(mod\ 7)$.

Conversely, let $\chi_{onc}(A_m^3) = 7$. In view of Theorem 2.5 and Lemma 2.10, we have $m \geqslant 7$. By observation, in the graph A_m^3, none of the vertices $v_{i-1}, v_i, v_{i+1}, u_{i-1}, u_i, w_i, w_{i+1}$, the suffix taken under modulo 7, can be given the same color in an open neighborhood coloring. Thus, if $m \not\equiv 0(mod\ 7)$, then seven colors are not sufficient to have an open neighborhood coloring of A_m^3 so that $m \equiv 0(mod\ 7)$. $\square$

Observation 2.13 Every integer $m \geqslant 11$, with $m \not\equiv 0(mod\ 7), m \neq 13, 17$ can be written as $m = 4k + 7l$ for integers $k \geqslant 1, l \geqslant 0$.

Theorem 2.14 *For* $m \geqslant 3$,

$$\chi_{onc}(A^3_m) = \begin{cases} 9, & \textit{if } m = 3, 6, \\ 7, & \textit{if } m \equiv 0 (mod\ 7), \\ 8, & \textit{otherwise.} \end{cases}$$

Proof The result holds for $m \leqslant 6$ and $m \equiv 0(mod\ 7)$ from Lemma 2.10 and Theorem 2.12. We now consider the case when $m \geqslant 3$ is an integer such that $m \neq 3, 6$ and $m \not\equiv 0(mod\ 7)$. In view of Theorem 2.12, $\chi_{once}(A^3_m) \geqslant 8$. Hence, it suffices to show that, in this case, $\chi_{once}(A^3_m) \leqslant 8$ for which we take various cases as follows.

Case 1. $m = 8$. Consider the coloring $c : V(A^3_8) \to \{1, 2, \cdots, 8\}$ defined by $c(v) = i + 1$ for $v = u_i, v = v_{i+6}$ or $v = w_{i+3}$ for $0 \leqslant i \leqslant 7$, the suffix taken under modulo 8. It is easy to verify that c is an open neighborhood coloring of A^3_8 using eight colors so that $\chi_{once}(A^3_8) \leqslant 8$.

Case 2. $m = 9$. Observing that the coloring of A^3_9 in Figure 9 is an open neighborhood coloring, $\chi_{once}(A^3_9) \leqslant 8$.

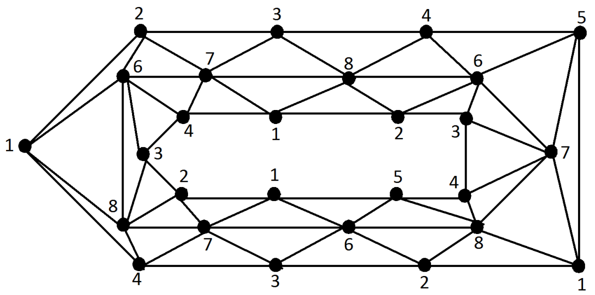

Figure 9. An open neighborhood coloring of A^3_9

Case 3. As seen from Figure 10, A^3_{10} can be colored with eight colors in an open neighborhood coloring so that $\chi_{once}(A^3_{10}) \leqslant 8$.

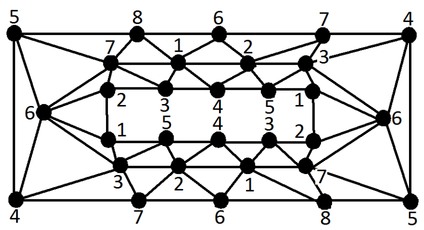

Figure 10. An open neighborhood coloring of A^3_{10}

Case 4. For $m = 13$ and $m = 17$, the coloring patterns are similar to Case 2.

Case 5. m is any other integer. Then, by Observation 2.13, $m = 4k + 7l$ for some integers $k \geqslant 1, l \geqslant 0$. Define a coloring $c : V(A_m^3) \to \{1,2,3,4,5,6,7,8\}$ as

$$c(u_i) = \begin{cases} 1, & \text{if } i \equiv 0 (\text{mod } 4) \text{ and } 0 \leq i \leq 4k-1 \text{ or } i - 4k \equiv 0 (\text{mod } 7) \text{ and } 4k \leq i \leq m-1 \\ 2, & \text{if } i \equiv 1 (\text{mod } 4) \text{ and } 0 \leq i \leq 4k-1 \text{ or } i - 4k \equiv 1 (\text{mod } 7) \text{ and } 4k \leq i \leq m-1 \\ 3, & \text{if } i \equiv 2 (\text{mod } 4) \text{ and } 0 \leq i \leq 4k-1 \text{ or } i - 4k \equiv 2 (\text{mod } 7) \text{ and } 4k \leq i \leq m-1 \\ 4, & \text{if } i \equiv 3 (\text{mod } 4) \text{ and } 0 \leq i \leq 4k-1 \text{ or } i - 4k \equiv 3 (\text{mod } 7) \text{ and } 4k \leq i \leq m-1 \\ 5, & i - 4k \equiv 4 (\text{mod } 7) \text{ and } 4k \leq i \leq m-1 \\ 6, & i - 4k \equiv 5 (\text{mod } 7) \text{ and } 4k \leq i \leq m-1 \\ 7, & \text{otherwise} \end{cases}$$

$$c(v_i) = \begin{cases} 1, & i - 4k - 2 \equiv 3 (\text{mod } 7) \text{ and } 4k+2 \leq i \leq m-1 \\ 2, & i - 4k - 2 \equiv 4 (\text{mod } 7) \text{ and } 4k+2 \leq i \leq m-1 \\ 3, & i - 4k - 2 \equiv 5 (\text{mod } 7) \text{ and } 4k+2 \leq i \leq m-1 \text{ or } i = 0 \\ 4, & i - 4k - 2 \equiv 6 (\text{mod } 7) \text{ and } 4k+2 \leq i \leq m-1 \text{ or } i = 1 \\ 5, & \text{if } i - 2 \equiv 0 (\text{mod } 4) \text{ and } 2 \leq i \leq 4k+1 \\ & \text{or } i - 4k - 2 \equiv 0 (\text{mod } 7) \text{ and } 4k+2 \leq i \leq m-1 \\ 6, & \text{if } i - 2 \equiv 1 (\text{mod } 4) \text{ and } 2 \leq i \leq 4k+1 \\ & \text{or } i - 4k - 2 \equiv 1 (\text{mod } 7) \text{ and } 4k+2 \leq i \leq m-1 \\ 7, & \text{if } i - 2 \equiv 2 (\text{mod } 4) \text{ and } 2 \leq i \leq 4k+1 \\ & \text{or } i - 4k - 2 \equiv 2 (\text{mod } 7) \text{ and } 4k+2 \leq i \leq m-1 \\ 8, & \text{otherwise} \end{cases}$$

and

$$c(w_i) = \begin{cases} 1, & \text{if } i \equiv 3 (\text{mod } 4) \text{ and } 3 \leq i \leq 4k+2 \\ & \text{or } i - 4k \equiv 3 (\text{mod } 7) \text{ and } 4k+3 \leq i \leq m-1 \\ 2, & \text{if } i \equiv 0 (\text{mod } 4) \text{ and } 3 \leq i \leq 4k+2 \\ & \text{or } i - 4k \equiv 4 (\text{mod } 7) \text{ and } 4k+3 \leq i \leq m-1 \\ 3, & \text{if } i \equiv 1 (\text{mod } 4) \text{ and } 3 \leq i \leq 4k+2 \\ & \text{or } i - 4k \equiv 5 (\text{mod } 7) \text{ and } 4k+3 \leq i \leq m-1 \\ 4, & \text{if } i \equiv 2 (\text{mod } 4) \text{ and } 3 \leq i \leq 4k+2 \\ & \text{or } i - 4k \equiv 6 (\text{mod } 7) \text{ and } 4k+3 \leq i \leq m-1 \\ 5, & i - 4k \equiv 0 (\text{mod } 7) \text{ and } 4k+3 \leq i \leq m-1 \text{ or } i = 0 \\ 6, & i - 4k \equiv 1 (\text{mod } 7) \text{ and } 4k+3 \leq i \leq m-1 \text{ or } i = 1 \\ 7, & \text{otherwise.} \end{cases}$$

It is easy to verify that c is an open neighborhood coloring of A_m^3 so that $\chi_{onc}(A_m^3) \leqslant 8$. $\square$

§3. Open Neighborhood Coloring of A_m^n

We recall that the *generalized antiprism* graph A_m^n is obtained by completing the generalized prism $C_m \times P_n$ by edges $\{v_{i,j+1}v_{i+1,j} : 1 \leqslant i \leqslant m-1, 1 \leqslant j \leqslant n-1\} \bigcup \{v_{m,j+1}v_{1,j} : 1 \leqslant j \leqslant n-1\}$ where $V(A_m^n) = V(C_m \times P_n) = \{v_{i,j} : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$ is the vertex set of A_m^n. Thus, $E(A_m^n) = E(C_m \times P_n) \bigcup \{v_{i,j+1}v_{i+1,j} : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n-1\}$ is the edge set of A_m^n, where i is taken modulo m.

Observation 3.1 For $n_2 \geqslant n_1 \geqslant 2$ and $m \geqslant 3$, $Q_m = A_m^2 \subseteq A_m^{n_1} \subseteq A_m^{n_2}$ so that $\chi_{onc}(Q_m) \leqslant \chi_{onc}(A_m^{n_1}) \leqslant \chi_{onc}(A_m^{n_2})$.

Lemma 3.2 *For the antiprism graph A_3^n and A_6^n, $n \geqslant 3$, $\chi_{onc}(A_3^n) = \chi_{onc}(A_6^n) = 9$.*

Proof By Lemma 2.10, we have $\chi_{onc}(A_3^3) = \chi_{onc}(A_6^3) = 9$. Thus, By Observation 3.1, for $n \geqslant 3, \chi_{onc}(A_3^n) = \chi_{onc}(A_6^n) \geqslant 9$.

To establish the reverse inequality, for $m = 3$ or 6, define a function $c : V(A_m^n) \to \{1, 2, \cdots, 9\}$ as $c(_{i,j}) = l$ with $i \equiv h(mod\ 3)$ & $j \equiv k(mod\ 3)$ and l corresponding to the respective hk^{th} entry in the following table.

$h \setminus k$	0	1	2
0	9	3	6
1	7	1	4
2	8	2	5

Table 1.

It is easy to verify that the above coloring is an open neighborhood coloring of A_m^n. Hence, $\chi_{onc}(A_m^n) \leqslant 9$.

To conclude, $\chi_{onc}(A_3^n) = \chi_{onc}(A_6^n) = 9$ for $n \geqslant 3$. □

Theorem 3.3 *For any integers $m \geqslant 3, n \geqslant 2$,*

$$\chi_{onc}(A_m^n) = \begin{cases} 7, & \text{if } m \equiv 0(mod\ 7), \\ 5, & \text{if } n = 2 \text{ and } m \equiv 0(mod\ 5), \\ 6, & \text{if } n = 2 \text{ and } m \neq 4, \\ 9, & \text{if } n \geqslant 3 \text{ and } m = 3, 6, \\ 8, & \text{otherwise.} \end{cases}$$

Proof In view of Theorems 2.5 and 2.14, the result holds for $n = 2$ and $n = 3$. For $n \geqslant 4$, we consider the following cases.

Case 1. For $m = 3, 6$, the result follows from Lemma 3.2.

Case 2. For $m = 5, 9, 10, 13, 17$, following the respective coloring patterns as in Lemma 2.10 and Theorem 2.14 yields an open neighborhood coloring with eight colors.

Case 3. For any other $m \geqslant 3$, following Observation 3.1 and Theorem 2.14, we see that $\chi_{onc}(A_m^n) \geqslant 8$. To prove the reverse inequality, define the coloring $c : V(A_m^n) \to \{1, 2, \cdots, 8\}$ as

$$(i)\quad c(v_{i,1}) = \begin{cases} 1, & \text{if } i \equiv 1 (\text{mod } 4) \text{ and } 0 \leq i \leq 4k \text{ or } i - 4k \equiv 1 (\text{mod } 7) \text{ and } 4k+1 \leq i \leq m \\ 2, & \text{if } i \equiv 2 (\text{mod } 4) \text{ and } 0 \leq i \leq 4k \text{ or } i - 4k \equiv 2 (\text{mod } 7) \text{ and } 4k+1 \leq i \leq m \\ 3, & \text{if } i \equiv 3 (\text{mod } 4) \text{ and } 0 \leq i \leq 4k \text{ or } i - 4k \equiv 3 (\text{mod } 7) \text{ and } 4k+1 \leq i \leq m \\ 4, & \text{if } i \equiv 0 (\text{mod } 4) \text{ and } 0 \leq i \leq 4k \text{ or } i - 4k \equiv 4 (\text{mod } 7) \text{ and } 4k+1 \leq i \leq m \\ 5, & i - 4k \equiv 5 (\text{mod } 7) \text{ and } 4k+1 \leq i \leq m \\ 6, & i - 4k \equiv 6 (\text{mod } 7) \text{ and } 4k+1 \leq i \leq m \\ 7, & \text{otherwise} \end{cases}$$

$$(ii)\quad c(v_{i,2}) = \begin{cases} 1, & i - 4k - 2 \equiv 4 (\text{mod } 7) \text{ and } 4k+3 \leq i \leq m \\ 2, & i - 4k - 2 \equiv 5 (\text{mod } 7) \text{ and } 4k+3 \leq i \leq m \\ 3, & i - 4k - 2 \equiv 6 (\text{mod } 7) \text{ and } 4k+3 \leq i \leq m \text{ or } i = 1 \\ 4, & i - 4k - 2 \equiv 0 (\text{mod } 7) \text{ and } 4k+3 \leq i \leq m \text{ or } i = 2 \\ 5, & \text{if } i \equiv 3 (\text{mod } 4) \text{ and } 3 \leq i \leq 4k+2 \text{ or } i - 4k \equiv 3 (\text{mod } 7) \text{ and } 4k+3 \leq i \leq m \\ 6, & \text{if } i \equiv 0 (\text{mod } 4) \text{ and } 3 \leq i \leq 4k+2 \text{ or } i - 4k \equiv 4 (\text{mod } 7) \text{ and } 4k+3 \leq i \leq m \\ 7, & \text{if } i \equiv 1 (\text{mod } 4) \text{ and } 3 \leq i \leq 4k+2 \text{ or } i - 4k \equiv 5 (\text{mod } 7) \text{ and } 4k+3 \leq i \leq m \\ 8, & \text{otherwise} \end{cases}$$

and (iii) $c(v_{i,j}) = c(v_{i-3,j-2})$ for $j \geqslant 3$ where i is taken under modulo m.

An illustration of the coloring c for the graph A_{29}^5 is given in Figure 11.

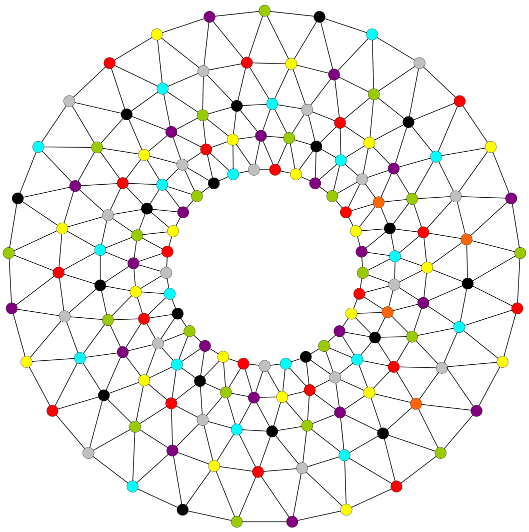

Figure 11. An open neighborhood coloring of A_{29}^5

It is easy to verify that the above coloring is an open neighborhood coloring of A_m^n. Hence, $\chi_{onc}(A_m^n) \leqslant 8$. $\square$

§4. Conclusion

The open neighborhood chromatic number of an antiprism graph Q_n has been determined in [8]. We have obtained this parameter for the generalized antiprism graph A_m^3 in this paper, by means of which, we have solved the problem of finding the open neighborhood chromatic number of A_m^n, $m \geqslant 3, n \geqslant 2$.

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Famous Words

Reality is merely an illusion, albeit a very persistent.

By Albert Einstein, an American theoretical physicist.

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